Fault tolerant on-board networks with priorities

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Abstract

We consider on-board networks in satellites interconnecting entering signals (inputs) to amplifiers (outputs). The connections are made via expensive switches, each of which has four available links. The paths connecting inputs to outputs should be link-disjoint. Some of the input signals, called priorities, must be connected to the amplifiers which provide the best quality of service (that is, to some specific outputs). In practice, amplifiers are prone to fail and the faults cannot be repaired. Therefore, extra outputs have to be built into the network to ensure that every input can be routed to operational outputs. Given three integers, n, p, and f, we would like to design a low cost network (where the network cost is proportional to the total number of switches) such that it is possible to route all n inputs to n operational amplifiers, and to route the p priorities to the p best quality amplifiers for any set of f faulty and p best-quality amplifiers. Let $R(n,p,f)$ be the minimum number of switches of such a network. We prove here that $R(n,p,f) \leq \frac{n+f}{2}\lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f)$ with g a function depending only on f. We then compute $R(n,p,f)$ exactly for a few small values of p and f.

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1 Introduction

Modern telecommunication satellites are very complex to design. Components are often prone to failure, and so providing robustness at the lowest possible cost is an important issue for the

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manufacturers. A key component of telecommunication satellites is an interconnection network which allows one to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. The network is made of expensive switches; so we want to minimize their number subject to the following conditions: each input (respectively output) is adjacent to exactly one link; each switch is adjacent to at most four links; there are n inputs (signals) and $n+f$ outputs (amplifiers); among the $n+f$ outputs, f are allowed to fail (all failures are irreversible). All the input signals should be sent to operational outputs via link-disjoint paths. Designing such networks is a complex problem that was proposed by Alcatel Space Industries and partial solutions are given in [4]and [6]. Alcatel Space Industries proposed also to consider the case where each signal needs a specific amplifier; that leads to design a network realizing any permutation from the inputs to the outputs (see [2, 3]). Recently it posed the following new problem. Out of the total of n inputs, p are called priorities and must be connected to the amplifiers providing the best quality of service (that is, to some specific outputs) and the other signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the best quality service change with the position of the satellite and so for any set of f failed outputs, the network should be able to route the signals to operational outputs such that it routes the p priorities to any given subset of p operational outputs.

This problem can be restated more formally as follows:

Definition 1 An (n, p, f)-*network* G is a graph (V, E) where the vertex set V is partitioned into four subsets P, I, O and S called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:

- there are p priorities, $n-p$ ordinary inputs and $n+f$ outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree at most 4.

An (n, p, f)-network is a *repartitor* if for any disjoint subsets F and Q of O with $|F| = f$ and $|Q| = p$, there exist n edge-disjoint paths in G such that p of them connect P to Q and the $n-p$ others connects I to $O \setminus (Q \cup F)$. The set F corresponds to the set of failures and Q to the set of amplifiers providing the best quality of service. We denote by $R(n, p, f)$ the minimum number of switches (i.e., the cardinality of S) of a (n, p, f)-repartitor. An (n, p, f)-repartitor with $R(n, p, f)$ switches will be called a *minimum repartitor*.

Problem 2 Determine $R(n, p, f)$ and construct minimum (or almost minimum) repartitors.

The problem is interesting for every integer n and f and for every p, $0 \leq p \leq n$. In this paper, we focus on fixed values of p and f; we give asymptotic bounds on $R(n, p, f)$ if n goes to infinity. In current application networks, the number of input signals n is around 15 and the numbers of priorities p and extra amplifiers f range from 0 to $n/2$. As technology progresses, satellite parts are getting smaller and more reliable, and so the number of inputs is expected to increase in the future followed by a modest increase in the number of priorities and tolerable amplifier failures.

As indicated above, the problem with no priority (that is, $p = 0$) has been considered in [4] and partially solved in [6]. In [4], it is shown that $R(n, 0, 1) = R(n, 0, 2) = n$. In [6], it is proved that $\frac{3n}{2} - O(\frac{n}{f}) \leq R(n, 0, f) \leq \frac{3n}{2} + g(f)$ with g a function depending only on f. The minimum repartitor size is also bounded for small values of f: $R(n, 0, 4) = n + \lceil \frac{n}{4} \rceil$,

$R(n,0,6) = n+\frac{n}{4}+\sqrt{\frac{n}{8}}+O(1)$, $R(n,0,8) = n+\frac{n}{3}+\frac{2}{3}\sqrt{\frac{n}{3}}+O(\sqrt[4]{n})$, $R(n,0,10) = n+\frac{3n}{8}+\Theta(\sqrt{n})$ and $R(n,0,12) = n+\frac{3n}{7}+O(\sqrt{n})$.

Repartitors with no priorities, also called *selectors*, are somewhat similar to concentrators [7]. An *(n,m)-concentrator*, $m \leq n$, is a directed acyclic graph with maximal degree 4 that has n distinguished *input* vertices and a disjoint set of m distinguished *output* vertices such that for any subset A of m inputs there exists a set of m vertex-disjoint paths connecting A to the outputs. Researchers have studied the minimum number of *edges* of an (n,m)-concentrator. In our problem, we minimize the number of *switches* of selectors, which is similar to minimizing the number of edges since every switch has degree at most 4, and so the number of edges of a minimum (p,n)-selector is at most twice the number of its switches. Havet [8] constructed repartitors based on techniques used for designing concentrators [1, 9]. However, these repartitors are far from optimal, especially when the number of inputs or priority inputs is small, because one step of this method emulates high degree vertices by small subnetworks with switches of degree 4.

In this paper, we study (n,p,f)-repartitors **when p is not zero**. We first give a general upper bound for R:

$$R(n,p,f) \leq \frac{n+f}{2}\lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f),$$

where g is a function of f. We then give optimal or near optimal bounds on $R(n,p,f)$ for small values of p and f:

$R(n,p,f)$	$p=1$	$p=2$	$p=3$	$p=4$
$f=0$	$n-1$	$\lceil \frac{3n}{2}-3 \rceil$	$\frac{3n}{2}+\frac{\sqrt{n}}{2}+\Theta(1)$	$\frac{7n}{4}+\Theta(1)$
$f=1$	$\lceil \frac{3n-1}{2} \rceil$	$\frac{3n}{2}+\frac{\sqrt{n}}{2}+\Theta(1)$	$\frac{7n}{4}+\Theta(1)$	
$f=2$	$\lceil \frac{3n-1}{2} \rceil$			

We present our general bounds in Section 2. Our lower bounds in Section 3 are based on a minimum cut maximum flow type criterion, a sensible classification of connected subgraphs, and elementary graph theory. In Section 4, we present a few explicit constructions and prove that they are indeed repartitors.

2 General upper bounds

In this section, we give an upper bound on $R(n,p,f)$. We first give an inductive construction of $(n,p,0)$-networks and then construct an (n,p,f)-repartitor from an $(n-p,0,f)$-repartitor and an $(n+f,p,0)$-repartitor.

Clearly, $R(n,p,f)$ grows as a function of f since an (n,p,f)-repartitor may be obtained from an (n,p,f')-repartitor (with $f' \geq f$) by deleting any set of $f'-f$ outputs. Moreover, priority signals and ordinary signals play symmetric roles: Swapping the ordinary inputs and the priorities is a one-to-one mapping between (n,p,f)-repartitors and $(n,n-p,f)$-repartitors.

Proposition 3 *(i) If $f \le f'$ then $R(n,p,f) \le R(n,p,f')$;*
(ii) $R(n,p,f) = R(n,n-p,f)$.

Proposition 4

$$R(n,1,0) \le n-1.$$

Proof. Let G be the graph with one priority p_1, ordinary input set $\{i_1, i_2, \dots, i_{n-1}\}$, output set $\{o_1, o_2, \dots, o_n\}$ and switch set $\{s_1, s_2, \dots, s_{n-1}\}$ such that $(p_1, s_1, s_2, \dots, s_{n-1}, o_n)$ is a path and for l, $1 \le l \le n-1$, i_l and o_l are adjacent to s_l. (See Figure 1.) We claim that G is

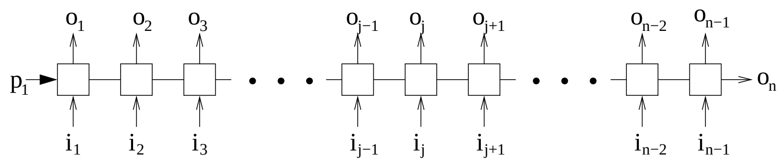

Figure 1: Minimum $(n,1,0)$-repartitor.

an $(n,1,0)$-repartitor. Indeed, suppose that o_j is the priority output. Since o_{n-1} and o_n are equivalent, we may suppose that $j \ne n$. The desired paths are $P_0 = (p_1, s_1, s_2, \dots, s_j, o_j)$, $P_j = (i_j, s_j, s_{j+1}, \dots, s_{n-1}, o_n)$ and $P_l = (i_l, s_l, o_l)$, for $l \ne j$, $1 \le l \le n-1$. ■

Lemma 5 *For $n \ge p \ge 1$,*

$$\begin{array}{lrcl}
(i) & R(2n,2p,0) & \le & 2R(n,p,0) + n - 1, \\
(ii) & R(2n+1,2p,0) & \le & R(n+1,p,0) + R(n,p,0) + n, \\
(iii) & R(2n,2p+1,0) & \le & R(n,p+1,0) + R(n,p,0) + n, \\
(iv) & R(2n+1,2p+1,0) & \le & R(n+1,p+1,0) + R(n,p,0) + n.
\end{array}$$

Proof. (i) Let G^1 and G^2 be two $(n,p,0)$-repartitors. For $i = 1,2$, let I^i, P^i, $O^i = \{o^i_1, o^i_2, \dots, o^i_n\}$ and S^i be the ordinary input set, priority set, output set and switch set of G^i. Let H be a graph defined as follows: its ordinary input set is $I = I^1 \cup I^2$, its priority set is $P = P^1 \cup P^2$, its switch set is $S^1 \cup S^2 \cup S$ with $S = \{s_1, s_2, \dots, s_{n-1}\}$ and its output set is $O = \{o_1, o_2, \dots, o_{n-1}\} \cup \{o'_1, o'_2, \dots, o'_{n-1}\} \cup \{o^1_n, o^2_n\}$; H contains the edges of G^1 and G^2 except the edges incident to o^1_j and o^2_j for $1 \le j \le n-1$. Furthermore, for every $1 \le j \le n-1$, s_j is linked to o_j, o'_j, the switch adjacent to o^1_j in G_1 and the switch adjacent to o^2_j in G_2. See Figure 2 (i) for an illustration.

We claim that H is a $(2n,2p,0)$-repartitor. Indeed, let Q be a $2p$-subset of O. Clearly, one can partition O into two n-sets O_1 and O_2 such that for $i = 1,2$, $o^i_n \in O_i$, $|Q \cap O_i| = p$ and $\forall\ 1 \le j \le n-1, |O_i \cap \{o_j, o'_j\}| = 1$. For $i = 1,2$, let $Q_i = Q \cap O_i$. Since G^1 and G^2 are $(n,p,0)$-repartitors, there are edge-disjoint paths joining P^i to Q_i and I^i to $O_i \setminus Q_i$. Hence, there are edge-disjoint paths joining P to Q and I to $O \setminus Q$.

The proofs of (ii), (iii) and (iv) are analogous and are omitted. Figure 2 (ii)—(iv) shows the schematic construction of H in these three cases. ■

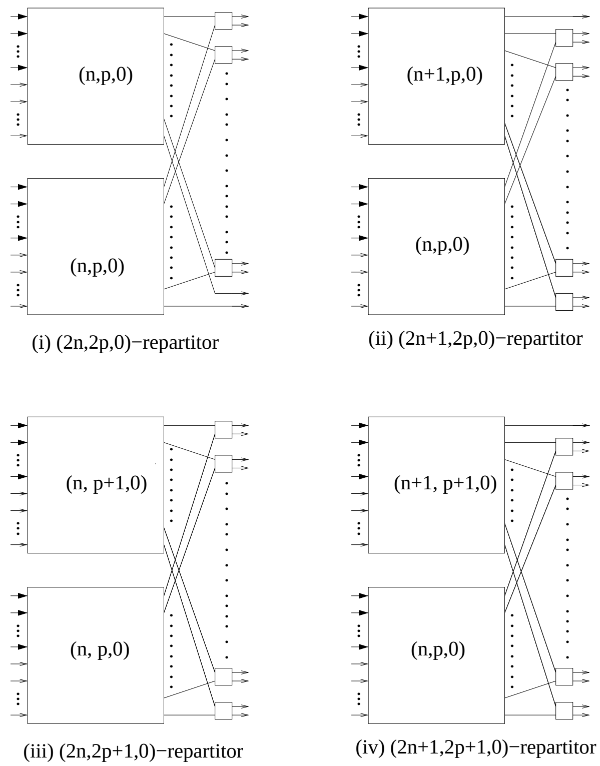

Figure 2: Construction of repartitors from smaller ones.

Theorem 6

$$R(n,p,0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p.$$

Proof. We proceed by induction on p. The inequality holds for $p = 1$ by Proposition 4.

Suppose that the inequality holds for every $p' < p$. Assume first that p and n are even. By Lemma 5 (i),

$$R(n,p,0) \leq 2R(n/2, p/2, 0) + n/2 - 1.$$

By the induction hypothesis,

$$R(n,p,0) \leq 2\left\{ \left\lfloor \frac{n}{4} \right\rfloor \lceil \log_2(p/2) \rceil + n/2 - p/2 \right\} + n/2 - 1.$$

Since $2\left\lfloor \frac{n}{4} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor$, we obtain

$$R(n,p,0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2(p/2) \rceil + n - p + n/2 - 1.$$

Because $\lceil \log_2(p/2) \rceil + 1 = \lceil \log_2(p) \rceil$, we obtain

$$R(n,p,0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p - 1.$$

If p or n is odd, we obtain the result analogously by Lemma 5 (ii), (iii) and (iv). ■

Lemma 7 *For $p \leq n$,*

$$R(n,p,f) \leq R(n-p,0,f) + R(n+f,p,0).$$

Proof. Let G_1 be an $(n-p,0,f)$-repartitor with output set $O^1 = \{o^1_1, o^1_2, \ldots, o^1_{n+f-p}\}$ and let G_2 be an $(n+f,p,0)$-repartitor with ordinary input set $I^2 = \{i^2_1, i^2_2, \ldots, i^2_{n+f-p}\}$. Let G be the network obtained from the union of G_1 and G_2 by replacing each pair $\{o^1_j, i^2_j\}$ by an edge between their neighboring switches. (See Figure 3.) We show that G is an (n,p,f)-repartitor.

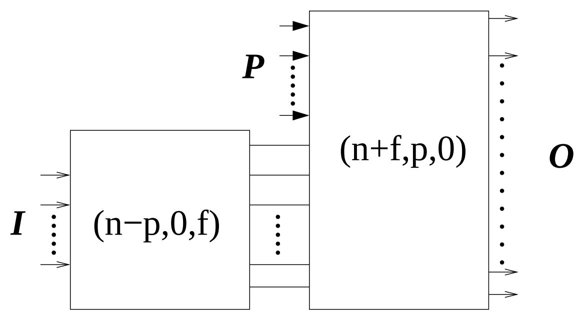

Figure 3: Construction of an (n,p,f)-repartitor.

Let $P = \{m_1, m_2, \ldots, m_p\}$ the set of priorities of G_2 be that of G, $I = \{i_1, i_2, \ldots, i_{n-p}\}$ the input set of G_1 be the ordinary input set of G, and $O = \{o_1, o_2, \ldots, o_{n+f}\}$ the output set of G_2 be that of G. Let Q and F be two disjoint subsets of O with cardinalities p and f respectively. Since G_2 is a repartitor there exists a set of edge-disjoint paths $\{P_1, P_2, \ldots, P_p\} \cup \{Q_1, Q_2, \ldots, Q_{n+f-p}\}$ such that for $1 \leq j \leq p$, P_j links m_j to an element of Q and for $1 \leq j \leq n+f-p$, Q_j links i^2_j to an element of $O \setminus Q$. Let J be the set of indices j such that Q_j ends in a vertex of F. Set $F^1 = \{o^1_j, j \in J\}$. Since G_1 is an $(n-p,0,f)$-repartitor, there are edge-disjoint paths R_l in G_1 $1 \leq l \leq n-p$ such that R_l links i_l to a vertex $o^1_{\phi(l)}$ that is not in F^1 for some function ϕ. Now the union of the paths R_l and $Q_{\phi(l)}$ induces a path P'_l in G which joins i_l to $o_{\phi(l)} \in O \setminus (Q \cup F)$. The paths P'_l, $1 \leq l \leq n-p$, and the paths P_j, $1 \leq j \leq p$, are obviously edge-disjoint. ∎

Theorem 8

$$R(n,p,f) \leq \frac{n+f}{2}\lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f),$$

where $g(f)$ is a function depending on f only.

Proof. By Lemma 7, $R(n,p,f) \leq R(n-p,0,f) + R(n+f,p,0)$. By Theorem 6, $R(n+f,p,0) \leq n+f-p+\frac{n+f}{2}\lceil \log_2 p \rceil$ and it is proved in [6] that $R(n-p,0,f) \leq \frac{3}{2}(n-p) + g(f)$. ∎

3 Lower bounds

Before we proceed with the lower bounds on $R(n,p,f)$, we make two observations on the structure of minimum repartitors. We first show that we can assume without loss of generality that in a minimum repartitor all switches, with the exception of at most one, have degree 4. Let $\epsilon(f) = 1$ if f is odd and 0 otherwise.

Proposition 9 *There is a minimum (n,p,f)-repartitor with $\epsilon(f)$ switches of degree 3 and all the others with degree 4.*

Proof. Let R be a minimum (n,p,f)-repartitor with the minimum number of switches with degree less than 4.

Obviously, R has no switches of degree 0 or 1 otherwise the network obtained by removing them is also a repartitor, which contradicts the minimality of R. Similarly, R has no switch of degree two. If a switch S had degree two, then we obtain a smaller valid repartitor by removing S and connecting its neighbours by an edge.

R has at most one switch of degree 3, otherwise the network obtained from R by adding an edge between two switches of degree 3 is an (n,p,f)-repartitor with fewer switches of degree less than 4. Since there is an even number of odd degree vertices in every graph, R has $\epsilon(f)$ switches of degree 3. ∎

In the remainder of Section 3, we assume that every (n,p,f)-repartitor has exactly $\epsilon(f)$ switches of degree 3. Next, we show that every switch is connected to at most one input.

Proposition 10 *In a minimum (n,p,f)-repartitor, a switch is connected to at most one ordinary input and at most one priority.*

Proof. Let R be an (n,p,f)-repartitor containing a switch s connected to at least two ordinary inputs i_1 and i_2. Let v_1 and v_2 be the two neighbours of s distinct from i_1 and i_2. Then the (n,p,f)-network obtained from R by removing s and adding the two edges (i_1,v_1) and (i_2,v_2) is also a repartitor and R is not minimum, a contradiction. The same proof works if both i_1 and i_2 are priority inputs. ∎

It follows that there are more switches than ordinary inputs.

Corollary 11

$$R(n,p,f) \geq n - p.$$

3.1 Cut criterion

One main tool to obtain lower bounds is to use the following cut criterion which gives necessary conditions for a network to be a repartitor. Let W be a set of vertices in a graph G. We denote by $in(W)$ (resp. $out(W)$, $pr(W)$) the cardinality of the set $In(W)$ (resp. $Out(W)$, $Pr(W)$) of ordinary inputs (resp. outputs, priorities) of W. We denote by $deg(W)$ the number of edges incident to exactly one vertex in W.

Proposition 12 (Cut criterion) *Let G be an (n,p,f)-repartitor and W be a set of vertices of G.*

(i) If $p + f \leq out(W)$ then $deg(W) \geq in(W) - out(W) + 2p + f - pr(W)$.

(ii) If $p \leq out(W) \leq p+f$ then $deg(W) \geq in(W) + p - pr(W)$.

(iii) If $out(W) \leq p$ then $deg(W) \geq in(W) + |out(W) - pr(W)|$.

(iv) $deg(W) \geq in(W) + pr(W) - \max(0, out(W) - f)$.

Proof.

(i) Suppose that p outputs of $Out(W)$ are in Q and f others are faulty. Then $in(W) - out(W) + p + f$ paths from an ordinary input to an output of $O \setminus Q$ leave W, and $p - pr(W)$ paths from priorities enter W. Since all these paths are edge-disjoint they go through different edges with an end in W and the other not in W. Thus, $deg(W) \geq in(W) - out(W) + 2p + f - pr(W)$.

(ii) Suppose that p outputs of $Out(W)$ are in Q and $out(W) - p$ others are faulty. Then $in(W)$ paths from an ordinary input to an output leave W and $p - pr(W)$ paths from priorities enter W. Thus, $deg(W) \geq in(W) + p - pr(W)$.

(iii) Suppose that $out(W)$ outputs of $Out(W)$ are in Q. Then $in(W)$ paths from an ordinary input to an output leave W. If $out(W) \geq pr(W)$ then $out(W) - pr(W)$ paths from priorities enter W and if $out(W) \leq pr(W)$ then $pr(W) - out(W)$ paths from priorities leave W. Thus, $deg(W) \geq in(W) + |out(W) - pr(W)|$.

(iv) Suppose that $\min(out(W), f)$ outputs of $Out(W)$ are faulty. Then $in(W) + pr(W) - \max(0, out(W) - f)$ paths leave W. ■

3.2 A bipartite graph on blocks and switches

We express the size of a minimum repartitor in terms of the number of switches with no inputs. By Proposition 10, we can distinguish two kinds of switches in a minimum (n, p, f)-repartitor: An *ordinary switch* is a switch adjacent to an ordinary input. A *usual switch* is a switch that is not an ordinary switch. Let $\mathcal{S}_o$ (resp. $\mathcal{S}_u$) denote the set of ordinary (resp. usual) switches and s_o (resp. s_u) their cardinality. Counting the ordinary inputs, we have $s_o = n - p$. Hence the total number of switches is

$$N = n - p + s_u.$$

Thus, a minimum (n, p, f)-repartitor is a repartitor with the fewest possible *usual switches.*

In order to obtain lower bounds for s_u, we consider the total number Σ of all edges incident to usual switches such that we count twice the edges between two vertices of $\mathcal{S}_u$. To avoid this weight problem, we insert a *link vertex* b_e in the middle of every edge e incident to two vertices of $\mathcal{S}_u$. Let $\tilde{G}$ be the graph obtained from G by replacing each edge $e = (s, s')$ between two usual switches by the path (s, b_e, s'). After this transformation, Σ is the number of edges of $\tilde{G}$ between S_u and $G - S_u$. Note that our *cut criterion* (Proposition 12) holds for $\tilde{G}$, too.

Following a definition of [6], let us call the connected components of $\tilde{G} - S_u$ *blocks.* A crucial observation in our argument is that Σ is the number of edges of a bipartite graph between S_u and the blocks. We distinguish two kinds of blocks, the *principal blocks* correspond to connected components of $G - S_u$ and *link blocks* correspond to a link vertex. Note that by the definition of block, every input or output incident to a switch in a block is also part of the block. We can deduce the following easy bound on the number of edges between a block B and S_u.

Proposition 13 *For every block B, $deg(B) \leq in(B) + 2 - out(B) - pr(B)$.*

Proof. The statement holds trivially for link blocks, since all of them are of degree 2. By definition, a principal block B has $in(B)$ switches. Let $e(B)$ be the number of edges connecting two switches of B. There are $in(B)$ (resp. $out(B)$, $pr(B)$) edges linking inputs (resp. outputs, priorities) to switches. Thus, $deg(B) = [\sum_{v \text{ switch}} deg(v)] - 2e(B) - in(B) - out(B) - pr(B)$. Since B is connected then $e(B) \geq in(B) - 1$. Since the degree of every switch is at most 4, we obtain $deg(B) \leq 4in(B) - 2e(B) - in(B) - out(B) - pr(B) \leq in(B) + 2 - out(B) - pr(B)$. ■

Remark 14 *Note that if equality holds in Proposition 13, then every switch has degree four.*

We list a few immediate consequences of the cut criterion (Proposition 12) for a block B.

Proposition 15 *Let B be a block of $\tilde{G}$.*

(i) $pr(B) \leq 1$.

(ii) If $p \geq 2$ *then* $out(B) \leq 1$.

(iii) If $f \geq 1$, *then* $out(B) + pr(B) \leq 1$.

Proof. (i) If B has two or more priorities, then by Proposition 13, $deg(B) \leq in(B) - out(B)$. This contradicts Proposition 12 (iv).

(ii) Suppose that B contains two or more outputs. We distinguish three cases: If $out(B) \geq p + f$, then $deg(B) \geq in(B) + 4 - out(B) - pr(B)$ by Proposition 12 (i) contradicting Proposition 13. If $p \leq out(B) \leq p + f$, then $deg(B) \geq in(B) + 2 - pr(B)$ by Proposition 12 (ii) contradicting Proposition 13. Finally, if $p \geq out(B)$, then $deg(B) \geq in(B) + out(B) - pr(B)$ by Proposition 12 (iii) contradicting Proposition 13.

(iii) If B has one priority and one output then by Proposition 13, $deg(B) \leq in(B)$. This contradicts Proposition 12.(iv) if $f \geq 1$. ■

In the remainder of this section, **we assume that either $p \geq 2$ or $p = 1$ and $f \geq 1$.** Indeed the case $p = 1$, $f = 0$ is already solved by Proposition 4 and Corollary 11.

We partition the blocks into the following four sets:

- $\mathcal{B}'_p$, the set of blocks having one priority and one output;
- $\mathcal{B}_p$, the set of blocks having one priority and no output;
- $\mathcal{B}_1$, the set the blocks having no priority and one output;
- $\mathcal{B}_0$, the set of blocks having no priority and no output (including the link blocks).

Remark 16 *Note that if $f \geq 1$, then $\mathcal{B}'_p$ is empty.*

Proposition 17 *Every block $B \in \mathcal{B}'_p$ satisfies $deg(B) = in(B)$. Every block $B \in \mathcal{B}_1 \cup \mathcal{B}_p$ satisfies $deg(B) = in(B) + 1$.*

Proof. It follows directly from Propositions 12 (iii) and 13. ∎

Remark 18 Note that in a block of $\mathcal{B}'_p \cup \mathcal{B}_p \cup \mathcal{B}_1$ every switch has degree four by Remark 14.

Proposition 19 *A block $B \in \mathcal{B}_0$ of a minimum (n,p,f)-repartitor satisfies $deg(B) = in(B)+2$ unless it contains the only vertex switch of degree 3 in which case $deg(B) = in(B)+1$.*

Proof. If B is a link block, then $deg(B) = 2 = in(B) + 2$. Assume now that B contains switches. Let $e(B)$ be the number of edges between pairs of switches of B. By Proposition 12:

$$deg(B) \geq in(B). \tag{a}$$

In the proof of Proposition 13, we had $deg(B) = \sum_{v \text{ switch}} deg(v) - 2e(B) - in(B)$. Thus, if one switch has degree 3 then

$$deg(B) = 3in(B) - 2e(B) - 1. \tag{b}$$

Since $e(B) \geq in(B) - 1$, Equations (a) and (b) yield $deg(B) = in(B) + 1$.

If every switch has degree four we obtain

$$deg(B) = 3in(B) - 2e(B). \tag{c}$$

Because $e(B) \geq in(B) - 1$, Equations (a) and (c) yield $deg(B) = in(B) + 2$ or $deg(B) = in(B)$. Now if $deg(B) = in(B)$, then the graph obtained by removing the switches of B and linking one to one the ordinary inputs of B to the neighbours of B is also an (n,p,f)-repartitor. This contradicts the minimality of G. So $deg(B) = in(B) + 2$. ∎

Let b_0, b_1, b_p and b'_p be the cardinalities of $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_p$ and $\mathcal{B}'_p$ respectively. We denote the number of ordinary inputs which are in a block of $\mathcal{B}_0$ (resp., $\mathcal{B}_1$, $\mathcal{B}_p$, and $\mathcal{B}'_p$) by n_0 (resp., n_1, n_p, and n'_p).

Let $\epsilon'(f) = 0$ if f is even, let $\epsilon'(f) = 1$ if f is odd and the switch of degree 3 is usual, and let $\epsilon'(f) = -1$ otherwise.

We summarize a few equations for further reference.

Proposition 20 *We have the following equalities:*

$$b_1 + b'_p = n + f. \tag{1}$$

$$b'_p = 0 \quad \text{if} \quad f \geq 1. \tag{2}$$

$$b_p + b'_p = p. \tag{3}$$

$$n_1 + n_0 + n_p + n'_p = n - p. \tag{4}$$

$$4s_u = 2n + f + 2b_0 - 2b'_p + \epsilon'(f). \tag{5}$$

Proof. (1) there is a one-to-one correspondence between blocks of $\mathcal{B}_1 \cup \mathcal{B}'_p$ and the outputs they contain.

(2) is Remark 16.

(3) there is a one-to-one correspondence between blocks of $\mathcal{B}_p \cup \mathcal{B}'_p$ and the priorities they contain.

(4) is obtained by noting that the $n-p$ ordinary inputs are in blocks.

(5) double counts the edges between usual switches and blocks. On one side, there are $4s_u$ edges (minus $\epsilon'(f)$ if the switch of degree 3 is a usual one). On the other side, we have by Proposition 17, n'_p edges incident to blocks of $\mathcal{B}'_p$, $n_p + b_p$ edges incident to blocks of $\mathcal{B}_p$ and $n_1 + b_1$ edges incident to blocks of $\mathcal{B}_1$ and by Proposition 19, $n_0 + 2b_0$ edges incident to blocks of $\mathcal{B}_0$ (or $n_0 + 2b_0 - 1$ if the switch of degree 3 is in a block of $\mathcal{B}_0$). Thus $4s_u = n'_p + n_p + b_p + n_1 + b_1 + n_0 + 2b_0 + \epsilon'(f)$. Then by (1), (3) and (4), we obtain the result. ∎

3.3 The lower bounds

Theorem 21 *If $f \geq 1$, then*

$$R(n,p,f) \geq \frac{3n}{2} - p + \frac{f+\epsilon(f)}{4}.$$

Proof. By (5) and (2), $s_u = n/2 + (f + 2b_0 + \epsilon'(f))/4$. Now if $\epsilon'(f) = -1$, by Remark 18, the switch of degree 3 is in a block of $\mathcal{B}_0$; hence $b_0 \geq 1$. Therefore $s_u \geq n/2 + (f+\epsilon(f))/4$. As $N = n - p + s_u$, we obtain $R(n,p,f) \geq \frac{3n}{2} - p + \frac{f+\epsilon(f)}{4}$. ∎

Theorem 22

$$R(n,2,0) \geq \frac{3n}{2} - 3.$$

Proof. By (5), $s_u \geq n/2 - b'_p/2 \geq n/2 - 1$. Then $R(n,2,0) \geq \frac{3n}{2} - 3$. ∎

In the remainder of this section, **we will suppose** $p \geq 3$ **or** $(p = 2$ **and** $f \geq 1)$**.**

Lemma 23 *For $p \geq 3$ or ($p = 2$ and $f \geq 1$), a usual switch is adjacent to at most two elements of $\mathcal{B}_1$.*

Proof. Suppose to the contrary that a usual switch S is adjacent to three blocks B_1, B'_1 and B''_1 of $\mathcal{B}_1$. Then $W = S \cup B_1 \cup B'_1 \cup B''_1$ satisfies $out(W) = 3$ and contradicts Proposition 12 (ii) if $p = 2$ and $f \geq 1$, or Proposition 12 (iii) if $p \geq 3$, as $deg(W) = in(W) + 1$ and $pr(W) = 0$. ∎

For $0 \leq i \leq 2$, let $\mathcal{S}_i$ be the set of usual switches adjacent to exactly i elements of $\mathcal{B}_1$ and s_i its cardinality. By Lemma 23, $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$ is a partition of $\mathcal{S}_u$. Moreover, from Proposition 20, one obtains the following corollary:

Corollary 24 *For $p \geq 3$ or ($p = 2$ and $f \geq 1$),*

$$\sum_{B \in \mathcal{B}_1} deg(B) = n_1 + b_1 = n_1 + n + f - b'_p = 2s_2 + s_1 \leq s_u + s_2. \tag{6}$$

Lemma 25 *If $p \geq 3$, then a block of $\mathcal{B}'_p$ and a switch of $\mathcal{S}_2$ cannot be adjacent.*

Proof. Suppose that a block B' of $\mathcal{B}'_p$ and a switch S_2 of $\mathcal{S}_2$ are adjacent. Let B_1 and B'_1 be the blocks of $\mathcal{B}_1$ adjacent to S_2. Then $W = B' \cup S_2 \cup B_1 \cup B'_1$ contradicts Proposition 12 (iii) as $deg(W) = in(W)$, $out(W) = 3$ and $pr(W) = 1$. ■

It follows from Remark 16 and Lemma 25 that a switch of $\mathcal{S}_2$ is adjacent to two blocks of $\mathcal{B}_1$ and two blocks of $\mathcal{B}_0 \cup \mathcal{B}_p$. Let H be the multigraph whose vertices correspond to the blocks of $\mathcal{B}_0 \cup \mathcal{B}_p$ and where two vertices are joined by μ edges if the corresponding blocks share μ neighbours in $\mathcal{S}_2$.

Lemma 26 *If $p \geq 2$ and $f \geq 1$, then H has no cycle of length at most $\left\lfloor \frac{f+3}{2} \right\rfloor$. If $p \geq 3$, then H is a simple graph (has no cycle of length 2).*

Proof. Suppose that in H there is a cycle C of length at most $\left\lfloor \frac{f+3}{2} \right\rfloor$ or a cycle of length 2 if $p \geq 3$. Let W be the union of the blocks $\mathcal{B}_0 \cup \mathcal{B}_p$ corresponding to the vertices of C, plus the switches of $\mathcal{S}_2$ corresponding to edges of C and the blocks of $\mathcal{B}_1$ adjacent to these switches. Then W contradicts Proposition 12 (i), (ii) or (iii), as $deg(W) = in(W) - pr(W)$. ■

Since a simple graph on n vertices has at most $n(n-1)/2$ edges, we have the following:

Corollary 27 *If ($p \geq 2$ and $f \geq 1$) or $p \geq 3$,*

$$s_2 \leq \frac{(b_0 + b_p)(b_0 + b_p - 1)}{2}. \tag{7}$$

Turán's theorem asserts that a simple graph on n vertices without a cycle of length 3 has at most $n^2/4$ edges. So we get:

Corollary 28 *For $p \geq 2$ and $f \geq 3$,*

$$s_2 \leq (b_0 + b_p)^2/4. \tag{8}$$

Theorem 29

$$R(n,2,2) \geq R(n,2,1) \geq \frac{3n}{2} + \frac{\sqrt{n+3}}{2} - 3.$$

$$R(n,3,0) \geq \frac{3n}{2} + \frac{\sqrt{n-3}}{2} - \frac{9}{2}.$$

Proof. Suppose $p = 2$ and $f = 1$. From (7) and (6), as $b'_p = 0$ thus $b_p = p = 2$, we obtain:

$$\frac{(b_0 + 2)(b_0 + 1)}{2} \geq s_2 \geq n_1 + n + 1 - s_u.$$

Replacing s_u by its value in (5),

$$\begin{aligned}
\frac{b_0^2+3b_0+2}{2} &\geq n_1+n+1-n/2-1/4-b_0/2-\epsilon'(1)/4,\\
b_0^2+4b_0+4 &\geq n+3,\\
b_0+2 &\geq \sqrt{n+3}.
\end{aligned}$$

Using this bound on b_0 in (5), we obtain $s_u \geq \frac{n}{2}+\frac{\sqrt{n+3}}{2}-1$. Thus

$$R(n,2,1) \geq \frac{3n}{2}+\frac{\sqrt{n+3}}{2}-3.$$

The proof is analogous if $p=3$ and $f=0$. We obtain $\frac{(b_0+b_p)(b_0+b_p-1)}{2} \geq n_1+n-b'_p-s_u$ and $(b_0+b_p)^2 \geq n-b'_p+b_p$. Hence $b_0 \geq \sqrt{n-b'_p+b_p}-b_p$. So $s_u \geq \frac{n}{2}+\frac{\sqrt{n-b'_p+b_p}}{2}-\frac{b_p+b'_p}{2} \geq \frac{n}{2}+\frac{\sqrt{n-3}}{2}-\frac{3}{2}$.

Finally $R(n,3,0) \geq \frac{3n}{2}+\frac{\sqrt{n-3}}{2}-\frac{9}{2}$. ■

Theorem 30

$$\text{If } f \geq 3, \text{ then } R(n,2,f) \geq \frac{3n}{2}+\frac{\sqrt{n}}{\sqrt{2}}+\Omega(f).$$

Proof. Suppose that $p=2$ and $f \geq 3$. In the same way as above, it follows from (5), (6) and (8) that

$$\begin{aligned}
(b_0+2)^2+2b_0 &\geq 2n+3f-\epsilon'(f),\\
b_0+3 &\geq \sqrt{2n+3f-\epsilon'(f)+5}.
\end{aligned}$$

Using this bound on b_0 in (5), we obtain

$$s_u \geq \frac{n}{2}+\frac{1}{2}\sqrt{2n+3f-\epsilon'(f)+5}+\frac{f+\epsilon'(f)}{4}-\frac{3}{2}.$$

Thus

$$R(n,2,f) \geq \frac{3n}{2}+\frac{1}{2}\sqrt{2n+3f-\epsilon'(f)+5}+\frac{f+\epsilon'(f)}{4}-\frac{7}{2}.$$

■

Lemma 31 *If $p \geq 4$ or ($p=3$ and $f \geq 1$), then a block of $\mathcal{B}_0 \cup \mathcal{B}_p$ is adjacent to at most one switch of $\mathcal{S}_2$.*

Proof. Suppose that a block $B \in \mathcal{B}_0 \cup \mathcal{B}_p$ is adjacent to two switches S_2 and S'_2 of $\mathcal{S}_2$. Let B_1 and C_1 and B'_1 and C'_1 be the elements of $\mathcal{B}_1$ adjacent to S_2 and S'_2, respectively. (See Figure 4, left.) Then the set $S_2 \cup S'_2 \cup B \cup B_1 \cup C_1 \cup B'_1 \cup C'_1$ contradicts Proposition 12 (iii) if $p \geq 4$, or Proposition 12 (ii) if $p=3$ and $f \geq 1$. ($deg(W) = in(W)+2-pr(W)$, $out(W)=4$ and $pr(W) \leq 1$.) ■

Let $\mathcal{B}'_0$ (resp. $\mathcal{B}''_0$) be the set of elements of $\mathcal{B}_0$ adjacent to exactly one (resp. no) switch in $\mathcal{S}_2$ and let b'_0 (resp. b''_0) be its cardinality.

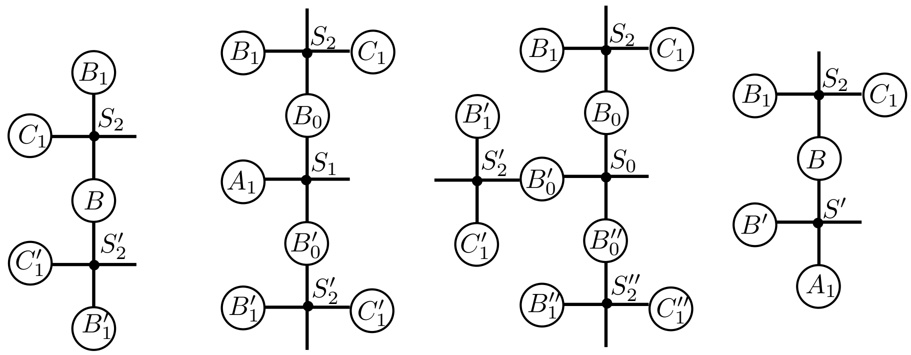

Figure 4: Illustration for Lemmas 31, 34, 35, and 38.

Corollary 32 *For $p \geq 4$ or ($p = 3$ and $f \geq 1$),*

$$b_0 + b_p \geq b_0' + b_p \geq 2s_2. \tag{9}$$

Theorem 33 *If $p \geq 4$ or ($p = 3$ and $f \geq 1$),*

$$R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f - \epsilon(f)}{8}.$$

Proof. By (6) and (9), we have $s_u \geq n + f - b_p' - (b_0 + b_p)/2$. By (5), $s_u = n/2 + (f + \epsilon'(f))/4 + (b_0 - b_p')/2$. Adding these two inequalities, we obtain:

$$2s_u \geq \frac{3n}{2} + \frac{5f + \epsilon'(f)}{4} - \frac{b_p + 3b_p'}{2}.$$

Hence, $R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f + \epsilon'(f)}{8}$. ■

Lemma 34 *If $p \geq 5$ or ($p = 4$ and $f \geq 1$), every switch of $\mathcal{S}_1$ is adjacent to at most one block of $\mathcal{B}_0'$.*

Proof. Suppose that $S_1 \in \mathcal{S}_1$ is a switch adjacent to two blocks B_0 and B_0' in $\mathcal{B}_0'$. Let S_2 (resp. S_2') be the switch of $\mathcal{S}_2$ adjacent to B_0 (resp. B_0'), and B_1 and C_1 (resp. B_1' and C_1') the two blocks of $\mathcal{B}_1$ adjacent to S_2 (resp. S_2'). Let A_1 be the block of $\mathcal{B}_1$ adjacent to S_1. (See Figure 4, middle left.) Then $W = A_1 \cup S_1 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S_2' \cup B_0' \cup B_1' \cup C_1'$ contradicts Proposition 12 as $deg(W) \leq in(W) + 3$ and $out(W) = 5$. ■

Lemma 35 *If $p \geq 6$ or ($p = 5$ and $f \geq 1$), every switch of $\mathcal{S}_0$ is adjacent to at most two blocks of $\mathcal{B}_0'$.*

Proof. Suppose that S_0 is a switch adjacent to three blocks B_0, B_0' and B_0'' in $\mathcal{B}_0'$. Let S_2 (resp. S_2', S_2'') be the switch of $\mathcal{S}_2$ adjacent to B_0 (resp. B_0', B_0'') and B_1 and C_1 (resp. B_1' and C_1', B_1'' and C_1'') the two blocks of $\mathcal{B}_1$ adjacent to S_2 (resp. S_2', S_2''). (See Figure 4, middle right.) Then $W = S_0 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S_2' \cup B_0' \cup B_1' \cup C_1' \cup S_2'' \cup B_0'' \cup B_1'' \cup C_1''$ contradicts Proposition 12 as $deg(W) \leq in(W) + 4$ and $out(W) = 6$. ■

Proposition 36 *Let $p \geq 2$. A block $B \in \mathcal{B}'_0$ has no switch of degree 3. Thus $deg(B) = in(B)+2$.*

Proof. Suppose that B has a switch of degree 3. Let S be the switch of $\mathcal{S}_2$ adjacent to B and B_1 and C_1 be the two blocks of B_1 adjacent to S. Then $W = B \cup S \cup B_1 \cup C_1$ contradicts Proposition 12 (iii), as $deg(W) = in(W) + 1$. ■

Let $\mathcal{S}'_1$ be the set of elements of $\mathcal{S}_1$ adjacent to exactly one block in $\mathcal{B}'_0$ and let s'_1 be its cardinality. Let n'_0 denote the number of inputs in blocks of $\mathcal{B}'_0$.

Corollary 37 *If $p \geq 6$ or ($p = 5$ and $f \geq 1$),*

$$n'_0 + b'_0 \leq s'_1 + 2s_0. \tag{10}$$

Proof. Let us count the number e of edges between $\mathcal{B}'_0$ and $\mathcal{S}'_1 \cup \mathcal{S}_0$.

On one side, every block B of $\mathcal{B}'_0$ is adjacent to exactly one element of $\mathcal{S}_2$ and no element in $\mathcal{S}_1 \setminus \mathcal{S}'_1$. Thus, by Proposition 36, there are $in(B)+1$ edges from B to $\mathcal{S}'_1 \cup \mathcal{S}_0$. Thus $e = n'_0 + b'_0$.

On the other side, there are s'_1 (resp. at most $2s_0$) edges joining $\mathcal{S}'_1$ (resp. $\mathcal{S}_0$) to $\mathcal{B}'_0$ according to the definition of $\mathcal{S}'_1$ and Lemma 34 (resp. Lemma 35). Thus $e \leq s'_1 + 2s_0$. ■

Lemma 38 *If $p \geq 4$, a block of $\mathcal{B}'_p$ is not adjacent to a switch of $\mathcal{S}'_1$.*

Proof. Suppose B' is a block of $\mathcal{B}'_p$ adjacent to S' in $\mathcal{S}'_1$. Let A_1 (resp. B) be the block of $\mathcal{B}_1$ (resp. $\mathcal{B}'_0$) adjacent to S'. Let S_2 be the switch of $\mathcal{S}_2$ adjacent to B, and B_1 and C_1 the two blocks of $\mathcal{B}_1$ adjacent to S_2. (See Figure 4, right.) Then $W = B' \cup S' \cup A_1 \cup B \cup S_2 \cup B_1 \cup C_1$ contradicts Proposition 12 (iii), as $deg(W) = in(W) + 1$, $out(W) = 4$ and $pr(W) = 1$. ■

Lemma 39 *If $p \geq 6$ or ($p = 5$ and $f \geq 1$), every element of $\mathcal{B}''_0 \cup \mathcal{B}_p$ is adjacent to at most one element of $\mathcal{S}'_1$.*

Proof. Let block $B \in \mathcal{B}''_0 \cup \mathcal{B}_p$ be adjacent to two switches S_1 and S'_1 of $\mathcal{S}'_1$. Let B_1 and B_0 (resp. B'_1 and B'_0) be the blocks of $\mathcal{B}_1$ and $\mathcal{B}'_0$ adjacent to S_1 (resp. S'_1). Let W_0 (resp. W'_0) be the union of the switch of $\mathcal{S}_2$ adjacent to B_0 (resp. B'_0) and its two adjacent blocks of $\mathcal{B}_1$. Then $W = B \cup B_1 \cup B_0 \cup W_0 \cup B'_1 \cup B'_0 \cup W'_0$ contradicts Proposition 12 (ii) or (iii), as $deg(W) = in(W) + 4 - pr(W)$, $out(W) = 6$ and $pr(W) \leq 1$. ■

Corollary 40 *If $p \geq 6$ or ($p = 5$ and $f \geq 1$),*

$$b''_0 + b_p \geq 2s'_1. \tag{11}$$

Theorem 41 *If $p \geq 6$ or ($p = 5$ and $f \geq 1$),*

$$R(n, p, f) \geq \frac{15n}{8} - \frac{15p}{8} + \frac{13f - \epsilon(f)}{16}.$$

Proof. We have $s_u = s_2 + s_1 + s_0$, then by (6),

$$s_u = n + n_1 + f - b'_p - s_2 + s_0,$$

$$2s_u \geq 2n + 2f - 2b'_p - 2s_2 + 2s_0.$$

By (9) and (10), $2s_0 \geq n'_0 + 2s_2 - s'_1 - b_p$, thus

$$2s_u \geq 2n + 2f + n'_0 - s'_1 - b_p - 2b'_p.$$

Then by (11),

$$2s_u \geq 2n + 2f + n'_0 - b''_0/2 - 3b_p/2 - 2b'_p,$$

$$4s_u \geq 4n + 4f - b''_0 - 3b_p - 4b'_p.$$

Since $b_0 = b'_0 + b''_0$, by (5) and (9), $2s_u \geq n + (f + \epsilon'(f))/2 - b'_p + b''_0 + 2s_2 - b_p$, so by adding the two inequalities one has

$$6s_u \geq 5n + 2s_2 + \frac{9f + \epsilon'(f)}{2} - 4b_p - 5b'_p.$$

Therefore, by (6),

$$8s_u \geq 7n + \frac{13f + \epsilon'(f)}{2} - 4b_p - 7b'_p.$$

■

4 Constructions

4.1 Useful constructions

In this subsection, we give general constructions that show how to merge several repartitors into a larger one. The first one is in the same vein as Lemma 7. The second one constructs an $(n-1, p, f)$-repartitor from an (n, p, f)-repartitor under certain conditions.

Lemma 42 *For* $0 \leq f' \leq f$,

$$R(n, p, f) \leq R(n + f', p + f', f - f') + R(p, p, f').$$

Proof. Let G_1 be an (p, p, f')-repartitor and G_2 be an $(n + f', p + f', f - f')$-repartitor. For $i = 1, 2$, let I_i (resp. P_i, O_i, S_i) be the sets of ordinary inputs (resp. priorities, outputs, switches) of G_i (note that I_1 is empty); and let $P_2 = \{p_1, p_2, \ldots, p_{p+f'}\}$ and $O_1 = \{o_1, o_2, \ldots, o_{p+f'}\}$. Let G be the network defined as follows:
Let $V(G) = P_1 \cup S_1 \cup I_2 \cup S_2 \cup O_2$; two vertices of $V(G)$ are joined by an edge if and only if $(u, v) \in E(G_1) \cup E(G_2)$ or there exists an $i \in \{0, 1\}$ such that $(u, o_i) \in E(G_1)$ and $(p_i, v) \in E(G_2)$; the ordinary input set of G is I_2, its priority set P_1, its output set O_2 and its switch set $S_1 \cup S_2$.

The proof that G is an (n, p, f)-repartitor is similar to that of Lemma 7. ■

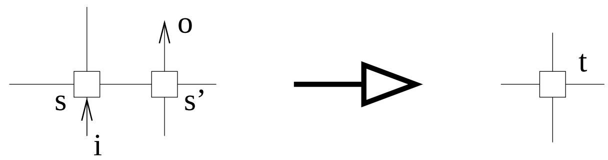

Figure 5: L-contraction.

Definition 43 Let G be a network. An edge (s, s') is said to be *linking* if there is an ordinary input i connected to s and an output o connected to s'.

The *L-contraction* of an edge (s, s') consists of contraction of the edge (s, s') into a vertex t and deletion of the input connected to s and the output connected to s'. (See Figure 5.)

Lemma 44 *Let G be an (n, p, f)-repartitor. The network obtained from G by the L-contraction of an edge is an $(n-1, p, f)$-repartitor.*

Proof. Let (s, s') be an edge of G and let i and o be the input and output linked to s and s', respectively. Let G' be the network obtained by the L-contraction of (s, s') into t.

Consider two disjoint subsets of outputs Q and F in G' such that $|Q| = p$ and $|F| = k$. Since G is a repartitor, there exists a set $\mathcal{P}$ of n edge-disjoint paths in G, connecting the p priorities to Q, and the $n-p$ ordinary inputs to $O \setminus (Q \cup F)$.

Case 1: A path $P \in \mathcal{P}$ connects i to o. Then replacing in the $n-1$ paths of $\mathcal{P} \setminus P$ the vertex s, s', or the succession of these two by t, we obtain the required set of $n-1$ paths in G'.

Case 2: A path P_1 of $\mathcal{P}$ connects an ordinary input $i' \neq i$ to o, and a path P_2 of $\mathcal{P}$ connects i to an output o'. Assume that these paths are $P_1 = (i', s'_1, s'_2, \ldots, s'_j, s', o)$ and $P_2 = (i, s, s_1, s_2, \ldots, s_l, o')$. Let W' be the walk $(i', s'_1, s'_2, \ldots, s'_j, s', s, s_1, s_2, \ldots, s_l, o')$ and let P' be the subpath of W' from i' to o'. We put $\mathcal{P}' = (\mathcal{P} \setminus \{P_1, P_2\}) \cup \{P'\}$. Then replacing in the $n-1$ paths of $\mathcal{P}'$ the vertex s, s', or the succession of these two by t, we obtain the required set of $n-1$ paths in G'. ∎

4.2 Minimum $(n, 1, 0)$-repartitors

Theorem 45

$$R(n, 1, 0) = n - 1.$$

Proof. By Proposition 4, $R(n, 1, 0) \leq n - 1$. By Corollary 11, $R(n, 1, 0) \geq n - 1$. ∎

We can, in fact, completely describe the set of minimum $(n, 1, 0)$-repartitors.

Let T be a binary tree with n leaves and $n-1$ internal nodes, one being the root r. Let us define the graph G_T associated with T in the following way: every leaf of T is an output of G_T; the internal nodes of T are the switches of G_T; each internal node of T is adjacent to an ordinary input, and furthermore the root is adjacent to the priority. (See Figure 6.)

Proposition 46 *A $(n, 1, 0)$-repartitor is minimum if and only if it is associated with a binary tree with n leaves.*

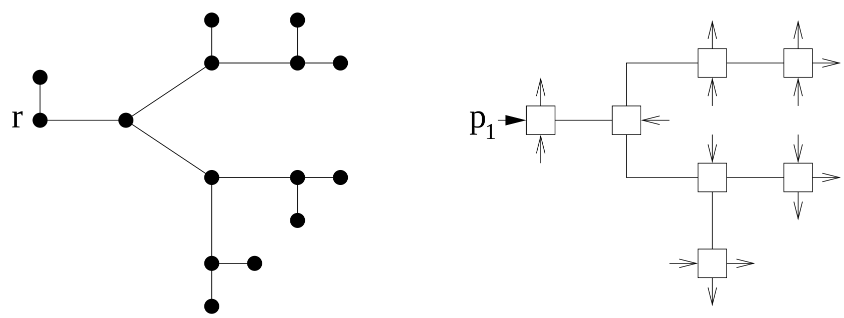

Figure 6: A binary tree with 8 leaves and the corresponding minimum $(8, 1, 0)$-repartitor.

Proof. It is easy to see that the graph associated with a binary tree is a minimum $(n, 1, 0)$-repartitor.

Let N be a minimum $(n, 1, 0)$-repartitor. It has $n-1$ switches and each of them is adjacent to exactly one ordinary input. Moreover n outputs and one priority are connected to the switches. Thus, there are exactly $(4(n-1) - 2n)/2 = n - 2$ edges connecting a switch to another. Since N is connected then it is a tree. Let T be the subtree of N induced by the switches and the outputs. It is clearly a binary tree. Indeed every output has degree one and every switch has degree 3 except the one that is connected to the priority that has degree 2 (and thus is the root). ■

4.3 Minimum $(n, 2, 0)$-repartitors

Theorem 47

$$R(n, 2, 0) = \left\lceil \frac{3n}{2} - 3 \right\rceil.$$

Proof. Theorem 22 yields $R(n, 2, 0) \geq \lceil \frac{3n}{2} - 3 \rceil$. Since $R(n, 1, 0) = n-1$, by the first inequality of Lemma 5, $R(n, 2, 0) \leq \lceil \frac{3n}{2} - 3 \rceil$. ■

Remark 48 Note that minimum $(n, 2, 0)$-repartitors are not necessarily formed from two minimum $(n/2, 1, 0)$-repartitors with the construction of Lemma 5. Indeed, let R be a minimum $(n, 2, 0)$-repartitor and let o and o' be two distinct outputs incident to the switches s and s', respectively. Let R' be the $(n+2, 2, 0)$-network obtained by removing o and o' and adding three switches t, t' and u such that t is adjacent to u, s and an ordinary input and an output, t' is adjacent to u, s' and an ordinary input and an output, and u is adjacent to two outputs. See Figure 7. It is easy to check that R' is a minimum $(n + 2, 2, 0)$-repartitor. Since t and t' are adjacent to an input and an output, R' may not be obtained via the previously mentioned construction.

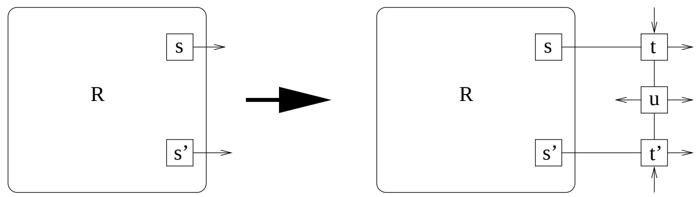

Figure 7: Extending a minimum $(n, 2, 0)$-repartitor to a minimum $(n + 2, 2, 0)$-repartitor.

4.4 Minimum $(n, 1, 1)$-repartitors and $(n, 1, 2)$-repartitors

Theorem 49

$$R(n, 1, 1) = \left\lceil \frac{3n - 1}{2} \right\rceil .$$

Proof. Theorem 21 yields $R(n, 1, 1) \geq \lceil \frac{3n-1}{2} \rceil$. According to Lemma 42, we can bound $R(n, 1, 1)$ by a sum: $R(n, 1, 1) \leq R(n + 1, 2, 0) + R(1, 1, 1)$. By Theorem 47, $R(n + 1, 2, 0) = \lceil \frac{3n+3}{2} - 3 \rceil$, and $R(1, 1, 1) = R(1, 0, 1) = 1$. Thus $R(n, 1, 1) \leq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$. ■

Theorem 50

$$R(n, 1, 2) = \left\lceil \frac{3n - 1}{2} \right\rceil .$$

Proof. Theorem 21 yields $R(n, 1, 2) \geq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$.

The network depicted in Figure 8 has $\lceil \frac{3n}{2} - \frac{1}{2} \rceil$ switches. Checking that this network is an

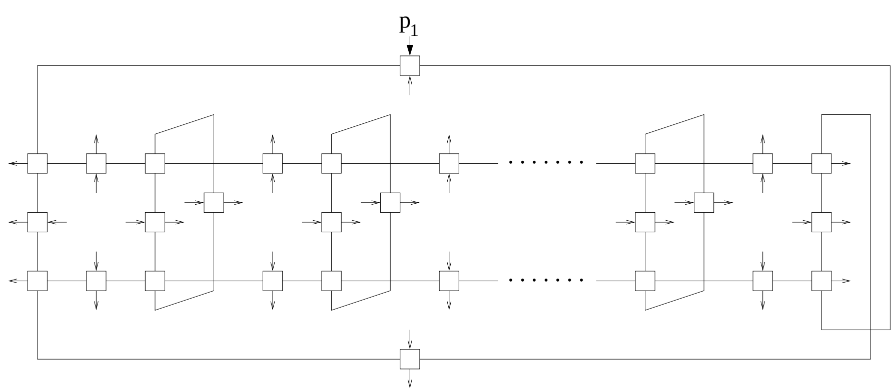

Figure 8: Minimum $(n, 1, 2)$-repartitor.

$(n, 1, 2)$-repartitor is not difficult but tedious since we must investigate various configurations and exhibit each time the corresponding edge-disjoint paths. The complete proof is available in [5]. ■

4.5 $(n, 3, 0)$-repartitors

Theorem 51

$$R(n, 3, 0) = \frac{3n}{2} + \frac{\sqrt{n}}{2} + c,$$

where $-5 \leq c \leq 9/4$.

Let G_l be the graph with vertex set $\{a_{i,j} | 1 \leq i \leq 3, 1 \leq j \leq l\}$ whose edge set is the union of the three paths $P_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,l})$ for $1 \leq i \leq 3$ and the l cycles $C_j = (a_{1,j}, a_{2,j}, a_{3,j}, a_{1,j})$ for $1 \leq j \leq l$. See Figure 9. So G_l has $3l$ vertices and $6l - 3$ edges.

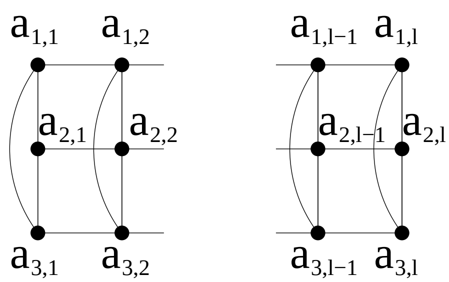

Figure 9: The graph G_l.

From G_l, we construct the network H_l as follows.

To each vertex $a_{i,j}$, $1 \leq i \leq 3$, $1 \leq j \leq l$, we associate a switch $s_{i,j}$. Each switch $s_{i,1}$, $1 \leq i \leq 3$, is connected to a priority input p_i and each switch $s_{i,l}$, $1 \leq i \leq 3$, is connected to an output o_i. We replace each edge e of G_l by a path $P(e)$ consisting of $n(e)$ switches with some ordinary input connected to each of them. In the bipartite graph $\tilde{H}_l$ defined previously, this corresponds to association of each edge e with a block $B_0(e)$ of $\mathcal{B}_0$. The size $n(e)$ of the paths $P(e)$ is $in(B_0(e))$ and will be specified later. For any two non-adjacent edges e and f in G_l, we add a switch $s_{e,f}$ in $\tilde{H}_l$: it is connected to an ordinary switch of $P(e)$, to an ordinary switch of $P(f)$, and two outputs $o_{e,f}$ and $o'_{e,f}$. (In $\tilde{H}_l$, $s_{e,f}$ is in $\mathcal{S}_2$.)

If $l \geq 3$, an edge e of G_l is adjacent to 6 other edges except the edges of $E_1 = \{(a_{i,1}, a_{i,2}) | 1 \leq i \leq 3\} \cup \{(a_{i,l-1}, a_{i,l}) | 1 \leq i \leq 3\}$, which are adjacent to 5 other edges and the edges of $E_2 = \{(a_{i,1}, a_{i+1,1}) | 1 \leq i \leq 3\} \cup \{(a_{i,l}, a_{i+1,l}) | 1 \leq i \leq 3\}$, which are adjacent to 4 edges. Hence the number $n(e)$ of switches of $P(e)$ is defined as follows: $n(e) = 6l - 8$ if $e \in E_2$, $n(e) = 6l - 9$ if $e \in E_1$ and $n(e) = 6l - 10$ otherwise.

In summary, H_l has for $l \geq 3$:

- 3 priority inputs,
- $n_l - 3 = 6(6l - 8) + 6(6l - 9) + (6l - 15)(6l - 10) = 36l^2 - 78l + 48$ ordinary inputs,
- n_l outputs,
- $n_l - 3$ ordinary switches (those of the $P(e)$),
- $(n_l - 3)/2$ switches of $\mathcal{S}_2$ (that is, $s_{e,f}$), and

- $3l$ switches of $\mathcal{S}_0$ (that is, $s_{i,j}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, l$).

Hence H_l has $N_l = \frac{3n_l}{2} + 3l - \frac{9}{2}$ switches. As $n_l = 36l^2 - 78l + 51 = (6l - \frac{13}{2})^2 + \frac{35}{4}$, then $N_l = \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l - \frac{35}{4}} - \frac{5}{4} \leq \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l} - \frac{5}{4}$.

Note that all the formulae are also valid for $l = 2$. Indeed in that case $n(e) = 4$ for any of the 9 edges and so $n_l - 3 = 36$.

Lemma 52 *H_l is a repartitor.*

We will reduce the proof of this lemma to the existence of some specific walks in the slightly modified graph G_l .

Definition 53 Let G'_l be the graph obtained from G_l by adding the set $R_3 = \{a_{1,l+1}, a_{2,l+1}, a_{3,l+1}\}$ and the three edges $(a_{i,l}, a_{i,l+1})$, $1 \leq i \leq 3$.

Let $E_3 = \{e_1, e_2, e_3\}$ be a set of distinct edges of G'_l. An E_3-*good set of walks* is a set of three edge-disjoint walks $\{W_1, W_2, W_3\}$ such that W_i contains e_i, where the set of initial vertices of the W_i's is $\{a_{1,1}, a_{2,1}, a_{3,1}\}$, and the set of terminal vertices of the W_i's is R_3.

Lemma 54 *If G'_l admits an E_3-good set of walks for any set E_3 of three distinct edges, two of which are non-adjacent, then H_l is a repartitor.*

Proof. Let us define an *assignment* as a mapping ϕ from the output set of H_l into the edges of G'_l such that:

- $\phi(o_i) = (a_{i,l}, a_{i,l+1})$ for $i \in \{1, 2, 3\}$;
- for the two outputs $o_{e,f}$ and $o'_{e,f}$ adjacent to the switch $s_{e,f}$, then $\phi(o_{e,f}) = e$ and $\phi(o'_{e,f}) = f$, or $\phi(o_{e,f}) = f$ and $\phi(o'_{e,f}) = e$.

Let $Q = \{q_1, q_2, q_3\}$ be any set of three outputs of H_l and let us denote $\{o_1, o_2, o_3\}$ by O_3. To prove that H_l is a repartitor, we have to find the desired edge-disjoint paths in H_l.

First let us show that there exists an assignment ϕ such that the three edges $\phi(q_i)$ are distinct and two of them are non-adjacent.

- If $|Q \cap O_3| \geq 2$, assume that $q_1 = o_{i_1}$ and $q_2 = o_{i_2}$. Then for any assignment ϕ, $\phi(q_1) = e_1 = (a_{i_1,l}, a_{i_1,l+1})$ and $\phi(q_2) = e_2 = (a_{i_2,l}, a_{i_2,l+1})$. Hence e_1 and e_2 are not adjacent and distinct from $\phi(e_3)$ since only q_1 (resp. q_2) may be mapped on e_1 (resp. e_2) by an assignment.
- If $|Q \cap O_3| = 1$, assume that $q_1 = o_{i_1}$ and q_2 (resp. q_3) are connected to s_{e_2,f_2} (resp. s_{e_3,f_3}). As all the edges adjacent to $\phi(q_1) = e_1 = (a_{1,l}, a_{1,l+1})$ are pairwise adjacent, one of $\{e_2, f_2\}$ is not adjacent to e_1. Let $\phi(q_2)$ be this edge and let $\phi(q_3)$ be an edge of $\{e_3, f_3\} \setminus \{e_2\}$. Then ϕ satisfies the requirement.

- If $|Q \cap O_3| = 0$, assume that q_i, $1 \leq i \leq 3$, are connected to s_{e_i,f_i}. By definition, $|\{e_i, f_i\} \cap \{e_j, f_j\}| \leq 1$ for every $1 \leq i, j \leq 3$.

  Suppose first that there are i, j, $i \neq j$, such that $|\{e_i, f_i\} \cap \{e_j, f_j\}| = 1$. Without loss of generality, we may assume that $e_1 = e_2$. Then let $\phi(q_1) = e_1$ and $\phi(q_2) = f_2$. It follows that $e_1 = e_2$ and f_2 are not adjacent. Let $\phi(q_3)$ be an edge of $\{e_3, f_3\} \setminus \{e_2, f_2\}$. (Such an edge exists since $|\{e_2, f_2\} \cap \{e_3, f_3\}| \leq 1$.)

  Suppose now that all the six edges $e_1, f_1, e_2, f_2, e_3, f_3$ are distinct. If one edge of $\{e_1, f_1\}$ is not adjacent to one edge of $\{e_2, f_2\}$, say e_1 is not adjacent to e_2, then let $\phi(q_i) = e_i$. Otherwise the four edges e_1, f_1, e_2, f_2 form a 4-cycle. Since there is no diagonal of a 4-cycle in G'_l, e_3 is adjacent to at most one edge of $\{e_1, f_1\}$, say e_3 is not adjacent to e_1. Then setting $\phi(q_i) = e_i$, we obtain the desired assignment.

Let us now exhibit the $n = 36l^2 - 78l + 51$ required edge-disjoint paths, 3 of them joining the priorities to Q, and the $n - 3$ remaining joining the ordinary inputs to outputs.

Consider an assignment ϕ as above and let $E_3 = \{\phi(q_1), \phi(q_2), \phi(q_3)\}$. By hypothesis, there exists an E_3-good set of walks $\{W_1, W_2, W_3\}$ such that W_i contains $\phi(q_i)$.

Let o be an output not in O_3. Let $e = \phi(o)$, $s_{e,f}(o)$ be the switch adjacent to o, $s_e(o)$ the vertex of $P(e)$ adjacent to $s_{e,f}(o)$, and $i(o)$ the input adjacent to $s_e(o)$. The *default path* associated with o is $P(o) = (i(o), s_e(o), s_{e,f}(o), o)$.

If $o \notin Q \cup O_3$, we choose its default path for joining the ordinary input $i(o)$ to o.

To each W_i corresponds a walk Q_i in $H(l)$ obtained by replacing an edge of G'_l by the corresponding path $P(e)$ and the edge $(a_{i,l}, a_{i,l+1})$ by the output o_i.

If $q_i \in O_3$ then choose Q_i which joins a priority to the priority output q_i.

If $q_i \notin O_3$, let $s_i = s_e(q_i)$ be the switch that is in both Q_i and the default path of q_i. Let D_i (resp. F_i) be the subpath of Q_i starting in $a_{j_i,0}$ (resp. at s_i) and ending at s_i (resp. in O_3). Then choose $(p_{j_i}, D_i, s_{e,f}(q_i), q_i)$ which joins a priority to a priority output, and (i_{q_i}, F_i) which joins an ordinary input to a non-priority output. All the paths we have constructed are edge-disjoint. ■

Proof of Lemma 52. By Lemma 54, it suffices to prove that for any set E_3 of three edges of G'_l with two non-adjacent edges, there is an E_3-good set of walks.

For $1 \leq i \leq 3$ and $1 \leq j \leq k \leq l+1$, we will denote by $L_i[j,k]$ the path $(a_{i,j}, a_{i,j+1}, \ldots, a_{i,k})$.

By symmetry of G'_l, it suffices to prove it for the following ten cases. All these cases and the corresponding covering walks are depicted in Figure 10. The walk W_1 (resp. W_2, W_3) is drawn as a dotted (resp. black, grey) line. The white (resp. black, grey) ellipses represent the possible positions of e_1 (resp. e_2, e_3).

(1) $e_1 = (a_{1,i_1}, a_{1,i_1+1})$, $e_2 = (a_{1,i_2}, a_{1,i_2+1})$ and $e_3 = (a_{1,i_3}, a_{1,i_3+1})$ with $0 \leq i_1 < i_2 < i_3 \leq l$. The covering walks are $W_1 = (L_1[1,i_2], a_{3,i_2}, L_2[i_2, l+1])$, $W_2 = (L_2[1,i_2], L_1[i_2,i_3], L_3[i_3, l+1])$, and $W_3 = (L_3[1,i_3], a_{2,i_3}, L_1[i_3, l+1])$.

(2) $e_1 = (a_{1,i_1}, a_{1,i_1+1})$, $e_2 = (a_{1,i_2}, a_{1,i_2+1})$ and $e_3 = (a_{3,i_3}, a_{3,i_3+1})$ with $0 \leq i_1 < i_2 \leq l$. The covering walks are $W_1 = (L_1[1,i_2], L_2[i_2, l+1])$, $W_2 = (L_2[1,i_2], a_{3,i_2}, L_1[i_2, l+1])$, and $W_3 = (L_3[1, l+1])$.

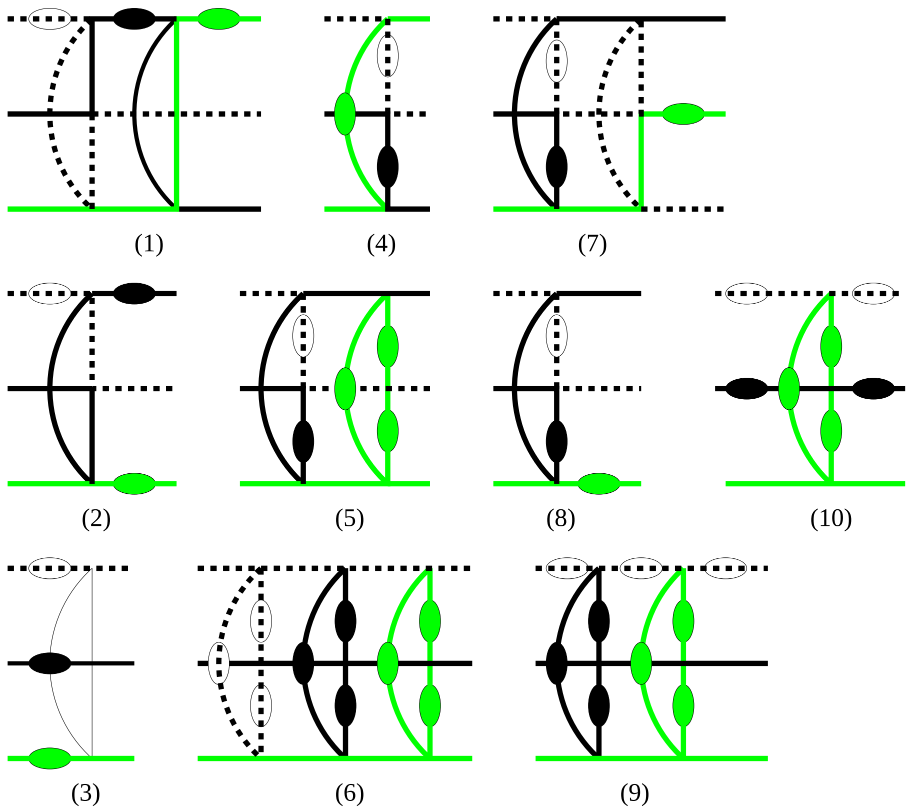

Figure 10: The covering walks in each case.

(3) $e_1 = (a_{1,i_1}, a_{1,i_1+1})$, $e_2 = (a_{2,i_2}, a_{2,i_2+1})$ and $e_3 = (a_{3,i_3}, a_{3,i_3+1})$. The covering walks are $W_1 = (L_1[1, l+1])$, $W_2 = (L_2[1, l+1])$, and $W_3 = (L_3[1, l+1])$.

(4) $e_1 = (a_{1,i}, a_{2,i})$, $e_2 = (a_{2,i}, a_{3,i})$ and $e_3 = (a_{3,i}, a_{1,i})$. $W_1 = (L_1[1, i], L_2[i, l+1])$, $W_2 = (L_2[1, i], L_3[i, l+1])$, and $W_3 = (L_3[1, i], L_1[i, l+1])$.

(5) $e_1 = (a_{1,i_1}, a_{2,i_1})$, $e_2 = (a_{2,i_1}, a_{3,i_1})$ and $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$ with $i_3 \neq i_1$. $W_1 = (L_1[1, i_1], L_2[i_1, l+1])$, $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$, and $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$.

(6) $e_1 \in \{(a_{1,i_1}, a_{2,i_1}), (a_{2,i_1}, a_{3,i_1}), (a_{3,i_1}, a_{1,i_1})\}$, $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$ and $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$ with $i_1 < i_2 < i_3$.
$W_1 = (L_1[1, i_1], a_{2,i_1}, a_{3,i_1}, L_1[i_1, l+1])$, $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$, and $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$.

(7) $e_1 = (a_{1,i_1}, a_{2,i_1})$, $e_2 = (a_{2,i_1}, a_{3,i_1})$ and $e_3 = (a_{2,i_3}, a_{2,i_3+1})$ with $i_1 < i_3$.
$W_1 = (L_1[1, i_1], L_2[i_1, i_3], a_{1,i_3}, L_3[i_3, l+1])$, $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$, and $W_3 = (L_3[1, i_3], L_2[i_3, l+1])$.

(8) $e_1 = (a_{1,i_1}, a_{2,i_1})$, $e_2 = (a_{2,i_1}, a_{3,i_1})$ and $e_3 = (a_{3,i_3}, a_{3,i_3+1})$. $W_1 = (L_1[1,i_1], L_2[i_1, l+1])$, $W_2 = (L_2[1,i_1], a_{3,i_1}, L_1[i_1, l+1])$, and $W_3 = (L_3[1, l+1])$.

(9) $e_1 = (a_{1,i_1}, a_{1,i_1+1})$, $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$ and $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$ with $i_2 < i_3$. $W_1 = (L_1[1, l+1])$, $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$, and $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$.

(10) $e_1 = (a_{1,i_1}, a_{1,i_1+1})$, $e_2 = (a_{2,i_2}, a_{2,i_2+1})$ and $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$. $W_1 = (L_1[1, l+1])$, $W_2 = (L_2[1, l+1])$, and $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$.

■

Proof of Theorem 51 By Theorem 29, $R(n,3,0) \geq \frac{3n}{2} + \frac{\sqrt{n}}{2} - 5$.

Let us now define the $(n,3,0)$-network $G_3(n)$ inductively as follows:
For $l \geq 2$, let $n_l = 36l^2 - 78l + 51$. If $n = n_l$ for some integer l, then $G_3(n) = H_l$. Otherwise let $n_{l-1} < n < n_l$. If n is odd then $G_3(n)$ is obtained from $G_3(n+2)$ by removing a switch $s_{e,f}$ and its two adjacent outputs, and deleting the two switches $s(e) \in P(e)$ and $s(f) \in P(f)$ adjacent to $s_{e,f}$ and their adjacent inputs, and adding an edge joining the two neighbours of $s(e)$ (resp. $s(f)$) in the input path $P(e)$ (resp. $P(f)$).
If n is even then $G_3(n)$ is obtained from $G_3(n+1)$ by an L-contraction of a linking edge $\{s(e), s_{e,f}\}$.

Since H_l is a repartitor, then if n is odd $G_3(n)$ is obviously an $(n,3,0)$-repartitor and if n is even, by Lemma 44, $G_3(n)$ is an $(n,3,0)$-repartitor. $G_3(n)$ has $N = N_l - \frac{3}{2}(n - n_l) + \varepsilon/2 = \frac{3n}{2} + \frac{\sqrt{n_l}}{2} - 5/4 + \varepsilon/2$ switches where $\varepsilon = 1$ if n is even. But as $n > n_{l-1}$, $\sqrt{n_l} - \sqrt{n} < \sqrt{n_l} - \sqrt{n_{l-1}} \leq 6$. So $N \leq \frac{3n}{2} + \frac{\sqrt{n}}{2} + \frac{9}{4}$. ■

Theorem 55

$$R(n,2,1) = \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + c,$$

where $-3 \leq c \leq \frac{23}{4}$.

Proof. From Lemma 42, we obtain $R(n,2,1) \leq R(n+1,3,0) + R(2,2,1)$. By Theorem 51, $R(n+1,3,0) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{15}{4}$. And by Proposition 3, $R(2,2,1) = R(2,0,1)$ and $R(2,0,1) = 2$ (see [4]). Hence $R(n,2,1) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{23}{4}$. Theorem 29 gives the other inequality. ■

4.6 $(n,4,0)$-networks

The aim of this subsection is to prove the following theorem :

Theorem 56

$$R(n,4,0) = \frac{7n}{4} + c$$

where $-7 \leq c \leq \frac{89}{4}$.

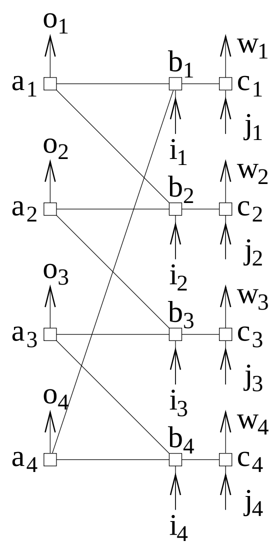

Figure 11: A brick.

We will construct a network $H(m)$, which is a $(16m + 4, 4, 0)$-repartitor.

We start with a basis brick B (see Figure 11). It contains 12 switches a_l, b_l, c_l for $1 \leq l \leq 4$. Switch a_l is connected to b_l and b_{l+1} (the indices being taken modulo 4), and b_l is connected to c_l. There are 8 inputs: for $1 \leq l \leq 4$, i_l is connected to b_l, and j_l connected to c_l. There are 8 outputs: for $1 \leq l \leq 4$, o_l connected to a_l and w_l connected to c_l.

Let $G(m)$ be the network obtained by concatenating m bricks B^k, $1 \leq k \leq m$, that is, by joining c_l^k to a_l^{k+1} for $1 \leq k \leq m-1$ and $1 \leq l \leq 4$.

Let $F(m)$ be the network obtained by taking two copies of $G(m)$, $G(m)$ and $\overline{G}_m$, deleting the $8m$ outputs w_l^k and $\bar{w}_l^k$, adding $4m$ switches s_l^k connected to c_l^k, $\bar{c}_l^k$ and two new outputs r_l^k and $\bar{r}_l^k$.

Finally let IS be an *input selector* with 4 inputs p_l, $1 \leq l \leq 4$, and 8 outputs α_l and $\bar{\alpha}_l$, $1 \leq l \leq 4$, such that for any set of four outputs, there are four edge-disjoint paths joining the inputs to these outputs.

Let OS be an *output selector* with 8 inputs γ_l and $\bar{\gamma}_l$, $1 \leq l \leq 4$, and 4 outputs t_l, $1 \leq l \leq 4$, such that for any set of four inputs of the form $C_1 \cup C_2$ and any partition $T_1 \cup T_2$ of $T = \{t_1, t_2, t_3, t_4\}$ with $|C_1| = |T_1|$, there are four edge-disjoint paths, $|C_1|$ of them joining C_1 to T_1 and $|C_2|$ of them joining C_2 to T_2.

Then $H(m)$ is obtained by concatenating IS, $F(m)$ and OS, that is, by identifying α_l (resp. $\bar{\alpha}_l$) with a_l^1 (resp. $\bar{a}_l^1$) and c_l^m (resp. $\bar{c}_l^m$) with γ_l (resp. $\bar{\gamma}_l$). The network $H(m)$ is depicted in Figure 12.

$H(m)$ has 4 priorities p_l, $1 \leq l \leq 4$, $16m$ ordinary inputs i_l^k, $\bar{i}_l^k$, j_l^k, $\bar{j}_l^k$ for $1 \leq k \leq m$ and $1 \leq l \leq 4$ and $n = 16m + 4$ outputs, o_l^k, $\bar{o}_l^k$, r_l^k, $\bar{r}_l^k$ for $1 \leq k \leq m$ and $1 \leq l \leq 4$ and t_l, $1 \leq l \leq 4$. It has $28m + N(IS) + N(OS)$ switches where $N(IS)$ (resp. $N(OS)$) is the minimum number of switches of an input (resp. output) selector. An input selector is nothing more than a $(4, 0, 4)$-repartitor and it is proved in [6] that $R(4, 0, 4) = 5$, so $N(IS) = 5$. An optimum

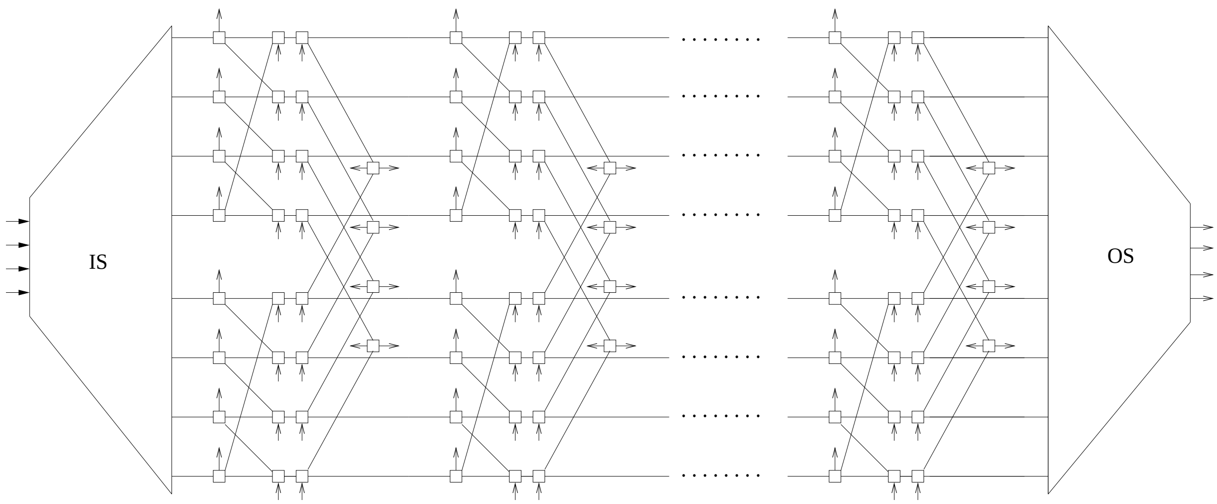

Figure 12: The network $H(m)$.

input selector is drawn in Figure 13 left. The network depicted in Figure 13 right is an output selector. Indeed, it is constructed from a Waksman network (see [2, 3]), which is able to realize any permutation of the 8 inputs into 8 outputs, by removing the useless outputs, switches and links (dotted on the figure). Hence $N(OS) \leq 13$.

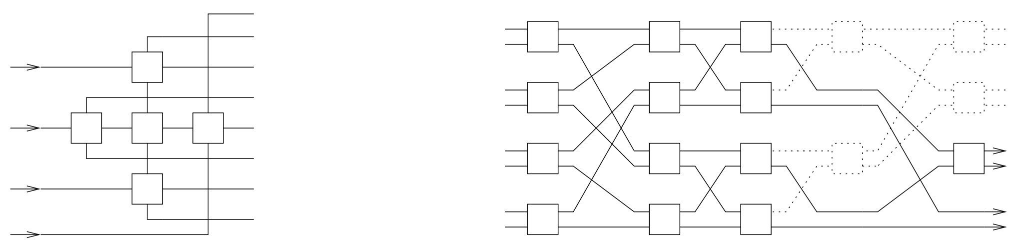

Figure 13: An optimum input selector and an output selector.

Lemma 57 *$H(m)$ is a $(16m+4,4,0)$-repartitor.*

Proof. In what follows we will denote $O = \bigcup_{k,l} o_l^k$, $\bar{O} = \bigcup_{k,l} \bar{o}_l^k$, $R = \bigcup_{k,l} r_l^k$, $\bar{R} = \bigcup_{k,l} \bar{r}_l^k$, $W = \bigcup_{k,l} w_l^k$, $\bar{W} = \bigcup_{k,l} \bar{w}_l^k$.

Let Q be any set of four outputs of $H(m)$. Let $Q_1 = Q \cap (O \cup \bar{O} \cup R \cup \bar{R})$ and $Q_2 = Q \cap T$ and $q_i = |Q_i|$, $1 \leq i \leq 2$.

To prove that $H(m)$ is a repartitor it suffices to find in $F(m)$ for any set $Q = Q_1 \cup Q_2$ a set of $16m+4$ edge-disjoint paths such that:
q_1 of them join a set A_1 of q_1 elements of $\bigcup_l \{a_l^1, \bar{a}_l^1\}$ to Q_1,
q_2 of them join a set A_2 of q_2 elements of $\bigcup_l \{a_l^1, \bar{a}_l^1\} \setminus A_1$ to a set C_2 of q_2 elements of $\bigcup_l \{c_l^m, \bar{c}_l^m\}$,

and the $16m$ other paths join the ordinary inputs to the set of outputs $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$ and a set C_1 of q_1 elements of $\bigcup_l \{c_l^m, \bar{c}_l^m\} \setminus C_2$.

Indeed IS being an input selector, we can connect the 4 priorities of $H(m)$ to the 4 outputs assigned to $A_1 \cup A_2$ with edge-disjoint paths; then one can link A_1 to Q_1 and A_2 to C_2, and the ordinary inputs either to outputs of $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$ or vertices of C_1; at last, OS being an output selector, we can find edge-disjoint paths from C_2 to Q_2, and from C_1 to $T \setminus Q_2$. Hence we have the $16m + 4$ required edge-disjoint paths of $H(m)$, 4 of them joining the p_j to Q and the remaining $16m$ joining the ordinary inputs to the outputs not in Q.

In fact, we can reduce the problem of finding the required set of paths in $F(m)$ to that of finding in $G(m)$ a Q_0-*good path set* defined as follows:

Let Q_0 be any set of at most 4 outputs of $G(m)$, $Q_0 \subseteq O \cup W$ and $|Q_0| = q_0 \leq 4$. A Q_0-*good path set* consists of $8m + 4$ edge-disjoint paths such that

- 4 of them link the vertices of $\bigcup_{1 \leq l \leq 4} a_l^1$ to Q_0 and a set D_1 of $4 - q_0$ vertices of $\bigcup_{1 \leq l \leq 4} c_l^m$, and

- the remaining $8m$ link the $8m$ ordinary inputs of $G(m)$ to $(O \cup W) \setminus Q_0$ and $D_2 = \bigcup_{1 \leq l \leq 4} c_l^m \setminus D_1$.

Let us define an *assignment* as a one-to-one mapping from $R \cup \bar{R}$ to $W \cup \bar{W}$ by associating to r_l^k one of w_l^k and $\bar{w}_l^k$, and to $\bar{r}_l^k$ the other one.

Let $Q_r = Q \cap (R \cup \bar{R})$. For any assignment ϕ, let $Q_\phi = (Q \cap O) \cup (\phi(Q_r) \cap W)$ and $\bar{Q}_\phi = (Q \cap \bar{O}) \cup (\phi(Q_r) \cap \bar{W})$.

If $G(m)$ contains a Q_ϕ-good path set and $\bar{G}(m)$ a $\bar{Q}_\phi$-good path set, then we obtain the required set of edge-disjoint paths of $F(m)$ by taking the paths in $G(m)$ and $\bar{G}(m)$ and replacing for $1 \leq k \leq m$ and $1 \leq l \leq 4$, the last edge (c_l^k, w_l^k) (resp. $(\bar{c}_l^k, \bar{w}_l^k)$) of the path ending in w_l^k (resp. $\bar{w}_l^k$) by the subpath $(c_l^k, s_l^k, \phi^{-1}(w_l^k))$ (resp. $(\bar{c}_l^k, s_l^k, \phi^{-1}(\bar{w}_l^k))$). Note that we actually get more paths than required; some of them joining some a_l^1 or $\bar{a}_l^1$ to some c_l^m or $\bar{c}_l^m$ are not needed.

Finally note that we have many choices of possible assignments and we can always choose an assignment ϕ such that Q_ϕ (resp. $\bar{Q}_\phi$) contains:

- either 2 outputs of W (resp. $\bar{W}$) and none of O (resp. $\bar{O}$),

- or 1 output of W (resp. $\bar{W}$) and at most 2 of O (resp. $\bar{O}$),

- or no outputs of W (resp. $\bar{W}$).

Indeed choose ϕ such that:

- if $|Q_r| = 4$, then $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 2$;

- if $|Q_r| = 3$ and $|Q \cap O| = 1$, then $|\phi(Q_r) \cap W| = 1$ and $|\phi(Q_r) \cap \bar{W}| = 2$;

- if $|Q_r| = 3$ and $|Q \cap \bar{O}| = 1$, then $|\phi(Q_r) \cap W| = 2$ and $|\phi(Q_r) \cap \bar{W}| = 1$;

- if $|Q_r| = 2$, then $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 1$;

- if $|Q_r| = 1$ and $|Q \cap O| > |Q \cap \bar{O}|$, then $|\phi(Q_r) \cap W| = 0$ and $|\phi(Q_r) \cap \bar{W}| = 1$;

- if $|Q_r| = 1$ and $|Q \cap O| < |Q \cap \bar{O}|$, then $|\phi(Q_r) \cap W| = 1$ and $|\phi(Q_r) \cap \bar{W}| = 0$.

In summary, to prove that $H(m)$ is a repartitor it suffices to prove the following lemma.

Lemma 58 *For any set Q_0 of outputs of $G(m)$ such that either $|Q_0 \cap W| = 2$ and $|Q_0 \cap O| = 0$, or $|Q_0 \cap W| = 1$ and $|Q_0 \cap O| \leq 2$, or $|Q_0 \cap W| = 0$, there exists a Q_0-good path set in $G(m)$.*

Proof. We call a *default path* for w_l^k, (resp. o_l^k) the path (j_l^k, c_l^k, w_l^k) (resp. $(i_{l+1}^k, b_{l+1}^k, a_l^k, o_l^k)$). Let L_l be the path $(a_l^1, b_l^1, c_l^1, a_l^2, b_l^2, c_l^2, \ldots, a_l^m, b_l^m, c_l^m)$ and let us denote by $L_l[x, y]$ the subpath of L_l with end vertices x and y. Let $O_l = \{o_l^k | 1 \leq k \leq m\}$ and $W_l = \{w_l^k | 1 \leq k \leq m\}$.

By symmetry of $G(m)$, we only need to prove the lemma for the following cases:

(i) $|Q_0 \cap (O_l \cup W_l)| \leq 1$ for $1 \leq l \leq 4$;

(ii) $Q_0 \cap (O \cup W) = \{w_1^{k_1}, w_1^{k_2}\}$ with $k_1 < k_2$;

(iii) $Q_0 \cap W = \{w_1^{k_1}\}$ and $Q_0 \cap O = \{o_1^{k_2}\}$;

(iv) $Q_0 \cap W = \{w_1^{k_0}\}$ and $Q_0 \cap O = \{o_{l_1}^{k_1}, o_{l_2}^{k_2}\}$ with $l_1 = 1$ or $l_1 = l_2$;

(v) $Q_0 \cap (W \cup O) = \{o_1^{k_1}, o_1^{k_2}, o_{l_3}^{k_3}, o_{l_4}^{k_4}\}$.

We only give here the proof in cases (i), (ii), (iii) and (v) with $l_3 = l_4 = 1$, the other cases being very similar.

(i) - if $|Q_0 \cap (O_l \cup W_l)| = 0$, take the path L_l;
- if $Q_0 \cap (O_l \cup W_l) = w_l^{k_0}$, take the paths $(L_l[a_l^1, c_l^{k_0}], w_l^{k_0})$ and $(j_l^{k_0}, L[c_l^{k_0}, c_l^m])$.
- if $Q_0 \cap (O_l \cup W_l) = o_l^{k_0}$, take the paths $(L_l[a_l^1, a_l^{k_0}], o_l^{k_0})$ and $(i_{l+1}^{k_0}, b_{l+1}^{k_0}, L[a_l^{k_0}, c_l^m])$.

These paths plus the default paths associated with the outputs not in Q_0 form a Q_0-good path set.

(ii) Consider the paths $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1})$, $(j_1^{k_1}, L_1[c_1^{k_1}, a_1^{k_2}], o_1^{k_2})$, $(L_2[a_2^1, b_2^{k_2}], L_1[a_1^{k_2}, c_1^{k_2}], w_1^{k_2})$, $(i_2^{k_2}, L_2[b_2^{k_2}, c_2^m])$, $(j_1^{k_2}, L_1[c_1^{k_2}, c_1^m])$, L_3 and L_4. These paths plus the default paths associated with the outputs not in $Q_0 \cup \{o_1^{k_2}\}$ form a Q_0-good path set.

(iii) Consider the paths $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1})$, $(j_1^{k_1}, L_1[c_1^{k_1}, c_1^m])$, $(L_2[a_2^1, b_2^{k_2}], a_1^{k_2}, o_1^{k_2})$, $(i_2^{k_2}, L_2[b_2^{k_2}, c_2^m])$, L_3 and L_4. These paths plus the default paths associated with the outputs not in Q_0 form a Q_0-good path set.

(v) Suppose that $l_3 = l_4 = 1$. Without loss of generality, we may assume that $k_1 < k_2 < k_3 < k_4$. Consider the paths $(L_1[a_1^1, a_1^{k_1}], o_1^{k_1})$, $(L_2[a_2^1, b_2^{k_2}], a_1^{k_2}, o_1^{k_2})$, $(L_3[a_3^1, b_3^{k_3}], a_2^{k_3}, b_2^{k_3}, a_1^{k_3}, o_1^{k_3})$, $(L_4[a_4^1, b_4^{k_4}], a_3^{k_4}, b_3^{k_4}, a_2^{k_4}, b_2^{k_4}, a_1^{k_4}, o_1^{k_4})$, $(i_2^{k_1}, b_2^{k_1}, L_1[a_1^{k_1}, c_1^m])$, $(i_2^{k_2}, L_2[b_2^{k_2}, a_2^{k_3}], o_2^{k_3})$, $(i_2^{k_3}, L_2[b_2^{k_3}, a_2^{k_4}], o_2^{k_4})$, $(i_3^{k_3}, L_3[b_3^{k_3}, a_3^{k_4}], o_3^{k_4})$, $(i_2^{k_4}, L_2[b_2^{k_4}, c_2^m])$, $(i_3^{k_4}, L_3[b_3^{k_4}, c_3^m])$, and $(i_4^{k_4}, L_4[b_4^{k_4}, c_4^m])$. See Figure 14.

These paths plus the default paths associated with the outputs not in $Q_0 \cup \{o_2^{k_3}, o_2^{k_4}, o_3^{k_4}\}$ form a Q_0-good path set.

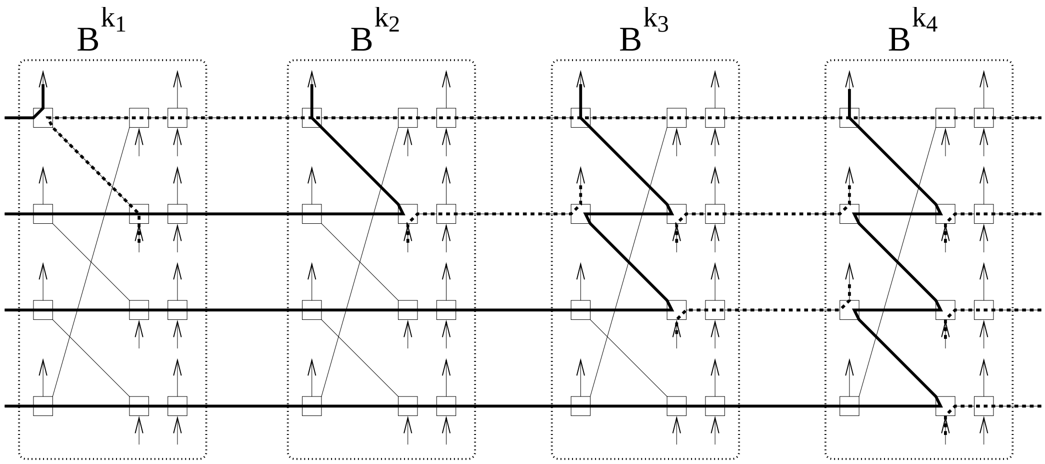

Figure 14: The non-default paths when $Q_0 \cap (W \cup O) = \{o_1^{k_1}, o_1^{k_2}, o_1^{k_3}, o_1^{k_4}\}$.

Proof of Theorem 56. By Theorem 33, $R(n, 4, 0) \geq \frac{7n}{4} - 7$. For $n = 16m + 4$, we have constructed an $(n, 4, 0)$-repartitor $H(m)$ with $28m + 18 = \frac{7n}{4} + 11$ switches. If n is not of the form $16m + 4$, let $n = 16m + 4 - h$ with $1 \leq h \leq 15$. Let $K(n)$ be the network obtained from $H(m)$ by the L-contraction of h linking edges. That is always possible as the edges (b_l^k, a_{l+1}^k) or (c_l^k, s_l^k) are linking. By Lemma 44, $K(n)$ is an $(n, 4, 0)$-repartitor. And $K(n)$ has $28m + 18 - h = \frac{7n}{4} + 11 + \frac{3h}{4} \leq \frac{7n}{4} + \frac{89}{4}$ switches. ■

Theorem 59

$$R(n, 3, 1) = \frac{7n}{4} + c,$$

where $-\frac{19}{4} \leq c \leq 27$.

Proof. From Lemma 42, we obtain $R(n, 3, 1) \leq R(n + 1, 4, 0) + R(3, 3, 1)$. By Theorem 56, $R(n+1, 4, 0) \leq \frac{7n}{4} + 24$. And $R(3, 3, 1) = R(3, 0, 1) = 3$. Hence $R(n, 3, 1) \leq \frac{7n}{4} + 27$. Theorem 33 gives the other inequality. ■

5 Conclusion

In this paper, we studied the minimum size $R(n, p, f)$ of an (n, p, f)-repartitor with n inputs, p of which are priorities, and $n + f$ outputs. We have obtained general upper bounds, lower bounds, and tight bounds for small values of p and f. It would be interesting to see if our lower bound for $p = 6$ and $f = 0$ is attained and to study the cases of $p = 1, 2, 3, 4$ and arbitrary f.

It would also be interesting to further explore the behaviour of the function $R(n, p, f)$. For example, we expect that $R(n, p, f)$ is monotone in p.

Conjecture 60 *If* $p \leq p' \leq \frac{n}{2}$ *then* $R(n, p, f) \leq R(n, p', f)$.

Our results confirm that this holds for small values of n, p and f. In view of this conjecture, the case $p = n/2$ seems particularly interesting. Very recently, Havet [8] discovered an asymptotically better upper bound than our Theorem 8: He proved that $R(n,p,f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n+f))$ if $p \leq \frac{n-f}{2}$, $R(n,p,f) \leq 18n + 34f + O(\log(n+f))$, if $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$, and $R(n,p,f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n+f))$ if $p \geq \frac{n+f}{2}$.

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References

[1] N. Alon and M. Capalbo, Smaller explicit superconcentrators, *Internet Math.* **1** (2003), 151–163.

[2] B. Beauquier and E. Darrot, Arbitrary size Waksman networks, *Actes des 1ères Rencontres Francophones sur les Aspects Algorithmiques des Télécommunications (AlgoTel'99)*, pp. 95-100, 1999, Editions INRIA.

[3] B. Beauquier and E. Darrot, Arbitrary size Waksman networks and their vulnerability, *Parallel Processing Letters* **12** (2002), 287–296.

[4] J.-C. Bermond, E. Darrot, and O. Delmas, Design of fault-tolerant networks for satellites (TWTA redundancy), *Networks* **40** (2002), 202–207.

[5] J.-C. Bermond, F. Havet, and D. Tóth, Fault tolerant on-board networks with priorities, INRIA Research Report RR-5363 (2004), `http://www-sop.inria.fr/rapports/sophia/RR-5363.html`.

[6] J.-C. Bermond, S. Pérennes, and D. Tóth, On the design of fault tolerant flow networks, Manuscript (2005).

[7] F. R. K. Chung, On concentrators, superconcentrators, generalizers, and nonblocking networks, *Bell System Tech. J.* **58** (1979), 1765–1777.

[8] F. Havet, Repartitors, selectors and superselectors, *Journal of Interconnection Networks*, submitted (2005).

[9] N. Pippenger, Superconcentrators, *SIAM J. Comput.* **6** (1977), 298–304.