TILING THREE-SPACE BY COMBINATORIALLY EQUIVALENT CONVEX POLYTOPES

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[Received 9 February 1983—Revised 4 September 1983]

ABSTRACT

The paper settles a problem of Danzer, Grünbaum, and Shephard on tilings by convex polytopes. We prove that, for a given three-dimensional convex polytope P, there is a locally finite tiling of the Euclidean three-space by convex polytopes each combinatorially equivalent to P. In general, face-to-face tilings will not exist.

1. *Introduction*

It is a well-known fact that, for a given n ($\geqslant 3$), the Euclidean plane can be tiled by convex n-gons, even in such a way that the intersection of any two tiles is either empty or a face of each (Grünbaum and Shephard [**8**]). In particular, for $n \leqslant 6$, there exist tilings by pairwise congruent convex n-gons (cf. Heesch [**10**]).

The analogous problem in higher dimensions is much more intractable. Given a convex d-polytope P one may ask whether the Euclidean space $\mathbb{E}^d$ can be tiled by convex d-polytopes each combinatorially equivalent to P. Such tilings may have the property of being *face-to-face*, which simply means that the intersection of any two tiles is either empty or a face of each. It is natural to restrict attention to *locally finite* tilings. A tiling is locally finite, if each point of $\mathbb{E}^d$ has a neighbourhood meeting only finitely many tiles.

It should be emphasized that we require combinatorial equivalence rather than congruence for the tiles. In fact, congruence requirements would strongly restrict the number of possibilities. For instance, it is not even known whether there is an upper bound for the number of 2-faces of a convex 3-polytope, of which congruent copies tile three-space face-to-face (cf. Delone [**4**], Grünbaum and Shephard [**7**], Engel [**5**]).

In a sense, a substitution for congruence is offered by the notion of normality (cf. Grünbaum and Shephard [**8**]). A tiling is *normal* if its tiles are uniformly bounded, that is, there exist two positive parameters r_1 and r_2 such that each tile contains a ball of radius r_1 and is contained in a ball of radius r_2. It has been established many times that for every normal tiling of the plane by convex n-gons we must have $n = 3, 4, 5$, or 6 (cf. Reinhardt [**15**], Niven [**13**], Grünbaum and Shephard [**8**]). In this paper only the 3-dimensional case is considered. The above-mentioned problem on tilings by isomorphic polytopes in three dimensions was originally posed by L. Danzer at the London Mathematical Society Symposium 'Relations between finite- and infinite-dimensional convexity' held in Durham in 1975, even with the stronger requirement that the tiling should be locally finite and face-to-face (cf. Danzer, Grünbaum, and Shephard [**3**], Larman and Rogers [**12**]). Recently there has been some progress in this direction.

In [**3**] Danzer, Grünbaum, and Shephard describe a simplicial 3-polytope which will not tile $\mathbb{E}^3$ both face-to-face and normally. Their example is inspired by the results of Perles and Shephard [14] on the existence of 3-dimensional non-facets. Recall that *non-facets* are convex polytopes, of which isomorphic copies will not fit together as the facets of a higher-dimensional polytope.

On the other hand, Grünbaum, Mani-Levitska, and Shephard prove in [9] that each simplicial 3-polytope admits a locally finite face-to-face tiling by isomorphic copies. The example of [3] shows, however, that this result cannot be strengthened by the requirement of normality.

Contrary to the general belief in a positive answer to Danzer's problem (cf. [9]) the author recently proved the existence of 3-dimensional non-tiles, that is, the existence of polytopes of which isomorphic copies will not give a tiling of $\mathbb{E}^3$ that is locally finite and face-to-face (cf. [17]). In fact, these polytopes can also serve as counter-examples, if we allow the isomorphic copies of P to be non-convex. Also, they provide non-facets in the sense of Perles and Shephard [14]. Furthermore, the results of [17] extend to higher dimensions (cf. [18]).

As the general answer to Danzer's problem is in the negative, it is interesting to ask whether tilings are possible, if we relax the condition that the tiling must be face-to-face. In fact, we shall give an affirmative answer in Theorem 1, with regard to the existence of non-tiles now known to be best possible. In other words, we show that for every convex 3-polytope P there exists a locally finite tiling of $\mathbb{E}^3$ by convex polytopes each combinatorially equivalent to P. However, these tilings are non-normal in general.

Although there seems to be little hope of characterizing those polytopes which give a tiling of three-space that is locally finite and face-to-face, one may look for special types of polytopes which admit such tilings. As already mentioned, the tilings exist, at least for simplicial 3-polytopes. However, we remark without proof that the same holds also for n-gonal convex pyramids and bipyramids (cf. [16], Grünbaum, Mani-Levitska, and Shephard [9]).

For notation and basic results the reader is referred to Grünbaum [6] for convex polytopes and Grünbaum and Shephard [8] for tilings. In § 2 we state some more or less technical results on convex 3-polytopes and describe the basic construction underlying the main theorem. The theorem itself is stated in § 3.

2. *Notation and basic constructions*

Throughout the whole paper we use the following notation. For two different points x and y in $\mathbb{E}^3$ the line passing through x and y, the ray issuing from x and passing through y, and the line segment joining x and y are denoted by $\overline{xy}$, $\overset{\mapsto}{xy}$, and $[xy]$ respectively. For a non-empty subset F of $\mathbb{E}^3$ let aff(F) be the affine hull of F. For x in $\mathbb{E}^3$ we write $C_x(F) := \bigcup_{y \in F} \overset{\mapsto}{xy}$ and $K_x(F) := \bigcup_{y \in F} [xy]$ and use this notation in the particular case where F is contained in a plane not containing x. For each positive r let $A_r := \{x \mid |x| > r\}$ where, as usual, $|x|$ denotes the Euclidean length of x.

Let F be a compact subset of $\mathbb{E}^3 \setminus \{0\}$ contained in a half-space determined by a hyperplane through 0 and let $\hat{F} := \{x/|x| \mid x \in F\}$. The spherical diameter of the smallest spherical cap containing $\hat{F}$ is the *spherical diameter* $d(F)$ *of* F. By definition $d(F) \leqslant \pi$. In particular, we shall make use of the following trivial fact. If $d(F) < \pi$, $r > 0$, and $F \subset A(r)$, then the convex hull of F lies in $A(r\cos(\frac{1}{2}d(F)))$.

Let P be a convex 3-polytope. A *P-tiling* $\mathscr{P}$ is a finite family of convex 3-polytopes each combinatorially isomorphic to P, such that no two polytopes have interior points in common and $\text{set}(\mathcal{P}) := \bigcup_{P' \in \mathcal{P}} P'$ is a closed topological ball. Note that the notion of a P-tiling is related to the combinatorial equivalence class of P rather than to P itself.

Let $\mathcal{P}$ be a P-tiling. Denote the boundary of set $(\mathcal{P})$ by $b(\mathcal{P})$. The points x in $b(\mathcal{P})$ can be distinguished by the number of pairwise non-planar plane segments, into which a sufficiently small neighbourhood of x in $b(\mathcal{P})$ can be dissected. It is easy to see that $b(\mathcal{P})$ can be decomposed into finitely many closed, connected plane segments each bounded by finitely many line segments. Later we shall be concerned mainly with those plane segments whose affine hull does not contain 0. The union of all these plane segments is denoted by $\hat{b}(\mathcal{P})$.

In the course of the construction we shall deal with a finite family Z of convex n-gons for a fixed n $(3 \leqslant n \leqslant 5)$, which provides a decomposition of $b(\mathcal{P})$ or a subset of $b(\mathcal{P})$ into n-gons, not necessarily face-to-face. A point in $b(\mathcal{P})$ or a subset of $b(\mathcal{P})$ is called a *vertex* or an *edge of* Z, respectively, if it is a vertex or an edge of a member of Z.

For the proof of the main theorem we adopt an idea, which has already been used by Grünbaum, Mani-Levitska, and Shephard in [**9**]. We shall always work with star-shaped P-tilings $\mathcal{P}$, that is, $\text{set}(\mathcal{P})$ is star-shaped relative to 0. In particular, this ensures that the boundary $b(\mathcal{P})$ of $\mathcal{P}$ is not too complicated.

It is well known that every convex 3-polytope P has at least one n-gonal facet and one k-valent vertex with $n, k \leqslant 5$ (cf. [**6**]). From Barnette and Grünbaum [**2**] we know that we can arbitrarily preassign the shape of one facet of a convex 3-polytope, or, more exactly, if G is an m-gonal facet of P with vertices $z_1, \ldots, z_m$ and F is an arbitrary convex m-gon in $\mathbb{E}^3$ with vertices $x_1, \ldots, x_m$, then there exists a convex 3-polytope P' combinatorially equivalent to P such that F is a facet of P' and the vertex x_i of P' corresponds to the vertex z_i of P for each i. Later we shall apply a refined version of this result to each n-gon of a suitable decomposition Z of a P-tiling $\mathcal{P}$. Note that, by the above remarks, we can restrict to $n \leqslant 5$.

At first we state some technical lemmas. We assume from now on that P is fixed.

LEMMA 1. *Let G be an m-gonal facet of P with vertices $z_1, \ldots, z_m$ (here and elsewhere the numbering is in cyclic order). Let F be a convex m-gon in $\mathbb{E}^3$ with vertices $x_1, \ldots, x_m$ $(x_{m+1} = x_1)$, let $y \in \mathbb{E}^3$ with $y \notin \text{aff}(F)$, and let $r > 0$ (see Fig.* 1). *Then there exists a polytope P' equivalent to P with the following properties:*

(a) *F is a facet of P' separating y and P', and the vertex x_i of P' corresponds to the vertex z_i of P $(i = 1, \ldots, m)$;*
(b) *$P' \subset C_y(F)$ and the planes $\text{aff}(y, x_i, x_{i+1})$ support P' in a facet or an edge, if $i = 1, m$ or $i = 2, \ldots, m-1$ respectively;*
(c) *all vertices of P' not in F and all edges joining two vertices not in F lie in $A(r)$.*

Proof. By the result of Barnette and Grünbaum we can preassign F for P'. Applying a suitable affine transformation leaving $\text{aff}(F)$ pointwise fixed we may assume that $\text{aff}(y, x_1, x_2)$ and $\text{aff}(y, x_1, x_m)$ support P' in a facet. A polytope with the desired properties arises if we take for large t the image of P' under the affine transformation, which leaves $\text{aff}(F)$ pointwise fixed and maps $x_1 + (x_1 - y)$ onto $x_1 + t(x_1 - y)$.

The next lemma is the dual version of the result of Barnette and Grünbaum and can be proved by applying their result to the dual polytope of P.

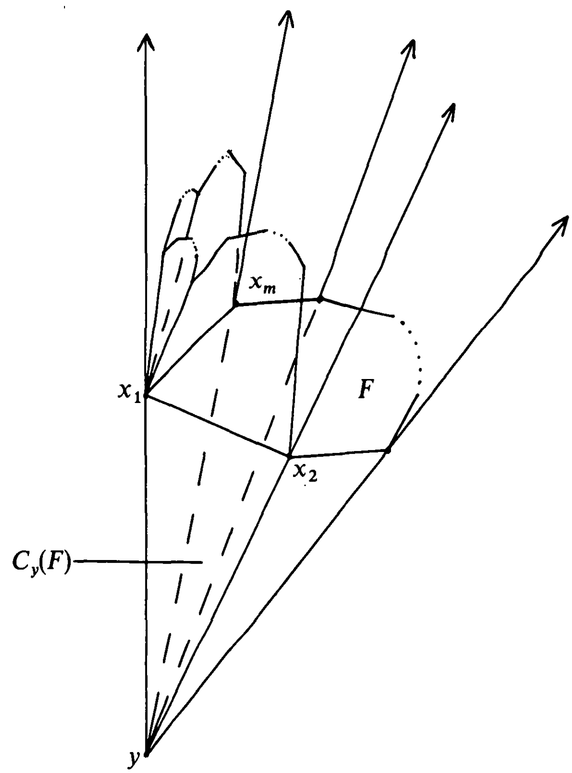

FIG. 1

LEMMA 2. *Let z be an l-valent vertex of P and let $G_1, \ldots, G_l$ be the facets of P containing z. Let $x \in \mathbb{E}^3$ and let Q be a convex l-gon with vertices $x_1, \ldots, x_l$ such that $x \notin \mathrm{aff}(Q)$. Then there exists a polytope P' combinatorially equivalent to P such that x is the vertex of P' corresponding to z, $C_x(Q)$ is the supporting cone for P' in x (that is, $C_x(Q) = C_x(P')$), and the facet of P' lying in $\mathrm{aff}(x, x_i, x_{i+1})$ corresponds to G_i for each $i = 1, \ldots, l$.*

Proof. Assume without loss of generality that 0 is an interior point of the cone $C_x(Q)$. For $i = 1, \ldots, l$ define u_i by $\mathrm{aff}(x, x_i, x_{i+1}) = \{u \mid \langle u, u_i \rangle = 1\}$, where $\langle \ , \ \rangle$ denotes the scalar product. Then, $u_1, \ldots, u_l$ are the vertices of a convex l-gon R lying in the plane $\{u \mid \langle u, x \rangle = 1\}$.

By the result of Barnette and Grünbaum we can preassign R for the dual P^* of P, that is, R is a 2-face of an isomorphic copy $(P^*)'$ of P^* with u_i corresponding to G_i for each i. By applying an affine transformation if need be, we may suppose that 0 is an interior point of $(P^*)'$.

Now, define P' to be the polar set of $(P^*)'$ (cf. Grünbaum [**6**, p. 47]). Obviously, P' and P are isomorphic convex polytopes. For each i, the facet of P' belonging to u_i is contained in the plane $\{u \mid \langle u, u_i \rangle = 1\} = \mathrm{aff}(x, x_i, x_{i+1})$. However, these planes intersect in the vertex x of P'. But this completes the proof.

The basic tool in the construction of P-tilings will be Lemma 4. It states that each simplicial cone is the supporting cone of a suitable star-shaped P-tiling.

At first we need some simple facts about dissecting a triangle into convex k-gons for a fixed k ($\leqslant 5$). Let $k = 4$ or $k = 5$. Fig. 2 (a, b) shows how an equilateral triangle Δ

with vertices z_1, z_2, and z_3 can be dissected face-to-face into convex k-gons $Q_1, \ldots, Q_3$ or $Q_1, \ldots, Q_{15}$ respectively such that the Q_i have the following property:

(*) *if* $i < j$, $(i, j) \neq (14, 15)$, *and* Q_i *and* Q_j *share an edge* e, *then* $z_1 \notin \mathrm{aff}(e)$ *and* Q_j *lies in the half-plane of* $\mathrm{aff}(\Delta)$ *determined by* $\mathrm{aff}(e)$ *and containing* z_1.

If Δ is an arbitrary triangle, then there is obviously a dissection into k-gons which is isomorphic to that of Fig. 2(a, b), that is, it also satisfies (*). Furthermore, if y_1, y_2, and y_3 are interior points of Δ such that y_1, y_2, y_3, z_2, and z_3 are the vertices of a convex 5-gon Q'_1, then there exists a dissection of Δ into 5-gons isomorphic to that of Fig. 2(b) such that Q'_1 corresponds to Q_1. In other words, we can arbitrarily preassign Q_1 in the case where $k = 5$.

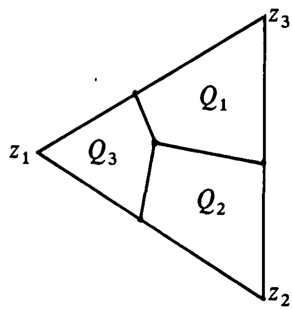

FIG. 2(a). The affine hull of $Q_1 \cap Q_2$ intersects $[z_1 z_3]$ in a relative interior point.

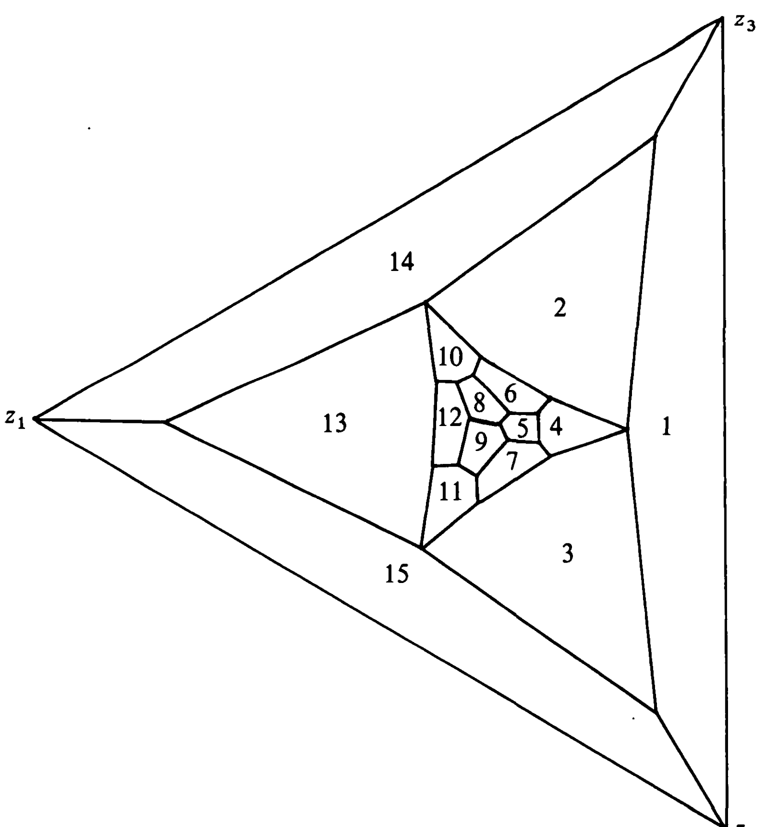

FIG. 2(b). The numbers 1, 2, ..., 15 give the indices for the Q_i. The affine hulls of $Q_5 \cap Q_7$ and $Q_8 \cap Q_9$ or $Q_5 \cap Q_6$ intersect $[z_1 z_3]$ or $[z_1 z_2]$ in a relative interior point, respectively.

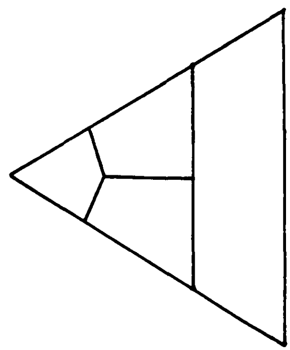

FIG. 2(c)

For the proof of Lemma 4 we need

LEMMA 3. *For $k = 4$ and $k = 5$ let $m_k = 3$ and $m_k = 15$, respectively. Assume that P has a k-valent vertex. Let Δ be a triangle with vertices z_1, z_2, z_3 and let $\{Q_1, \ldots, Q_{m_k}\}$ be a dissection of Δ into convex k-gons isomorphic to that illustrated in Fig.* 2(a, b) (*that is,* (*)) *holds). Suppose that $x \notin \operatorname{aff}(\Delta)$ and that, for $i = 1, \ldots, m_k$, P_i is a convex polytope equivalent to P, for which x is a vertex and $C_x(Q_i)$ is the corresponding supporting cone in x. If, for given i and j, the k-gons Q_i and Q_j share an edge, we write $F_{i,j}$ for the facet of P_i which contains x and is contained in $\operatorname{aff}(x, Q_i \cap Q_j)$. We assume that the P_i satisfy the following property:*

(**) *if $i < j$ and Q_i and Q_j share an edge, then $F_{j,i} \subset F_{i,j}$*

(see Fig. 3). Define $\mathscr{P} := \{P_1, \ldots, P_{m_k}\}$ and $F := \operatorname{set}(\mathscr{P}) \cap \operatorname{aff}(x, z_2, z_3)$. Then there is a point y_0 in $\overrightarrow{z_1 x}$ not in $[z_1 x]$ such that $\operatorname{set}(\mathscr{P}) \cup K_y(F)$ is star-shaped relative to y for every y in $[x y_0]$.

REMARK. Obviously, by Lemma 2, there exists a family $\{P_i \mid i = 1, \ldots, m_k\}$ of polytopes satisfying all the assumptions of Lemma 3 apart from the star-shape condition (**). If we replace each polytope P_i by its image under a suitable dilation with centre x (depending on i) if need be, we can obtain a family $\mathscr{P}$ with all the desired properties. In fact, we can even require that $F = \operatorname{set}(\mathscr{P}) \cap \operatorname{aff}(x, z_2, z_3)$ lies in any preassigned set.

Proof. For $i = 1, \ldots, m_k$ let $\mathscr{P}_i := \{P_1, \ldots, P_i\}$, so that $\mathscr{P}_{m_k} = \mathscr{P}$. We prove the statement for $\mathscr{P}_i$ instead of $\mathscr{P}$ inductively, thus establishing the result for $\mathscr{P}$ after finitely many steps.

For $i = 1, \ldots, m_k$ let T_i denote the intersection of all open half-spaces containing P_i and determined by a plane which supports P_i in a facet not containing x. Obviously, a sufficiently small neighbourhood U of x lies in T_i for each i. We choose y_0 in $U \cap (\overrightarrow{z_1 x} \setminus [z_1 x])$ and show that for every y in $[x y_0]$ and every ray s issuing from y the intersection of s and $\operatorname{set}(\mathscr{P}_i) \cup K_y(F)$ is either empty or a line segment containing y. Let y and s be given such that the intersection is non-empty. The case where $s = \overrightarrow{yx}$ is trivial and so we exclude it.

Assume that s intersects a fixed polytope P_j. By the choice of y_0 the ray s meets the star $P_j(x)$ of x in P_j either in one point or in a line segment in the exceptional cases

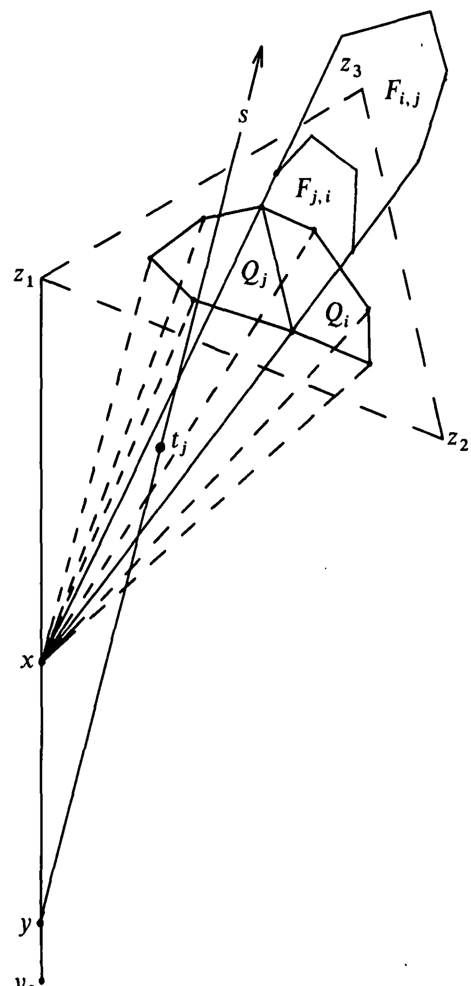

FIG. 3

where $k = 5$, $j = 14$ or 15, $s \subset \text{aff}(x, Q_{14} \cap Q_{15})$, or in at least one, but at most two, points in all other cases. Let t_j denote the first point of contact of s and $P_j(x)$ (see Fig. 3). Let us exclude the exceptional case for a moment. Adopting the beneath-beyond terminology of Grünbaum [6, p. 78] we note that t_j lies in a facet of P_j containing x, for which y is beyond (with respect to P_j). By (*) and (**) each facet of P_j in $P_j(x)$, for which y is beyond, lies either in $\text{aff}(x, z_2, z_3)$ or in a facet of a certain P_l with $l < j$.

Now let us proceed inductively. For $i = 1$ we have $\text{set}(\mathscr{P}_1) = P_1$ and so the result is an immediate consequence of the convexity of P_1. Suppose that the result holds for a given i and that s intersects $\text{set}(\mathscr{P}_{i+1})$. If s does not meet P_{i+1}, then the induction hypothesis applies, and so we may assume that $s \cap P_{i+1} \neq \emptyset$. We exclude the exceptional case for a moment.

As already mentioned, the point t_{i+1} of $P_{i+1}(x)$ lies either in $\text{aff}(x, z_2, z_3)$ or in a facet of a certain P_l with $l < i+1$. In the former case (that is when $k = 4$, $i = 1$) the convexity of P_{i+1} proves the statement. In the latter case, t_{i+1} is in $\text{set}(\mathscr{P}_i)$ and, by the induction hypothesis, is the endpoint of the line segment $s \cap (\text{set}(\mathscr{P}_i) \cup K_y(F))$. But t_{i+1} is also the starting point of the line segment (or point) $s \cap P_{i+1}$, and consequently, $s \cap (\text{set}(\mathscr{P}_{i+1}) \cup K_y(F))$ is also a line segment containing y.

Finally, we consider the exceptional case; in particular, $i+1 = 14$ or 15. In both cases $t_{i+1} \in \operatorname{aff}(x, Q_{13} \cap Q_{14} \cap Q_{15})$. If $i+1 = 14$ or 15 respectively, then

$$t_{i+1} \in F_{14,13} \subset F_{13,14} \subset \operatorname{set}(\mathscr{P}_{13}) = \operatorname{set}(\mathscr{P}_i)$$

or

$$t_{i+1} \in F_{15,14} \subset F_{14,15} \subset \operatorname{set}(\mathscr{P}_{14}) = \operatorname{set}(\mathscr{P}_i).$$

Hence, t_{i+1} is the endpoint and starting point respectively of the line segment $s \cap (\operatorname{set}(\mathscr{P}_i) \cup K_y(F))$ and $s \cap F_{14,15}$. But the union of both sets is just

$$s \cap (\operatorname{set}(\mathscr{P}_{i+1}) \cup K_y(F))$$

and that completes the proof.

The next step is to turn the P-tiling of Lemma 3 into a P-tiling which is actually star-shaped relative to 0. This will be done in Lemma 4.

The boundary $b(\mathscr{P})$ of the P-tiling in Lemma 3 consists of finitely many plane segments which are either convex polygons or difference sets of such. Of course, each can be triangulated without adding new vertices. By dissecting each triangle into n-gons ($n = 4, 5$), as illustrated in Fig. 2(b, c), one gets a dissection into convex n-gons (not necessarily face-to-face). In particular, if each triangle has one vertex in $A(r)$, then the dissection can be arranged in such a way that all new vertices lie in $A(r)$. Recall that $\hat{b}(\mathscr{P})$ denotes the union of all plane segments in $b(\mathscr{P})$ whose affine hull does not contain 0.

LEMMA 4. *Let $k, m, n \in \mathbb{N}$ and $k, n \leqslant 5$. Assume that P has at least one n-gonal facet and also one m-gonal facet which is incident with a k-valent vertex of P. Let Δ be a triangle with vertices z_1, z_2, and z_3, let $0 \notin \operatorname{aff}(\Delta)$, and let x be a relative interior point of $[0z_1]$. Suppose that F is a convex m-gon with vertex x such that the two edges of F incident with x lie on $\overrightarrow{xz_2}$ or $\overrightarrow{xz_3}$ respectively (see Fig. 4). Let $r > 0$.*

Then there exists a P-tiling $\mathscr{P}$ with the following properties:

(a) *x is a vertex of each member of $\mathscr{P}$;*
(b) *$C_x(\Delta)$ is the supporting cone for $\mathscr{P}$, that is, $C_x(\Delta) = C_x(\operatorname{set}(\mathscr{P}))$;*
(c) *F is a facet of one member of $\mathscr{P}$ and $\operatorname{set}(\mathscr{P}) \cap \operatorname{aff}(x, z_2, z_3) = F$;*
(d) *$\operatorname{set}(\mathscr{P}) \cup K_0(F)$ is star-shaped relative to 0, and if H is one of the boundary hyperplanes of $C_0(F)$ not containing $\overline{0x}$, then $\operatorname{set}(\mathscr{P}) \cap H$ is an edge of F;*
(e) *there exists a dissection Z of $\hat{b}(\mathscr{P}) \setminus \operatorname{relint}(F)$ into convex n-gons such that each vertex of Z not in F and each edge of Z joining two vertices not in F lies in $A(r)$.*

Proof. First, for $k = 3, 4, 5$, we shall construct a convex polytope P_1 equivalent to P and a P-tiling $\mathscr{P}'$ including P_1. Here, F will be a facet of P_1 separating 0 and P_1, x will be a k-valent vertex of P_1, and $P_1 \subset C_0(F)$. If $k = 3, 4$ or $k = 5$, then the planes $\operatorname{aff}(0, x, z_2)$ and $\operatorname{aff}(0, x, z_3)$ will support P_1 in a facet or an edge belonging to F, respectively. Furthermore, there will exist a point y_0 in $\overrightarrow{z_1x}$ not in $[z_1x]$ such that $\operatorname{set}(\mathscr{P}') \cup K_y(F)$ is star-shaped relative to y for every y in $[xy_0]$.

By Lemma 1 the existence of P_1 is clear for $k = 3, 4$. Let $k = 5$ and let z be a relative interior point of Δ. By Lemma 1, there is a polytope P_1' equivalent to P, for which F is a facet and x is a k-valent vertex, and which is contained in $C_x(\operatorname{conv}\{z, z_2, z_3\})$. The

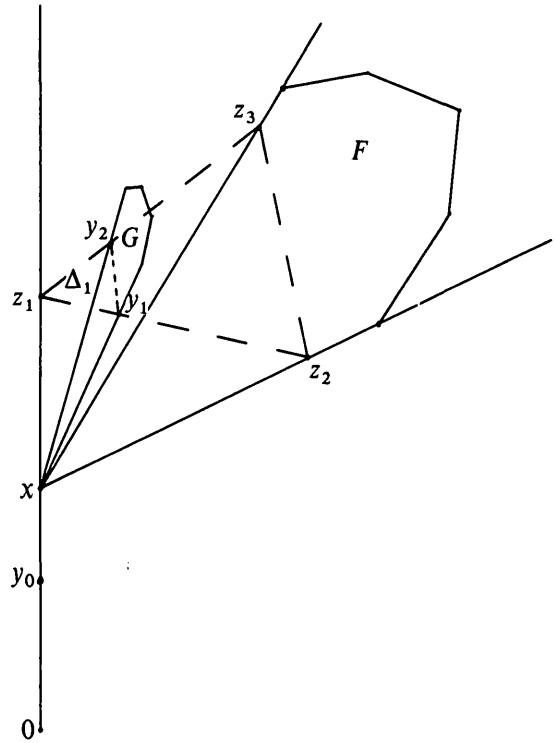

FIG. 4

image P_1 of P'_1 under a suitable affine transformation, which leaves aff(F) pointwise fixed and maps z_1 onto tz_1 has the required properties.

If $k = 3$, then $C_x(\Delta)$ is the supporting cone for P_1. Define $\mathscr{P}' := \{P_1\}$. If we choose y_0 in the intersection of $\overrightarrow{x0} \setminus \{x\}$ and all open half-spaces containing P_1, and belonging to a plane which supports P_1 in a facet not containing x, then of course, set$(\mathscr{P}') \cup K_y(F)$ is star-shaped relative to y for every y in $[xy_0]$.

Let $k = 4$ or $k = 5$. Each ray issuing from x and containing an edge of P_1 incident with x meets Δ. Denote the intersection points different from z_2, z_3 by $y_1, \ldots, y_{k-2}$.

Let $k = 4$ and let G be the facet of P_1 which contains x and is contained in the supporting plane aff(x, y_1, y_2) of P_1 (cf. Fig. 4). Let Δ_1 be the triangle with vertices z_1, y_1, and y_2. By Lemma 3 and its associated remark applied to Δ_1, there exists a P-tiling $\mathscr{P}_1$ with the following properties. Let $F_1 :=$ set$(\mathscr{P}_1) \cap$ aff(x, y_1, y_2). Then, $F_1 \subset G$ and there is a point y_0 in $\overrightarrow{z_1x}$ not in $[z_1x]$ such that set$(\mathscr{P}_1) \cup K_y(F_1)$ is star-shaped relative to y for every y in $[xy_0]$. Define $\mathscr{P}' := \{P_1\} \cup \mathscr{P}_1$. By replacing y_0 by a suitable point in $[xy_0] \setminus \{x\}$ if need be, we can even show that set$(\mathscr{P}') \cup K_y(F)$ is star-shaped relative to y for every y in $[xy_0]$. In fact, by our choice, $F_1 \subset G$.

Let $k = 5$. As already mentioned, it is possible to preassign the 5-gon Q_1 in the dissection of Δ (see Fig. 2(b)). So, by Lemma 3 and its associated remark applied to Δ and F, there is a P-tiling $\mathscr{P}'$ with the following properties. The member of $\mathscr{P}'$ corresponding to the 5-gon Q_1 with vertices z_2, z_3, y_1, y_2, and y_3 is just P_1 and, again, there is a y_0 with the required properties.

The last step is to use an affine transformation to turn $\mathscr{P}'$ into a P-tiling $\mathscr{P}$ which is star-shaped relative to 0. If we apply, for large t, the affine transformation φ_t which leaves aff(F) pointwise fixed and maps z_1 onto tz_1, then $\mathscr{P}_t := \{\varphi_t(P') \mid P' \in \mathscr{P}'\}$ will do. Furthermore, $\mathscr{P}_t$ has the additional property of (d).

By the construction, $\hat{b}(\mathscr{P}_t)$ consists of finitely many convex polygons and difference sets of such. For sufficiently large t, all vertices of members of $\mathscr{P}_t$ not in F lie in $A(r)$. Hence, by the above considerations, there exists a dissection Z of $\hat{b}(\mathscr{P}_t) \setminus \text{relint}(F)$ into convex n-gons with the same property. By increasing t if need be, we can also ensure that each edge joining two vertices not in F lies in $A(r)$. This completes the proof.

The basic step in the construction is described in Lemma 5. Its proof is now a consequence of Lemmas 1 and 4.

LEMMA 5. *Assume that $n \leqslant 5$ and that P has at least one n-gonal facet. Let F be a convex n-gon with $0 \notin$ aff(F) and let $r > 0$. Then there is a P-tiling $\mathscr{P}$ such that*

(a) *F is a facet of one member of $\mathscr{P}$ separating* set($\mathscr{P}$) *and* 0, *and also* set($\mathscr{P}$) $\subset C_0(F)$,

(b) set($\mathscr{P}$) $\cup K_0(F)$ *is star-shaped relative to* 0, *and if H is one of the boundary hyperplanes of $C_0(F)$, then $H \cap$* set($\mathscr{P}$) *is a polygon sharing an edge with F*,

(c) *there is a dissection Z of $\hat{b}(\mathscr{P}) \setminus$* relint($F$) *into convex n-gons such that each vertex and each edge of Z lies in $A(r)$.*

Proof. As mentioned earlier, P has a k-valent vertex z with $k \leqslant 5$. Choose m so that P has an m-gonal facet incident with z. Let k and m be fixed. Denote the vertices of F by $x_1, \dots, x_n$ (see Fig. 5).

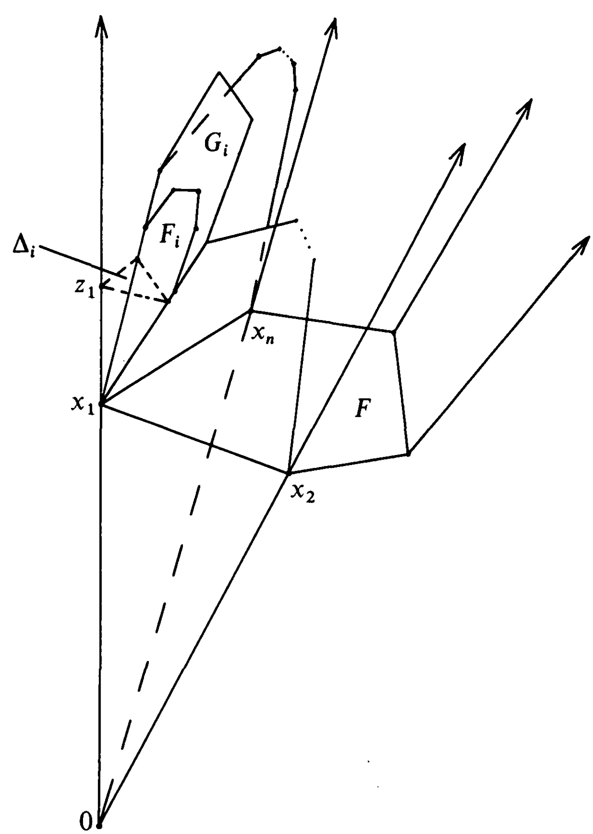

FIG. 5

We shall construct P-tilings $\mathscr{P}_1,\dots,\mathscr{P}_n$ and, for each $i = 1,\dots,n$, a dissection Z_i of $\hat{b}(\mathscr{P}_i)\setminus\operatorname{relint}(F)$ into n-gons, such that $\mathscr{P}_1 \subset \mathscr{P}_2 \subset \dots \subset \mathscr{P}_n$, $\operatorname{set}(\mathscr{P}_i) \cup K_0(F)$ is star-shaped relative to 0, and each vertex of Z_i different from $x_{i+1},\dots,x_n$ and each edge of Z_i joining two of these vertices lie in $A(r)$ $(i = 1,\dots,n)$. Finally, $\mathscr{P} = \mathscr{P}_n$ will have all the required properties. We begin with $\mathscr{P}_1$.

By Lemma 1, there exists a polytope P_1 equivalent to P with the following properties. The face F is a facet of P_1 separating 0 and P_1, and P_1 is contained in $C_0(F)$. All vertices of P_1 not in F and all edges joining two vertices not in F lie in $A(r)$. The planes $\operatorname{aff}(0,x_i,x_{i+1})$ support P_1 in a facet or an edge, if $i = 1,n$ or $i = 2,\dots,n-1$ respectively. Let l denote the valence of x_1 in P_1 and let $G_1,\dots,G_l$ denote the facets of P_1 incident with x_1, where $G_1 = F$, $G_2 \subset \operatorname{aff}(0,x_1,x_2)$, and $G_l \subset \operatorname{aff}(0,x_1,x_n)$. Fig. 5 illustrates the case where $l = 4$.

If $l = 3$, we define $\mathscr{P}_1 := \{P_1\}$ and take a suitable dissection of $\hat{b}(\mathscr{P}_1)\setminus\operatorname{relint}(F)$ for Z_1. Then this suffices.

Let $l \geqslant 4$. For $i = 3,\dots,l-1$ let F_i denote a convex m-gon such that $F_i \subset G_i$, both edges of F_i incident with x_1 lie on edges of G_i incident with x_1, and all vertices of F_i different from x_1 and all edges of F_i joining two vertices different from x_1 lie in $A(r)$. Let $z_1 \in \overleftrightarrow{0x_1}\setminus[0x_1]$ and let Δ_i $(i = 3,\dots,l-1)$ be a triangle, whose vertices are z_1 and two points in $(G_{i-1}\cap G_i)\setminus\{x_1\}$ and $(G_i \cap G_{i+1})\setminus\{x_1\}$ respectively. Applying Lemma 4 to each triangle Δ_i and m-gon F_i we get a P-tiling $\mathscr{P}_{1,i}$ with the properties of Lemma 4. Define $\mathscr{P}_1 := \{P_1\} \cup \bigcup_{i=3}^{l-1}\mathscr{P}_{1,i}$. Since $F_i \subset G_i$ $(i = 3,\dots,l-1)$, $\operatorname{set}(\mathscr{P}_1)\cup K_0(F)$ is star-shaped relative to 0. The set $\hat{b}(\mathscr{P}_1)$ consists of F, the facets of P_1 not incident with x_1, the difference sets $G_i\setminus F_i$, and the sets $\hat{b}(\mathscr{P}_{1,i})\setminus\operatorname{relint}(F_i)$ $(i = 3,\dots,l-1)$. Obviously, there is a dissection Z_1 of $\hat{b}(\mathscr{P}_1)\setminus\operatorname{relint}(F)$ into convex n-gons, so that all vertices of Z_1 different from $x_2,\dots,x_n$ and all edges of Z_1 joining two of these vertices lie in $A(r)$. This establishes the existence of $\mathscr{P}_1$.

We shall use P-tilings of type $\mathscr{P}_1$ as building blocks for the desired P-tiling of Lemma 5. Since $\mathscr{P}_1$ is attached to the preassigned n-gon F, the vertex x_1 of F, and the parameter r, we shall say for convenience that $\mathscr{P}_1$ belongs to (F,x_1,r).

Now let us consider the n-gons H of Z_1 containing x_2. For each H there is a P-tiling $\mathscr{P}_{2,H}$ that belongs to (H,x_2,r). Let $\mathscr{P}_2$ denote the union of $\mathscr{P}_1$ and all P-tilings $\mathscr{P}_{2,H}$. A dissection Z_2 arises from Z_1 and the dissections of the sets $\hat{b}(\mathscr{P}_{2,H})\setminus\operatorname{relint}(H)$. Then, Z_2 has the property that all vertices different from $x_3,\dots,x_n$ and all edges joining two of these vertices lie in $A(r)$.

Next we proceed with x_3 and the corresponding n-gons in Z_2 which contain x_3, get a P-tiling $\mathscr{P}_3$ and a dissection Z_3 analogous to $\mathscr{P}_2$ and Z_2, proceed with x_4, and so on. Finally, the P-tiling $\mathscr{P}_n$ and the dissection Z_n have all the required properties. This completes the proof.

3. *Tiling three-space by isomorphic polytopes*

By the results of § 2 we are now able to prove the theorem in question.

THEOREM 1. *For every convex 3-polytope P there exists a locally finite tiling of $\mathbb{E}^3$ by convex polytopes each combinatorially equivalent to P.*

Proof. The polytope P has an n-gonal facet with $n \leqslant 5$. Let n be fixed (of course, we choose n minimal).

We shall construct two infinite sequences $(\mathscr{P}_i)_{i\in\mathbb{N}}$ of P-tilings and $(Z_i)_{i\in\mathbb{N}}$ of corresponding dissections such that

(1) $\mathscr{P}_i \subset \mathscr{P}_{i+1}$ $(i \in \mathbb{N})$,

(2) 0 is an interior point of set($\mathscr{P}_i$) and set($\mathscr{P}_i$) is star-shaped relative to 0 $(i \in \mathbb{N})$,

(3) Z_i is a dissection of $\hat{b}(\mathscr{P}_i)$ into convex n-gons, and each vertex of Z_i lies in $A(i)$ $(i \in \mathbb{N})$,

(4) there is a constant c, with $c < \pi$, such that the spherical diameter $d(F)$ is at most c for each i in $\mathbb{N}$ and each F in Z_i.

At first we note that $\mathscr{T} := \bigcup_{i\in\mathbb{N}} \mathscr{P}_i$ will be a locally finite tiling of $\mathbb{E}^3$ by convex polytopes each equivalent to P. Indeed, by the remark at the beginning of § 2, the boundary $b(\mathscr{P}_i)$ is contained in $A(i\cos(\frac{1}{2}c))$ for each i. Since c does not depend on i, $\mathscr{T}$ provides a locally finite tiling of $\mathbb{E}^3$.

We shall construct the sequences inductively. For $\mathscr{P}_1$ we take the P-tiling consisting of exactly one translate P_1 of P with $0 \in \text{int}(P_1)$ and $\partial P_1 \subset A(1)$. Let Z_1 be a dissection of $\hat{b}(\mathscr{P}_1)$ $(= b(\mathscr{P}_1))$ into convex n-gons. Each n-gon F of Z_1 has spherical diameter $d(F)$ less than π. Define $c := \max(d(F) \mid F \in Z_1)$. Then of course, $\mathscr{P}_1$ and Z_1 satisfy (1)–(4). The constant c will also serve for the constant in (4) for general i. Indeed, we shall construct the P-tilings in such a way that each polytope different from P_1 lies in one of the cones $C_0(F)$ with $F \in Z_1$.

Assume now that $\mathscr{P}_i$ and Z_i have been obtained so far and that $\mathscr{P}_{i+1}$ and Z_{i+1} are to be constructed.

Applying Lemma 5 to each F in Z_i and letting $r = i+1$ provides a P-tiling $\mathscr{P}_F$ with the properties of Lemma 5. Define $\mathscr{P}_{i+1} := \bigcup_{F\in Z_i} \mathscr{P}_F \cup \mathscr{P}_i$. Then of course, $\mathscr{P}_{i+1}$ has the properties (1) and (2). The set $\hat{b}(\mathscr{P}_{i+1})$ is the union of the sets $\hat{b}(\mathscr{P}_F)\setminus\text{relint}(F)$ for F in Z_i. If we take for Z_{i+1} the union of all dissections Z_F of $\hat{b}(\mathscr{P}_F)\setminus\text{relint}(F)$ with F in Z_i, then (3) and (4) hold too. But this establishes the theorem.

By a slight modification of the arguments we also get the following corollary.

COROLLARY. *Let $n \leqslant 5$ and let P have an n-gonal facet. Assume that $\mathscr{P}$ is a P-tiling which covers 0 and is star-shaped relative to 0. Let Z be a dissection of $\hat{b}(\mathscr{P})$ into convex n-gons. Then, there exists a locally finite tiling $\mathscr{T}$ of $\mathbb{E}^3$ by convex polytopes each equivalent to P, such that $\mathscr{P} \subset \mathscr{T}$ and each polytope not in $\mathscr{P}$ is contained in one of the cones $C_0(F)$ with F in Z.*

Unfortunately, the methods will not provide normal tilings in general. In fact, the following problem is undecided even for n-gonal pyramids $(n \geqslant 6)$ and simplicial polytopes.

Given a convex 3-polytope P, is there a normal tiling of $\mathbb{E}^3$ by convex polytopes each combinatorially equivalent to P?

The example of Danzer, Grünbaum, and Shephard (cf. [3]) shows that such a tiling cannot be face-to-face in general.

As stated in the introduction, locally finite face-to-face tilings do not exist in general. However, there might be some hope for a positive result at least for some special types of polytopes. As the simplicial case is solved, the next class of interest might be the class of cubical polytopes.

Given a cubical convex 3-polytope P, is there a locally finite face-to-face tiling of $\mathbb{E}^3$ by convex polytopes each isomorphic to P?

The chance of a positive answer seems to be good, since the existence of 3-valent vertices is guaranteed by Euler's theorem (cf. Grünbaum [6]). However, the author does not believe that the answer to the analogous question for simple polytopes is in the affirmative.

It is noteworthy that in some instances the isomorphic copies of P may be chosen to be affinely or projectively equivalent to P. This is due to the fact that any triangle or quadrangle can be arbitrarily preassigned as a facet (or vertex-figure) of an affinely or projectively equivalent copy of the polytope. So, if P has a triangular facet and a 3-valent vertex, then P admits a tiling by affinely equivalent polytopes. If P has an n-gonal facet and a k-valent vertex with $n, k \leqslant 4$, then projectively equivalent copies of P will tile $\mathbb{E}^3$.

In conclusion, we may ask to what extent Theorem 1 holds in higher dimensions. The author suspects that the result remains true. Among other things this is supported by the fact that it is not even possible to preassign the facets of higher-dimensional polytopes (cf. Barnette and Grünbaum [2], Barnette [1], Kleinschmidt [11]). So the result of Barnette and Grünbaum, a basic tool in the construction for three dimensions, will not hold any longer.

Finally, I would like to thank Professors L. Danzer and B. Grünbaum for discussion on the subject of this paper.

References

1. D. Barnette, 'The triangulation of the 3-sphere with up to 8 vertices', *J. Combin. Theory*, 14 (1973), 37–52.
2. D. Barnette and B. Grünbaum, 'Preassigning the shape of a face', *Pacific J. Math.*, 32 (1970), 299–306.
3. L. Danzer, B. Grünbaum, and G. C. Shephard, 'Does every type of polyhedron tile three-space?', *Structural Topology*, 8 (1983), 3–14.
4. B. N. Delone, 'A proof of the fundamental theorem of the theory of stereohedra', *Dokl. Akad. Nauk SSSR*, 138 (1961), 1270–1272; *Soviet Math. Dokl.*, 2 (1961), 812–815.
5. P. Engel, 'Über Wirkungsbereichseinteilungen mit kubischer Symmetrie', *Z. Krist.*, 154 (1981), 199–215.
6. B. Grünbaum, *Convex polytopes* (Interscience, New York, 1967).
7. B. Grünbaum and G. C. Shephard, 'Tilings with congruent tiles', *Bull. Amer. Math. Soc.*, 3 (1980), 951–973.
8. B. Grünbaum and G. C. Shephard, *Tilings and patterns* (Freeman, San Francisco, 1984).
9. B. Grünbaum, P. Mani-Levitska, and G. C. Shephard, 'Tiling three-dimensional space with polyhedral tiles of a given isomorphism type', *J. London Math. Soc.* (2), 29 (1984), 181–191.
10. H. Heesch, *Reguläres Parkettierungsproblem* (Westdeutscher Verlag, Köln-Opladen, 1968).
11. P. Kleinschmidt, 'On facets with non-arbitrary shape', *Pacific J. Math.*, 65 (1976), 97–101.
12. D. G. Larman and C. A. Rogers, 'Durham symposium on the relations between infinite-dimensional and finitely-dimensional convexity', *Bull. London Math. Soc.*, 8 (1976), 1–33.
13. I. Niven, 'Convex polygons that cannot tile the plane', *Amer. Math. Monthly*, 85 (1978), 785–792.
14. M. A. Perles and G. C. Shephard, 'Facets and nonfacets of convex polytopes', *Acta Math.*, 119 (1967), 113–145.
15. K. Reinhardt, 'Über die Zerlegung der Ebene in Polygone', Ph. D. thesis, Univ. Frankfurt a. M., Noska, Borna-Leipzig, 1918.
16. E. Schulte, 'Pflasterungen des Raumes mit Pyramiden und Doppelpyramiden', *Elem. Math.*, to appear.
17. E. Schulte, 'The existence of non-tiles and non-facets in three dimensions', *J. Combin. Theory Ser. A*,
18. E. Schulte, 'Non-tiles and non-facets for the Euclidean space, convex polytopes and spherical complexes', manuscript, Dortmund, 1983.

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