Group chromatic number of planar graphs of girth at least 4

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Abstract

Jeager *et al* introduced a concept of group connectivity as an generalization of nowhere zero flows and its dual concept group coloring, and conjectured that every 5-edge connected graph is Z_3-connected. For planar graphs, this is equivalent to that every planar graph with girth at least 5 must have group chromatic number at most 3. In this paper we show that if G is a plane graph with girth at least 4 such that all 4-cycles are independent, every 4-cycle is a facial cycle and the distance between every pair of a 4-cycle and a 5-cycle is at least 1, then the group chromatic number of G is at most 3. As a special case, we show that the conjecture above holds for planar graphs. We also prove that if G is a connected $K_{3,3}$-minor free graph with girth at least 5, then the group chromatic number is at most 3.

Keywords: Group connectivity; group coloring; planar graphs; group chromatic number

1 Introduction

Our terminology is standard as in [1] except otherwise defined. Let G and H be two graphs. Denote $H \subseteq G$ if H is a subgraph of G. If H can be obtained from G by contracting some edges of G, then G is *contractible* to H. If G contains a subgraph which is contractible to Γ, then Γ is a *minor* of G. A set of subgraphs of G is said to be *independent* if no two of them have a common vertex.

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A *k-path* (*k-cycle*) denotes a path (cycle) of length k. The *distance* of 4-cycle $v_1v_2v_3v_4v_1$ and 5-cycle $u_1u_2u_3u_4u_5u_1$ in a graph G is $\min\{d_G(v_i, u_j)|1 \leq i \leq 4, 1 \leq j \leq 5\}$, where $d_G(u, v)$ denotes the length of a shortest (u, v)-path in G. The *girth* of a graph G is the length of a shortest cycle of G. For a plane graph, the unique unbounded face is called the *outer face*. If C is a cycle in a planar graph G, then *int(C)* is the set of vertices and edges inside C; if $int(C) = \emptyset$, then C is *facial*. If the outer face is bounded by a cycle, we call it the *outer cycle*. A cycle C is *separating cycle* in G if G has at least one vertex outside C and at least one vertex inside C. Throughout this paper, Z_3 denotes the cyclic group of order 3.

Jeager *et al* [6] introduced a concept of group connectivity as an generalization of nowhere zero flows and its dual concept group coloring. The results about nowhere zero flows can be found in [5, 14]. Let A denote an (additive) Abelian group and $F(G, A)$ denote the set of all functions from $E(G)$ to A. For $f \in F(G, A)$, an (A, f)-coloring of G under an orientation D is a function $c : V(G) \mapsto A$ such that for every edge $e = uv$ from u to v, $c(u) - c(v) \neq f(uv)$. G is *A-colorable* under an orientation D if for every function $f \in F(G, A)$, G has an (A, f)-coloring. It is known ([6]) that whether G is A-colorable is independent of the choice of the orientation. The *group chromatic number* of a graph G is defined to be the smallest positive integer m for which G is A-colorable for every Abelian group A of order at least m under a given orientation D, and is denoted by $\chi_g(G)$.

Let H be a subgraph of a graph G. Given an $f \in F(G, A)$, if for an $(A, f|_{E(H)})$-coloring c_0 of H, there is an (A, f)-coloring c of G such that c is an extension of c_0, then we say that c_0 is *extended to c*. If every $(A, f|_{E(H)})$-coloring c_0 of H can be extended to an (A, f)-coloring c, then we say that (G, H) is *(A, f)-extensible*. If for every $f \in F(G, A)$, (G, H) is (A, f)-extensible, then (G, H) is *A-extensible*.

Jaeger, *et al* [6] proved that if G is a simple planar graph, then $\chi_g(G) \leq 6$. It is shown (see [8, 10]) that if G is a simple graph without a K_5-minor or without a $K_{3,3}$-minor, then $\chi_g(G) \leq 5$. Jaeger, *et al* [6] also proved that if G is a simple planar graph with girth at least 4, then $\chi_g(G) \leq 4$. In this paper we prove the following results.

Theorem 1.1 *Suppose that G is a simple planar graph with girth at least 4 such that all 4-cycles are independent and every 4-cycle is facial. If the distance between every pair of a 4-cycle and a 5-cycle is at least 1, then $\chi_g(G) \leq 3$.*

Theorem 1.2 *If G is a $K_{3,3}$-minor free graph with girth at least 5, then $\chi_g(G) \leq 3$.*

Král *at al* [7] constructed a family of bipartite planar graphs with girth 4 and group chromatic number 4. Thus Theorem 1.1 is best possible in the sense that the condition on the girth in Theorem 1.1 cannot be relax. The proofs of Theorems 1.1 and 1.2 are in Section 2 and Section 3, respectively.

For a given orientation of a graph G and for a vertex $v \in V(G)$, let $E^-(v) = \{(u,v) \in E(G) : u \in V(G)\}, E^+(v) = \{(v,u) \in E(G) : u \in V(G)\}$ and $E(v) = E^+(v) \cup E^-(v)$.

Throughout this paper, let A denote a nontrivial Abelian group and let $A^* = A - \{0\}$. Define

$$F^*(G, A) = \{f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the boundary of f is a function $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. Denote

$$Z(G, A) = \{b : V(G) \mapsto A \text{ such that } \sum_{v \in V(G)} b(v) = 0\}.$$

A graph G is *A-connected* if G has an orientation D such that for every function $b \in Z(G, A)$ there is a function $f \in F^*(G, A)$ such that $b = \partial f$. The following conjecture is due to Jaeger *et al* [6].

Conjecture 1.3 *Every 5-edge connected graph is Z_3-connected.*

Let G be a connected plane graph, G^* be the geometric dual of G, and A be an Abelian group. Jeager *et al* [6] showed that G is A-connected if and only if G^* is A-colorable. An algorithmic proof of this fact can be found in [2]. By Theorem 1.1, Conjecture 1.3 thus holds for planar graphs.

Corollary 1.4 *Every 5-edge connected planar graph is Z_3-connected.*

2 A Z_3-Coloring Theorem

Let $\mathcal{F}$ denote the set of connected graphs such that a graph $G \in \mathcal{F}$ if and only if each of the following holds.
(F1) G is a plane graph with girth at least 4 and every 4-cycle is facial;
(F2) all 4-cycles are independent; and
(F3) the distance between every pair of a 4-cycle and a 5-cycle is at least 1.

In the discussions below, we assume that for each $G \in \mathcal{F}$, G is embedded in the plane with an orientation.

Theorem 2.1 *Suppose $G \in \mathcal{F}$ and $f \in F(G, Z_3)$. Let W be a subset of vertices on the outer face of G such that*
(W1) either $G[W]$ is edgeless or
(W2) $G[W]$ has exactly one edge $e = xy$ and G has no 2-path from x to other vertex in W.
(W3) if xy is an edge of $G[W]$, then $G[\{\lambda_1, \lambda_2, x, y\}]$ is not a 4-cycle for every

pair of distinct vertices λ_1, λ_2 of the out face of G.
Assume that each of the following holds:
(a) each vertex $w \in W$ is associated with an element $b_w \in Z_3$,
(b) $u, v \notin W$ are two adjacent vertices on the out face of G (assume that the edge uv is oriented from u to v),
(c) $a_u, a_v \in Z_3$ with $a_u - a_v \neq f(uv)$.
Define $c_1 : \{u, v\} \mapsto Z_3$ by $c_1(u) = a_u, c_1(v) = a_v$. Then c_1 can be extended to $c : V(G) \mapsto Z_3$ such that $c\,|_{\{u,v\}} = c_1$ and
(i) $c(w) \neq b_w$ for every vertex $w \in W$,
(ii) $c(x') - c(y') \neq f(x'y')$ for any edge $x'y' \in E(G)$ oriented from x' to y'.

Remark. Condition (W3) can not be relaxed. Let $C = x_1x_2x_3x_4x_1$ be a 4-cycle. Assume that $W = \{x_3, x_4\}$, $b_{x_3} = 1$ and $b_{x_4} = 1$, and that C is oriented from x_i to x_{i+1}, $1 \leq i \leq 3$ and from x_4 to x_1. Define $f \in F(C, Z_3)$ as follows: $f(e) = 0$ if $e \in E(C) - \{x_4x_1\}$ and $f(x_4x_1) = -1$. Define $c_1 : \{x_1, x_2\} \mapsto Z_3$ by $c_1(x_1) = 1, c_1(x_2) = 0$. Then c_1 can not be extended to $c : V(C) \mapsto Z_3$ such that $c\,|_{\{x_1,x_2\}} = c_1$.

We need some preparations before presenting the proof of Theorem 2.1.

Lemma 2.2 *Let $G \in \mathcal{F}$ and let $C : x_1x_2 \ldots x_5x_1$ be a 5-cycle. Assume that $f \in F(G, Z_3)$ and x_ix_{i+1} is oriented from x_i to x_{i+1}, $1 \leq i \leq 5$ (indices taken mod 5). For every (Z_3, f)-coloring c_1, there is some $i \in \{1, 2, \ldots 5\}$ (indices taken mod 5) such that*

$$c_1(x_i) - f(x_ix_{i+1}) \neq c_1(x_{i+2}) + f(x_{i+1}x_{i+2}).$$

Proof.. By contradiction, suppose that

$$c_1(x_i) - f(x_ix_{i+1}) = c_1(x_{i+2}) + f(x_{i+1}x_{i+2}) \qquad (1)$$

for every $1 \leq i \leq 5$ (indices taken mod 5). Since Z_3 is an Abelian group, by (1) we have $f(x_1x_2) + f(x_2x_3) + \ldots + f(x_5x_1) = 0$. Thus we have

$$\begin{aligned} c_1(x_1) &= f(x_1x_2) + c_1(x_3) + f(x_2x_3) \\ &= f(x_3x_4) + c_1(x_5) + f(x_4x_5) + f(x_2x_3) + f(x_1x_2) \\ &= -f(x_5x_1) + c_1(x_5). \end{aligned}$$

It follows that $c_1(x_5) - c_1(x_1) = f(x_5x_1)$, a contradiction. □

Theorem 2.1 implies the following Corollary 2.3. We shall argue by induction on $|V(G)|$ to prove Theorem 2.1. Our induction hypotheses is the assumption that both Theorem 2.1 and Corollary 2.3 hold for smaller values of $|V(G)|$.

Corollary 2.3 *Let $G \in \mathcal{F}$ with outer cycle $C : x_1x_2 \ldots x_5x_1$ and let $f \in F(Z_3, G)$. If $c_1 : V(C) \mapsto Z_3$ is a (Z_3, f)-coloring, then c_1 can be extended to a (Z_3, f)-coloring c of G such that $c\,|_{V(C)} = c_1$.*

Proof. Since $G \in \mathcal{F}$, the 5-cycle C has no chord. Hence, $G[V(C)] = C$. Assume that C is oriented from x_i to $x_{i+1}, 1 \le i \le 5$ (indices mod 5). By Lemma 2.2, we assume that $c_1(x_5)+f(x_4x_5) \neq c_1(x_3)-f(x_3x_4)$. Set $W = \{x_3, x_5\}$. Since c_1 is a (Z_3, f)-coloring, $c_1(x_2)-c_1(x_3) \neq f(x_2x_3)$ and $c_1(x_5)-c_1(x_1) \neq f(x_5x_1)$. We pick $b_{x_3} \in Z_3-\{c_1(x_2)-f(x_2x_3), c_1(x_3)\}, b_{x_5} \in Z_3-\{c_1(x_1)+f(x_5x_1), c_1(x_5)\}$. By Theorem 2.1, $c_1 : \{x_1, x_2\} \mapsto Z_3$ can be extended to $c : V(G) \mapsto Z_3$ such that $c\mid_{\{x_1,x_2\}}= c_1$ and $c(w) \neq b_w$ for every $w \in W$. By the choice of b_{x_3} and b_{x_5}, $c(x_3) \in \{c_1(x_2) - f(x_2x_3), c_1(x_3)\}, c(x_5) \in \{c_1(x_1) + f(x_5x_1), c_1(x_5)\}$. Hence $c(x_3) = c_1(x_3), c(x_5) = c_1(x_5)$.

Since $c(x_4) - c(x_5) \neq f(x_4x_5)$ and $c(x_3) - c(x_4) \neq f(x_3x_4)$, $c(x_4) \in Z_3 - \{c(x_5)+f(x_4x_5), c(x_3)-f(x_3x_4)\} = Z_3-\{c_1(x_5)+f(x_4x_5), c_1(x_3)-f(x_3x_4)\}$. Since $c_1 : V(C) \mapsto Z_3$ is a (Z_3, f) coloring, $c_1(x_4) \in Z_3-\{c_1(x_5)+f(x_4x_5), c_1(x_3)-f(x_3x_4)\}$. Thus $c_1(x_4) = c(x_4)$. □

In order to prove Theorem 2.1, we first prove some lemmas. The following lemmas have the same hypotheses of Theorem 2.1 with additional assumptions that

$$G \text{ is a counterexample to Theorem 2.1,} \tag{2}$$

and

$$|V(G)| \text{ is minimized.} \tag{3}$$

Since c_1 can be easily extended to a (Z_3, f)-coloring for every forest and for a 4-cycle which satisfy the conditions of Theorem 2.1, we may assume that G is connected with $|V(G)| \ge 5$ and that G contains a cycle.

Lemma 2.4 $\kappa(G) \ge 2$. *Moreover, if* $z \in V(G) - W$ *and* $z \notin \{u, v\}$, *then* $d_G(z) \ge 3$.

Proof. If G is not 2-connected, then G has a block B containing the edge uv. By (3), c_1 can be extended to a (Z_3, f)-coloring of B. Let B_1 be the block which has a common vertex w_1 with B and pick its adjacent vertex w_2 in the outer face of B_1. Assume that the edge w_1w_2 is oriented from w_1 to w_2 and put $c_1(w_2) \in Z_3$ such that $c_1(w_1) - c_1(w_2) \neq f(w_1w_2)$. By (3) again, $c_1\mid_{\{w_1,w_2\}}$ can be extended to a (Z_3, f)-coloring of B_1. We continue this procedure and finally c_1 can be extended to a (Z_3, f)-coloring of G. This contradicts to (2). Therefore, G is 2-connected and $d(z) \ge 2$ for every vertex $z \in V(G)$.

Suppose, to the contrary, that there is $z_0 \in V(G)-W$ such that $z_0 \notin \{u, v\}$ and $d_G(z_0) = 2$. Let $G_1 = G-z_0$ and denote $N(z_0) = \{z_1, z_2\}$. Assume that the edge z_0z_1 is oriented from z_0 to z_1 and the edge z_0z_2 is oriented from z_0 to z_2. By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G_1. Define $c : V(G) \mapsto Z_3$ by

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{z_0\}, \\ a \in Z_3 - \{c(z_1) + f(z_0z_1), c(z_2) + f(z_0z_2)\} & \text{if } z = z_0 \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, violating (2). □

In the following lemmas, by Lemma 2.4, we assume that G is 2-connected and that $C : x_1x_2 \dots x_mx_1$ is the outer cycle of G oriented from x_i to $x_{i+1}, 1 \le i \le m$ (indices mod m) and for every $z \in N(x_j) - V(C)$, the edge x_jz is oriented from x_j to z. Let $u = x_1$ and $v = x_2$. If $G[W]$ has an edge xy, we can assume that $y = x_i, x = x_{i+1}$, where $3 \le i \le m-1$.

Lemma 2.5 *C has no chords.*

Proof. Suppose, to the contrary, that such a chord $u'v'$ exists. Let C_1 be the cycle in $C \cup \{u'v'\}$ containing $u'v'$ and uv and C_2 the cycle in $C \cup \{u'v'\}$ containing $u'v'$ but not uv. Then both $C_1 \cup int(C_1) \in \mathcal{F}$ and $C_2 \cup int(C_2) \in \mathcal{F}$. It follows that both $C_1 \cup int(C_1)$ and $C_2 \cup int(C_2)$ satisfy the hypotheses of Theorem 2.1. By (3), we can extend c_1 to a (Z_3, f)-coloring of $C_1 \cup int(C_1)$ and the coloring of u' and v' can be extended to a (Z_3, f)-coloring of $C_2 \cup int(C_2)$, violating (2). □

Throughout the proof of Theorem 2.1, we need to redefine W' to replace W in order to apply induction hypotheses. In each occasion a set W' is introduced, we only indicate how to assign the new value b_z for some $z \in W'$ including all $z \in W' - W$, while leaving b_z unchanged for those $z \in W \cap W'$ if b_z need not to be redefined.

Lemma 2.6 *If G has a 2-path $u'v'w$ for $v' \in V(G) - V(C), u' \in V(C)$ and $w \in W$, then all of the following properties hold.*
(1) uv and xy lie in the different components of $G - \{u', v', w\}$, and
(2) $G[\{u', v', y, x\}]$ is a 4-cycle where $u' \notin W$.

Proof. Suppose, to the contrary, that such a 2-path $u'v'w$ exists such that at least one of (1) and (2) does not satisfied. Assume first that $G[\{u', v', y, x\}]$ is not a 4-cycle. By (W2), $G[\{w, v', y, x\}]$ is not a 4-cycle. Define C_1 and C_2 as follows: Let C_1 be the cycle in $C \cup \{u'v', v'w\}$ containing $u'v', v'w$ and uv and C_2 the cycle in $C \cup \{u'v', v'w\}$ containing $u'v', v'w$ but not uv. It follows that both $C_1 \cup int(C_1)$ and $C_2 \cup int(C_2)$ satisfy the hypotheses of Theorem 2.1. By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of $C_1 \cup int(C_1)$. We recall that the edge $u'v'$ is oriented from u' to v' and that the edge $v'w$ is oriented from w to v'.

Since c_2 is a (Z_3, f)-coloring of $C_1 \cup int(C_1)$ and $c_2(w) \ne b_w$, we redefine $b_w \in Z_3 - \{c_2(w), c_2(v') + f(wv')\}$. By (3), $c_2 \mid_{\{u', v'\}}$ can be extended to a (Z_3, f)-coloring c_3 of $C_2 \cup int(C_2)$ such that $c_3 \mid_{\{u', v'\}} = c_2$. Since $c_3(w) \ne b_w$, it follows that $c_3(w) = c_2(w)$. Thus we obtain a required (Z_3, f)-coloring of G, contrary to (2).

Assume then that $G[\{u', v', y, x\}]$ is a 4-cycle. By (W2), the edges xy and uv lie in the same component of $G - \{u'v'w\}$. Note that the 4-cycle $yxu'v'y$ is a facial cycle. That is, $int(yxu'v'y) = \emptyset$. By Lemma 2.5, we further assume that $G[\{u', v', y, x\}]$ is a 4-cycle $yxu'v'y$ where $u' = x_{i+2}$. Let C_1 be the cycle in $C \cup \{yv'w\}$ containing uv and $wv'y$. By (3), c_1 can be extended to a(Z_3, f)-coloring c_2 of $C_1 \cup int(C_1)$. Let $c_2(x) \in Z_3 - \{c_2(y) - f(yx), b_x\}, c_2(u') \in Z_3 - \{c_2(x) - f(xu'), c_2(v') + f(u'v')\}$. Let C_2 be the cycle in $C \cup \{u'v'w\}$ not containing uv nor xy. We define $b_w \in Z_3 - \{c_2(w), c_2(v') + f(wv')\}$. By (3), $c_2 \mid_{\{u',v'\}}$ can be extended to a (Z_3, f)-coloring c_3 of $C_2 \cup int(C_2)$ such that $c_3(w) \neq b_w$. It follows that $c_3(w) = c_2(w)$. By combining c_2 and c_3, c_1 can be extended to a(Z_3, f)-coloring of G, contrary to (2). $\Box$

Lemma 2.7 *G has no 3-path $w_1u'v'w_2$ with $w_1, w_2 \in W$ and $u', v' \in V(G) - V(C)$.*

Proof. Suppose, to the contrary, that such a 3-path exists. By Lemma 2.6, we may assume that $G[\{u', v', y, x\}]$ is not a 4-cycle. By (W2), none of $G[\{w_1, u', y, x\}]$ and $G[\{w_2, v', y, x\}]$ is a 4-cycle. Assume that $u'v'$ is oriented from u' to v'. Define C_1 to be the cycle in $C \cup \{w_1u', u'v', v'w_2\}$ containing $w_1u', u'v', v'w_2$ and uv, and C_2 to be the cycle in $C \cup \{w_1u', u'v', v'w_2\}$ containing $w_1u', u'v'$ and $v'w_2$ but not uv. It follows that both $C_1 \cup int(C_1)$ and $C_2 \cup int(C_2)$ satisfy the hypotheses of Theorem 2.1. By (3), $c_1 : \{u, v\} \mapsto Z_3$ can be extended to a (Z_3, f)-coloring c_2 of $C_1 \cup int(C_1)$ such that $c_2 \mid_{\{u,v\}} = c_1$.

Since c_2 is a (Z_3, f)-coloring of $C_1 \cup int(C_1)$ and since $c_2(w_1) \neq b_{w_1}, c_2(w_2) \neq b_{w_2}$, we redefine $b_{w_1} \in Z_3 - \{c_2(w_1), c_2(u') + f(w_1u')\}, b_{w_2} \in Z_3 - \{c_2(w_2), c_2(v') + f(w_2v')\}$. By (3) again, $c_2 \mid_{\{u',v'\}}$ can be extended to a (Z_3, f)-coloring c_3 of $C_2 \cup int(C_2)$ such that $c_3 \mid_{\{u',v'\}} = c_2$ and $c_3(w_1) \neq b_{w_1}, c_3(w_2) \neq b_{w_2}$. It follows that $c_3(w_1) = c_2(w_1)$ and $c_3(w_2) = c_2(w_2)$ and $c_3(x') - c_3(y') \neq f(x'y')$ for every directed edge $x'y' \in E(G_2)$. A desired (Z_3, f)-coloring of G extending c_1 can be obtained by combining c_2 and c_3, contrary to (2). $\Box$

Lemma 2.8 *G has no separating 5-cycle.*

Proof. If G has a separating 5-cycle C', then by (3), we can extend c_1 to an group coloring of $G - int(C')$. By Corollary 2.3, c_1 can be extended to a group coloring of $C' \cup int(C')$. This contradiction proves Lemma 2.8. $\Box$

Lemma 2.9 *(i) G has no a vertex $q \in V(G) - V(C)$ such that $G[\{x_i, x_{i+1}, x_{i+2}, q\}]$ is a 4-cycle.*

(ii) G has no a vertex $q \in V(G) - V(C)$ such that $G[\{x_{i-1}, x_i, x_{i+1}, q\}]$ is a 4-cycle.

Proof. (i) Assume that G has such a 4-cycle. Define $G' = G - \{x_i, x_{i+1}, x_{i+2}, q\}$ and $W' = W \cup N(x_i) \cup N(x_{i+1}) \cup N(x_{i+2}) \cup N(q) - \{x_i, x_{i+1}, x_{i+2}, q\}$. Take $a_{x_i} \in Z_3 - \{b_{x_i}\}, a_{x_{i+1}} \in Z_3 - \{b_{x_{i+1}}, a_{x_i} - f(x_ix_{i+1})\}, a_{x_{i+2}} \in Z_3 - \{a_{x_{i+1}} - f(x_{i+1}x_{i+2})\}, a_q \in Z_3 - \{a_{i+2} - f(x_{i+2}q), a_{x_i} - f(x_iq)\}$. Let $b_z = a_{x_j} - f(x_jz)$ for every $z \in N(x_j) - \{x_i, x_{i+1}, x_{i+2}, q\}, j \in \{i, i+1, i+2\}$ and $b_z = a_q - f(qz)$ for every $z \in N(q) - \{q, x_i, x_{i+2}\}$. By (F1) and (F2), b_z is well-defined. If there is a 2-path x_iqx_j where $x_j \in V(C)$, by Lemma 2.6, $x_j \in \{x_3, x_4, \ldots, x_{i-3}\}$. By Lemma 2.7, G has no 3-path $x_iqq'x_j$ where $q, q' \in V(G) - V(C)$ and $x_j \in W$. Thus G' may have some cut vertices. We will distinguish the following two cases.

Case 1. There is a 2-path x_iqx_j where $x_j \in V(C), j \in \{3, 4, \ldots, i-4\}$.

Then G' can be decomposed into blocks $B_1, B_2, \ldots, B_k$ such that
(1) B_1 contains the edge uv and B_k contains $x_{i-1}x_{i-2}$, and
(2) $V(B_i) \cap V(B_{i+1})$ is a cut vertex.

Now we claim that c_1 can be extended to a (Z_3, f)-coloring of B_1. Suppose that $V(B_1) \cap V(B_2) = \{z\}$. Then $B_1[W']$ contains at most one edge zw where $w \in W \cap V(B_1)$. By Lemma 2.6, there is no a 2-path from z to any vertex of $B_1[W']$. By (3), c_1 can be extended to a (Z_3, f)-coloring of B_1.

We claim again that if c_1 can be extended to a (Z_3, f)-coloring of $B_1 \cup B_2 \cup \ldots \cup B_i$, then c_1 can be extended to a (Z_3, f)-coloring c_2 of $B_1 \cup B_2 \cup \ldots \cup B_{i+1}$. Observe the block B_{i+1}. $B_{i+1}[W']$ contains at most two edges one of those contains vertices of $V(B_i) \cap V(B_{i+1})$ and the other contains vertices of $V(B_{i+1}) \cap V(B_{i+2})$. Suppose that $V(B_i) \cap V(B_{i+1}) = \{z_1\}$. We define z_2 as follows. If $B_{i+1}[W']$ does not contain an edge one end of which is z_1, let $z_2 \in V(B_{i+1}) \cap V(C)$ such that $z_1z_2 \in E(G)$. If $B_{i+1}[W']$ contains an edge one end of which is z_1, let z_2 be the other end of such edge. Assume that z_1z_2 is oriented from z_1 to z_2. Define $c_1(z_2) \in Z_3 - \{b_{z_2}, c_1(z_1) - f(z_1z_2)\}$ if $z_2 \in W'$; $c_1(z_3) \in Z_3 - \{c_1(z_1) - f(z_1z_2)\}$ if $z_2 \notin W'$.

Now we consider the vertex, say z_1', of $V(B_{i+1}) \cap V(B_{i+2})$. Then $B_{i+1}[W']$ maybe contains an edge one end of which is z_1'. If $B_{i+1}[W']$ indeed contains such edge, let z_2' be the other end of that edge. By Lemma 2.6, there is no 2-path from z_1' to any vertex of $V(B_{i+1}) \cap W'$. Therefore z_1z_2 plays the role of uv in G and $z_1'z_2'$ plays the role of xy in G if such $z_1'z_2'$ exists. By (3), $c_2 \mid_{\{z_1,z_2\}}$ can be extended to a (Z_3, f)-coloring of $B_1 \cup B_2 \cup \ldots \cup B_{i+1}$.

Thus we assume that c_1 can be extended to a (Z_3, f)-coloring c' of G'. Define

$$c(z) = \begin{cases} c'(z) & \text{if } z \in V(G'), \\ a_{x_j} & \text{if } z = x_j, j \in \{i, i+1, i+2\} \\ a_q & \text{if } z = q. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, contrary to (2).

Case 2. There is no a 2-path x_iqx_j where $x_j \in \{x_3, x_4, \ldots, x_{i-3}\}$.

By Lemmas 2.6 and 2.7, $G'[W']$ contains at most one edge $x_{i-1}x_{i-2}$. If $G'[W']$ indeed contains that edge, by Lemma 2.6 there is no 2-path from x_{i-1} to any vertex of W'' and hence x_{i-1} plays the role of x in G. By (3), c_1 can be extended to a (Z_3, f)-coloring c' of G'. Define

$$c(z) = \begin{cases} c'(z) & \text{if } z \in V(G'), \\ a_{x_j} & \text{if } z = x_j, j \in \{i, i+1, i+2\}, \\ a_q & \text{if } z = q. \end{cases}$$

Then c is a (Z_3, f)-coloring of G, contrary to (2).

The proof of (ii) is similar.□

If u or v (say u) is in W, then we will replace W by $W - u$. Therefore we assume that

$$\{u, v\} \subset V(C) - W. \tag{4}$$

Lemma 2.10 *If $G[W]$ has the edge xy, then $i \geq 4$.*

Proof. Suppose, to the contrary, that $i = 3$. Let $G' = G - \{y, x\}$ and $W' = W \cup N(x) \cup N(y) - \{x, y, v\}$. Take $a_y \in Z_3 - \{b_y, c_1(v) - f(vy)\}$, $a_x \in Z_3 - \{b_x, a_y - f(yx)\}$. Let $b_z = a_x - f(xz)$ for every $z \in N(x) - \{y\}$ and let $b_z = a_y - f(yz)$ for every $z \in N(y) - \{x, v\}$. By (F2), b_z is well-defined.

By (W2), $x_6 \notin W$. If $G[N(x) \cup N(y)]$ contains no 4-cycles, by Lemma 2.6, $G[W']$ is edgeless and hence both G' and W' satisfy the conditions of Theorem 2.1. Assume that $G[N(x) \cup N(y)]$ contains a 4-cycle xq_1q_2yx, where $q_1 \in N(x)$ and $q_2 \in N(y)$. By Lemma 2.9, we assume that $q_2 \neq v$ and $q_1 \neq x_{i+2}$. By Lemma 2.6 and (F2), $G'[W']$ contains at most one edge q_1q_2. If G' has a 2-path $q_2q_3q_4$ where $q_4 \in W'$, by Lemma 2.6, $q_3 \neq q_1$. By (F2) and (F3), $q_4 \notin N(x_3) \cup N(x_4)$. Thus $q_4 \in W$, contrary to Lemma 2.7. By Lemma 2.6 and (F2), $G[\{\lambda_1, \lambda_2, q_1, q_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Therefore both G' and W' satisfy the conditions of Theorem 2.1 with q_2 playing the role of x in G in this case.

By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G' such that $c_2 |_{\{u,v\}} = c_1$ and $c_2(w') \neq b_{w'}$ for each vertex $w' \in W'$. Define $c : V(G) \mapsto Z_3$ by

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x, y\}, \\ a_x & \text{if } z = x, \\ a_y & \text{if } z = y. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G such that $c(w) \neq b_w$ for each vertex $w \in W$ and $c |_{\{u,v\}} = c_1$, contrary to (2). □

Lemma 2.11 *If $G[W]$ has the edge yx, then $x_{i-2} \in W$ and hence $i \geq 5$.*

Proof. Since $G[W]$ has only one edge, $x_{i-1} \notin W$. By contradiction, suppose that $x_{i-2} \notin W$, (if $z \in \{u, v\}$, then by (4), $z \notin W$).

Define $a_y \in Z_3 - \{b_y\}$, $a_x \in Z_3 - \{b_x, a_y - f(yx)\}$, $G' = G - \{x, y\}$ and $W' = W \cup N(x) \cup N(y) - \{x, y\}$. By (W2) and by Lemma 2.4, $x_{i+3} \notin W'$.

If $G[N(x) \cup N(y)]$ contains no 4-cycles, then by Lemma 2.6 and (W2), $G[W']$ is edgeless. So assume that $G[N(x) \cup N(y)]$ contains a 4-cycle xq_1q_2yx, where $q_1 \in N(x)$ and $q_2 \in N(y)$. By Lemma 2.9, $q_2 \neq x_{i-1}$ and $q_1 \neq x_{i+2}$. By Lemma 2.6 and (F2), $G'[W']$ contains only one edge q_1q_2. Suppose that G' has a 2-path $q_2q_3q_4$, where $q_4 \in W'$. By Lemma 2.6, $q_3 \neq q_1$. By (F2) and (F3), $q_4 \in W - N(x) \cup N(y)$, contrary to Lemma 2.7. By Lemma 2.6 and (F2), $G'[\{q_1, q_2, \lambda_1, \lambda_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Therefore both G' and W' satisfy the hypotheses of Theorem 2.1 (with q_2 playing the role of x in G if $q_1q_2 \in E(G'[W'])$.

Define $b_z = a_x - f(xz)$ if $z \in N(x) - W$ and $b_z = a_y - f(yz)$ if $z \in N(y) - (W \cup \{x_{i-1}\})$. By (F1) b_z is well defined. By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G' such that $c_2(w) \neq b_w, w \in W$. Define $c : V(G) \mapsto Z_3$ by

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x, y\}, \\ a_x & \text{if } z = x, \\ a_y & \text{if } z = y. \end{cases}$$

Then c is a required (Z_3, f)-coloring satisfying $c(w) \neq b_w$ for each vertex $w \in W$ and extending c_1, contrary to (2). $\Box$

Lemma 2.12 (i) *G has no 3-path $x_{i+2}u'v'x_j$ for $j \in \{i+4, \ldots, m\}$ nor 3-path $x_{i-1}u'v'x_j$ for $j \in \{3, \ldots i-3\}$ where $u', v' \in V(G) - V(C)$ and $x_j \in W$.*
(ii) *G has no 2-path $x_{i+2}u'x_j$ for $j \in \{i+4, \ldots, m\}$ nor 2-path $x_{i-1}u'x_j$ for $j \in \{3, \ldots i-4\}$ where $u' \in V(G) - V(C)$ and $x_j \notin W$.*

Proof. We only prove that G has no 3-path $x_{i+2}u'v'x_j$ for $j \in \{i+4, \ldots, m\}$ where $u', v' \in V(G) - V(C)$ and $x_j \in W$. The proofs for the other three cases are similar.

Assume that such $P = x_{i+2}u'v'x_j$ exists. Let C_1 be the cycle in $C \cup P$ containing $u'v'$ and xy and C_2 the cycle in $C \cup P$ containing $u'v'$ but not xy. Let $G_i = C_i \cup int(C_i), i = 1, 2$. By (3), we can extend c_1 to a (Z_3, f)-coloring c_2 of G_1. Assume that $u'v'$ is oriented from u' to v'.

Let $W'' = (W \cap V(G_2)) \cup \{x_{i+2}\}$. By (W2), $x_{i+3} \notin W$. By Lemma 2.4, $G_2[W'']$ is edgeless. Then G_2 and W'' satisfy the conditions of Theorem 2.1. Define

$$b''_z = \begin{cases} b_z & \text{if } z \in W'' - \{x_{i+2}, x_j\}, \\ b_{x_{i+2}} \in Z_3 - \{c_2(x_{i+2}), c_2(u') + f(x_{i+2}u')\} & \text{if } z = x_{i+2}, \\ b_{x_j} \in Z_3 - \{c_2(x_j), c(v') + f(x_j v')\} & \text{if } z = x_j. \end{cases}$$

By (3), $c_2 \mid_{\{u', v'\}}$ can be extended to a (Z_3, f)-coloring c_3 of G_2 such that $c_3(w'') \neq b_{w''}$ for every $w'' \in W''$. It follows that $c_3(x_{i+2}) = c_2(x_{i+2})$ and $c_3(x_j) = c_2(x_j)$. Combining c_2 and c_3, we get a required (Z_3, f)-coloring of G extending c_1 such that $c(z) \neq b_z$ for each vertex $z \in W$, contrary to (2). $\Box$

Lemma 2.13 *$G[W]$ has no edge.*

Proof. Suppose that $G[W]$ has the edge xy where $x = x_{i+1}, y = x_i$. By Lemma 2.11 and (W2), $x_{i-1} \notin W$ and $x_{i-2} \in W$. We consider the following two cases.

Case 1 $G[N(x) \cup N(y)]$ contains no 4-cycles.

Let $G' = G - \{x_{i-1}, x_i, x_{i+1}\}$ and $W' = W \cup N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1}) - \{x_{i-1}, x_i, x_{i+1}\}$. Let $a_{i-1} = b_{x_{i-2}} - f(x_{i-2}x_{i-1}), a_i \in Z_3 - \{b_{x_i}, a_{i-1} - f(x_{i-1}x_i)\}$ and $a_{i+1} \in Z_3 - \{b_{x_{i+1}}, a_i - f(x_i x_{i+1})\}$. Let $b_z = a_j - f(x_j z)$ for every vertex $z \in (W' - W) \cap (N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1})), j \in \{i-1, i, i+1\}$. Since $G[N(x) \cup N(y)]$ contains no 4-cycle, b_z is well defined. Suppose that $G'[W']$ has an edge q_1q_2. By Lemmas 2.6 and 2.5, $q_1, q_2 \in W' - W \subset N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1})$. We will distinguish the following two subcases.

Subcase 1.1 $q_2 \in N(x_{i-1}), q_1 \in N(x_{i+1})$.

Then G has a 5-cycle $q_1q_2x_{i-1}x_ix_{i+1}q_1$. By (F3) and Lemmas 2.6 and 2.8, q_1q_2 is the only edge in $G'[W']$. By Lemmas 2.4 and 2.8, $q_2 \neq x_{i-2}$. Assume that G' has a path $q_1q_3q_4$ where $q_4 \in W'$. By Lemma 2.6, $q_3 \neq q_2$.

We claim that $q_4 \in W$. By contradiction, suppose $q_4 \in W' - W$. If $q_4 \in N(x_{i-1}) - \{q_2\}$, then G has two 5-cycles $x_{i-1}x_ix_{i+1}q_1q_2x_{i-1}$ and $x_{i-1}q_4q_3q_1q_2x_{i-1}$.

By Lemma 2.8, $d_G(q_2) = 2$, contrary to Lemma 2.4. By Lemma 2.8, $q_4 \notin N(x_i)$. By (F3), $q_4 \notin N(x_{i+1})$. Thus $q_4 \in W$.

By Lemmas 2.5, 2.6 and 2.7, $q_1 = x_{i+2}, q_3 = x_{i+3}$ and $q_4 = x_{i+4}$. Note that $q_2 \neq x_{i-2}$. We shall verify that both G' and W' satisfy the hypotheses of Theorem 2.1.

We now assume that G' has a path $q_2q_5q_6$ where $q_6 \in W'$. By (F3), $q_6 \notin N(x_{i-1})$. By Lemma 2.8, $q_6 \notin N(x_i)$ and $q_6 \notin N(x_{i+1})$. Hence $q_6 \in W$. If $q_5 \notin V(C)$, then there exist two 3-paths $x_{i+2}q_2q_5q_6$ and $x_{i-1}q_2q_5q_6$ where $q_6 \in W$. By (F3), $q_6 \neq x_{i-2}$. Thus $q_6 \in W \cap (\{x_{i+4}, \ldots, x_m\} \cup \{x_3, \ldots, x_{i-3}\})$, contrary to Lemma 2.12. If $q_5 \in V(C)$, then there exist two 2-paths $x_{i-1}q_2q_5$ and $x_{i+2}q_2q_5$, where $q_2 \in V(G) - V(C), q_5 \in V(C) - W$ because $G[W]$ contains at most one edge. By (F1) and (F3), $q_5 \notin \{x_{i-2}, x_{i-3}\}$. Thus $q_5 \in \{x_{i+4}, \ldots, x_m\} \cup \{x_3, \ldots, x_{i-4}\}$, contrary to Lemma 2.12. Thus there is no 2-path from q_2 to a vertex of W' in G'.

By Lemma 2.6 and (F3), $G'[\{\lambda_1, \lambda_2, q_1, q_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Therefore both G' and W' satisfies the hypotheses of Theorem 2.1 with q_2 playing the role of x in G. Thus c_1 can be extended to a (Z_3, f)-coloring $c_2 : V(G') \mapsto Z_3$ such that $c_2 |_{\{u,v\}} = c_1$ and $c_2(w') \neq b_{w'}$ for every vertex $w' \in W'$. Define $c : V(G) \mapsto Z_3$ by

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x_{i-1}, x_i, x_{i+1}\}, \\ a_j & \text{if } z = x_j, j \in \{i-1, i, i+1\}. \end{cases}$$

Then c is a required (Z_3, f)-coloring satisfying $c(w) \neq b_w$ for every vertex $w \in W$ and extending c_1, contrary to (2).

Subcase 1.2 $q_2 \in N(x_{i-1}), q_1 \in N(x_i)$.

By Lemma 2.4 and (F1), $q_2 \neq x_{i-2}$. Thus $q_2 \in V(G) - V(C)$. Since $G[N(x) \cup N(y)]$ contains no 4-cycles, $q_1 \neq x_{i+1}$. If G' has a 2-path $q_1q_3q_4$, where $q_4 \in W'$, by (F3) and (F1) $q_4 \notin N(x_{i+1}) \cup N(x_i) \cup N(x_{i-1})$. Thus $q_4 \in W$, contrary to Lemma 2.7.

By Lemma 2.6 and (F2), $G'[\{\lambda_1, \lambda_2, q_1, q_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Therefore both G' and W' satisfy the conditions of Theorem 2.1 with q_1 playing the role of x of G. Thus c_1 can be extended to a (Z_3, f)-coloring $c_2 : V(G') \mapsto Z_3$ such that $c_2 |_{\{u,v\}} = c_1$ and $c_2(w') \neq b_{w'}$ for every vertex $w' \in W'$. Define $c : V(G) \mapsto Z_3$ by

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x_{i-1}, x_i, x_{i+1}\}, \\ a_j & \text{if } z = x_j, j \in \{i-1, i, i+1\}. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G such that $c(w) \neq b_w$ for every vertex $w \in W$, contrary to (2).

Case 2 $G[N(x) \cup N(y)]$ contains a 4-cycle.

Assume that xq_1q_2yx is a 4-cycle in $G[N(x) \cup N(y)]$, where $q_1 \in N(x_{i+1}), q_2 \in N(x_i)$. By lemma 2.9, $q_1 \neq x_{i+2}, q_2 \neq x_{i-1}$.

Let $G' = G - \{x_{i-1}, x_i, x_{i+1}\}$ and $W' = W \cup N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1}) - \{x_{i-1}, x_i, x_{i+1}\}$. Let $a_{i-1} = b_{x_{i-2}} - f(x_{i-2}x_{i-1}), a_i \in Z_3 - \{b_{x_i}, a_{i-1} - f(x_{i-1}x_i)\}$ and $a_{i+1} \in Z_3 - \{b_{x_{i+1}}, a_i - f(x_ix_{i+1})\}$. Let $b_z = a_j - f(x_jz)$ for every vertex $z \in (W' - W) \cap (N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1})), j \in \{i-1, i, i+1\}$. By (F2), b_z is well defined. By (W2), $x_{i+3} \notin W$. It follows that $x_{i+2}x_{i+3} \notin E(G'[W'])$. By Lemmas 2.5 and 2.6 and by (F2) and (F3), $G'[W']$ has only one edge q_1q_2.

We claim that G' has no 2-path from q_2 to any vertex of W'. Suppose, to the contrary, that G' has a path $q_2q_3q_4$ where $q_4 \in W'$. By Lemma 2.6, $q_3 \neq q_1$. By (F2) and (F3), $q_4 \notin N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1})$ and so $q_4 \in W$, contrary to Lemma 2.7. By Lemma 2.6 and (F2), $G'[\{\lambda_1, \lambda_2, q_1, q_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Thus both G' and W' satisfy the conditions of Theorem 2.1 with q_2 playing the role of x of G. By (3), c_1 can be extended to a (Z_3, f) coloring c_2 of G'. Define

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x_{i-1}, x_i, x_{i+1}\}, \\ a_j & \text{if } z = x_j, j \in \{i-1, i, i+1\}. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, contrary to (2).□

Lemma 2.14 $x_3 \notin W$.

Proof Suppose, to the contrary, that $x_3 \in W$. By Lemma 2. 13, $x_4 \notin W$. To prove our lemma, we need to prove two claims at first.

Claim 1 $x_5 \in W$ and G contains 4-cycle $x_3q_3q_4x_4x_3$, where $q_3, q_4 \in V(G) - V(C)$.

Proof. Suppose, to the contrary, that G has one of the following properties:

(1) $x_5 \notin W$;

(2) $x_5 \in W$ and $G[N(x_3) \cup N(x_4)]$ does not contain a 4-cycle; or

(3) $x_5 \in W$ and G contains 4-cycle $x_3q_3q_4x_4x_3$, where $|\{q_3, q_4\} \cap (V(G) - V(C))| \leq 1$.

Let $G' = G - x_3$ and $W' = W \cup N(x_3) - \{x_2, x_3\}$. Pick $a_{x_3} \in Z_3 - \{b_{x_3}, c_1(x_2) - f(x_2x_3)\}$. Define $b_z = a_{x_3} - f(x_3z)$ if $z \in N(x_3) - \{x_2, x_3\}$. Then b_z is well defined. If $x_5 \notin W$, by Lemmas 2.5, 2.6, $G'[W']$ is edgeless. If $x_5 \in W$, then x_4x_5 is only edge in $G'[W']$. If G contains 4-cycle $x_3q_3q_4x_4x_3$, by lemma 2.4, $q_4 \neq x_5$. By Lemma 2.5, $q_4 \in V(G) - V(C)$. By the assumption that either $G[N(x_3) \cup N(x_4)]$ does not contain a 4-cycle or G contains 4-cycle $x_3q_3q_4x_4x_3$ and $q_3 = x_2$ and by Lemma 2.6, there is no 2-path from x_4 to any vertex in W' Since $x_2 \notin W$. Applying the induction of hypotheses with x_4 playing the role of x to G' and W', we can extend c_1 to a (Z_3, f)-coloring of G'. Moreover, we can obtain a (Z_3, f)-coloring of G, contrary to (2). □

Claim 2. For $j = 1, 2, \cdots, \lceil \frac{m-1}{2} \rceil - 1$, G contains 4-cycles $x_{2j+1}q_{2j+1}q_{2j+2}x_{2j+2}x_{2j+1}$, where $x_{2j+1} \in W$ and $q_{2j+1}, q_{2j+2} \in V(G) - V(C)$.

Proof. By Claim 1, we may assume that for $k = 2, 3, \cdots, \lceil \frac{m-1}{2} \rceil - 1$, G contains 4-cycles

$x_{2k-1}q_{2k}q_{2k}x_{2k}x_{2k-1}$, where $x_{2k-1} \in W$ and $q_{2k-1}, q_{2k} \in V(G) - V(C)$ and

G does not contain the 4-cycle $x_{2k+1}q_{2k+1}q_{2k+2}x_{2k+2}x_{2k+1}$, where $x_{2k+1} \in W$ and $q_{2k+1}, q_{2k+2} \in V(G) - V(C)$. Let $G' = G - \{x_{2k}, x_{2k+1}\}$ and $W' = W \cup N(x_{2k}) \cup N(x_{2k+1} - \{x_{2k}, x_{2k+1}\}$. Pick $a_{x_{2k}} = b_{x_{2k-1}} - f(x_{2k-1}x_{2k}), a_{x_{2k+1}} \in Z_3 - \{b_{x_{2k+1}}, a_{x_{2k}} - f(x_{2k}x_{2k+1})\}$. By Lemma 2.13, $x_{2k+2} \notin W$. Thus, $G'[W']$ contains at most one edge $x_{2k+2}x_{2k+3}$. If $G'[W']$ indeed contains such edge, by the assumption that G does not contain a 4-cycle containing edge $x_{2k+2}x_{2k+3}$. By Lemma 2.6, there is no 2-path from x_{2k+2} to any vertex of W'. By Lemma 2.5, $G'[\lambda_1, \lambda_2, x_{2k+2}, x_{2k+3}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'. Thus, both G' and W' satisfy the hypotheses of Theorem 2.1, by the induction, c_1 can be extended to a (Z_3, f)-coloring of G'. Moreover, we get a (Z_3, f)-coloring of G, contrary to (2).

Thus, G contains 4-cycle $x_{2k+1}q_{2k+1}q_{2k+2}x_{2k+2}x_{2k+1}$. By Lemma 2.3, $q_{2k+2} \neq x_{2k+3}$. By (F3), $q_{2k+1} \neq x_{2k+1}$. Therefore, $q_{2k+1}, q_{2k+2} \in V(G) - V(C)$. $\square$

We are ready to complete our proof of Lemma 2.14. By Claim 2, we may assume that for $j = 1, 2, \cdots, \frac{m}{2} - 1$, G contains 4-cycles $x_{2j+1}q_{2j+1}q_{2j+2}x_{2j+2}x_{2j+1}$, where $x_{2j+1} \in W$ and $q_{2j+1}, q_{2j+2} \in V(G) - V(C)$.

When m is odd, $x_m \in W$. By (F2), $G[N(x_m) \cup N(x_{m-1})]$ does not contain a 4-cycle. By symmetry and by Claim 1, we obtain a contradiction.

When m is even, $x_{m-1} \in W$ and $x_m \notin W$. Let $G' = G - \{x_{m-1}, x_m\}$ and $W' = W \cup N(x_{m-1}) \cup N(x_{m-2}) - \{x_{m-2}, x_{m-1}\}$. Pick $a_{x_{m-2}} = b_{x_{m-3}} - f(x_{m-3}x_{m-2})$ and $a_{x_{m-1}} \in Z_3 - \{b_{x_{m-1}}, a_{x_{m-2}} + f(x_{m-2}x_{m-1})\}$. Define $b_z = a_{x_j} - f(x_j z)$ if $z \in N(x_j), j = m-1, m-2$. By $G \in \mathcal{F}$, b_z is well defined. It is easy to check that $G'[W']$ contains edgeless. By the induction of hypotheses of Theorem 2.1, c_1 can be extended to a (Z_3, f)-coloring c_2 of G'. Define

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x_{m-2}, x_{m-1}\}, \\ a_{x_j} & \text{if } z = x_j, j \in \{m-2, m-1\}. \end{cases}$$

Then c is a (Z_3, f)-coloring of G, contrary to (2).

Lemma 2.15 *$x_4 \in W$.*

Proof. Suppose that $x_4 \notin W$. By Lemma 2.14, $x_3 \notin W$. Assume first that $x_5 \notin W$. Let $G' = G - x_3$ and $W' = W \cup N(x_3) - \{x_2, x_3\}$. Pick $a_{x_3} \in Z_3 - \{c_1(x_2) - f(x_2x_3)\}$. Let $b_z = a_{x_3} - f(x_3 z)$ if $z \in N(x_3) - x_2$. Then b_z is well defined. By Lemmas 2.5, 2.6 and 2.13, $G'[W']$ contains no edges.

Assume then that $x_5 \in W$. Let $G' = G - \{x_3, x_4\}$ and let $W' = W \cup N(x_3) \cup N(x_4) - \{x_2, x_3, x_4, x_5\}$. Take $a_{x_4} = b_{x_5} + f(x_4x_5), a_{x_3} = Z_3 - \{c(x_2) - f(x_2x_3), a_{x_4} + f(x_3x_4)$. Let $b_z = a_{x_j} - f(x_j z)$ if $z \in N(x_3) \cup N(x_4) - \{x_2, x_3, x_4, x_5\}$. Then b_z is well defined. By Lemma 2.6 and (F1), $G'[W']$ contains at most one edge. If $G'[W']$ indeed contains that edge q_1q_2, let $q_1 \in N(x_3)$ and $q_2 \in N(x_4)$. By (4), we may assume that $q_1 \neq v$. Let $q_1q_3q_4$ be a 2-path. By (F2) and (F1), $q_4 \notin W' - W$.

Now we prove that $q_4 \notin W$. Suppose, to the contrary, that $q_4 \in W$. Then $q_4 \in \{x_6, x_7, \ldots, x_m\}$. Let C_1 be a cycle containing the path $x_3q_1q_3q_4$ and the edge uv and C_2 the cycle containing the path $x_3q_1q_3q_4$ but not uv. Let

$G_i = int(C_i) \cup C_i, i = 1, 2$. By (3), we can extend c_1 to a (Z_3, f)-coloring of G. Assume that the edge q_1q_3 is oriented from q_1 to q_3.

Let $W' = (W \cap V(G_2)) \cup \{x_3\}$. By the assumption that $x_4 \notin W$ and by Lemma 2.13, $G_2[W']$ has no edges. Therefore G_2 and W' satisfy the hypotheses of Theorem 2.1. Define

$$b'_z = \begin{cases} b_z & \text{if } z \in W' - \{x_3, q_4\} \\ b_{x_3} \in Z_3 - \{c_2(x_3), c_2(q_1) + f(x_3q_1)\} & \text{if } z = x_3, \\ b_{q_4} \in Z_3 - \{c_2(q_4), c_2(q_3) + f(q_4q_3)\} & \text{if } z = q_4. \end{cases}$$

By (3) $c_2 \mid_{\{q_1,q_3\}}$ can be extended to a (Z_3, f)-coloring c_3 of G_2 such that $c_3(w') \neq b_{w'}$ for any vertex $w' \in W'$. It follows that $c_3(x_3) = c_2(x_2), c_3(q_4) = c_2(q_3)$. By using c_2 and c_3 together, we get a required (Z_3, f)-coloring of G, contrary to (2).

Hence both G' and W' satisfy the hypotheses of Theorem 2.1 (with q_1 playing the role of x of G if $q_1q_2 \in E(G'[W'])$). By (3), G' has a (Z_3, f)-coloring and hence c_1 can be extended to a required (Z_3, f)-coloring of G, contrary to (2). □

Lemma 2.16 $|C| \geq 6$.

Proof. Suppose that $|C| = 5$. Let $G' = G - \{x_3, x_4\}$ and $W' = W \cup N(x_3) \cup N(x_4) - \{x_2, x_3, x_4\}$. By Lemma 2.15, $x_4 \in W$. Let $a_{x_3} \in Z_3 - \{c_1(x_2) - f(x_2x_3)\}$ and $a_{x_4} \in Z_3 - \{b_{x_4}, a_{x_3} - f(x_3x_4)\}$. Put $b'_z = a_{x_i} - f(x_iz)$ where $z \in N(x_i) - \{x_2, x_3, x_4\}, i = 3, 4$. By (F1), b_z is well defined. By (F3), $G[N(x_3) \cup N(x_4)]$ contains no 4-cycles. Thus by Lemma 2.6 $G'[W']$ is edgeless. Hence both G' and W' satisfy the hypotheses of Theorem 2.1. By (3), c_1 can be extended to a (Z_3, f)-coloring of G', say c_2. Define

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x, y\}, \\ a_{x_3} & \text{if } z = x_3, \\ a_{x_4} & \text{if } z = x_4. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, contrary to (2). □

Lemma 2.17 $x_6 \in W$.

Proof. Suppose that $x_6 \notin W$. By Lemmas 2.14 and 2.15, $x_3, x_5 \notin W$ and $x_4 \in W$. Let $G' = G - \{x_4\}$ and $W' = W \cup N(x_4) - \{x_4\}$. Take $a_{x_4} \in Z_3 - \{b_{x_4}\}$ and let $b_z = a_{x_4} - f(x_4z)$ if $z \in N(x_4)$. Then b_z is well defined. By Lemmas 2.5 and 2.6, $G'[W']$ is edgeless. Therefore both G' and W' satisfy the conditions of Theorem 2.1. By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G'. Define

$$c(z) = \begin{cases} c_2(z) & \text{if } z \in V(G) - \{x, y\}, \\ a_{x_4} & \text{if } z = x_4. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, violating (2). □

Proof of Theorem 2.1. By Lemmas 2.13, 2.15 and 2.17, $x_4, x_6 \in W$ and $x_5 \notin W$. Define $G' = G - \{x_4, x_5\}$ and $W' = W \cup N(x_4) \cup N(x_5) - \{x_4, x_5\}$.

pick $a_{x_5} = b_{x_6} + f(x_5x_6), a_{x_4} \in Z_3 - \{b_{x_4}, a_{x_5} + f(x_4x_5)\}$. Let $b_z = a_{x_j} - f(x_jz)$ if $z \in N(x_j), 4 \le j \le 5$. By (F1), b_z is well defined.

If $G[N(x_4) \cup N(x_5)]$ contains no 4-cycles, by Lemmas 2.5 and 2.6, $G'[W']$ is edgeless. Assume that $G[N(x_4) \cup N(x_5)]$ contains a 4-cycle $x_4q_2q_1x_5x_4$. By (F2) and Lemmas 2.5 and 2.6, $G'[W']$ contains only edge q_1q_2, where $q_2 \in N(x_4)$ and $q_1 \in N(x_5)$. By Lemma 2.4 and (F1), $q_1 \ne x_6$. Let $q_2q_3q_4$ be a 2-path where $q_4 \in W'$. By (F2) and (F3), $q_4 \in W$, contrary to Lemma 2.7 if $q_2 \ne x_3$ or contrary to Lemma 2.6 if $q_2 = x_3$. Thus, q_2 plays the role of x of G if $q_1q_2 \in E(G'[W']$. By (F2), $G'[\{\lambda_1, \lambda_2, q_1, q_2\}]$ is not a 4-cycle for any pair of distinct vertices λ_1, λ_2 of the out cycle of G'

Therefore, both G' and W' satisfy the conditions of Theorem 2.1 (with q_2 playing the role of x of G if $q_1q_2 \in E(G'[W'])$). By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G'. Define

$$c(z) = \begin{cases} c_2(z), & \text{if } z \in V(G) - \{x, y\}, \\ a_{x_3} & \text{if } z = x_3, \\ a_{x_4} & \text{if } z = x_4. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, contrary to (2). This completes the proof of the theorem. □

Corollary 2.18 *Let $G \in \mathcal{F}$ be a simple planar graph and let $H = K_2$. Then (G, H) is Z_3-extensible.*

Proof. Let $f \in F(G, Z_3)$ and $V(H) = \{v_1, v_2\}$ and let $c_0 : V(H) \mapsto Z_3$ is a (Z_3, f)-coloring of H. We may assume that in a plane embedding of G, the only edge in $E(H)$ is on the outer face of G. By Theorem 2.1, c_0 can be extended to a (Z_3, f)-coloring of G. □

3 Z_3-coloring of $K_{3,3}$-minor free graphs

Let G_1 and G_2 be two vertex disjoint bridgeless graphs and $u_1, v_1 \in V(G_1), u_2, v_2 \in V(G_2)$. Obtain G by identifying u_1 with u_2 to get a new vertex u, and v_1 with v_2 to get a new vertex v. Then G is called a *2-sum* of G_1 and G_2. $\{u, v\}$ are called a *separating set* of this 2-sum of G_1 and G_2. If $u \in V(G_1)$ and $v \in V(G_2)$, and if G is obtained from G_1 and G_2 by identifying u with v, then G is called a 1-sum of G_1 and G_2.

Hall [4] characterized the graphs excluding $K_{3,3}$ as a minor. There are several versions of this result (see [3, 11],). The following theorem is due to Hall [4].

Theorem 3.1 *(Hall [4]) Let G be a graph without $K_{3,3}$ minor. One of the followings must hold.*
(1) G is a planar graph;
(2) $G \cong K_5$, or
(3) G can be constructed recursively by the i-sums of planar graphs and copies of K_5 where $i \in \{1, 2\}$.

Lemma 3.2 *Let G be a graph with a vertex cut S such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. If $S = \{u\}$ and both G_1 and G_2 are (Z_3, f)-coloring, then G is a (Z_3, f)-coloring.*

Proof. Let $f \in F(G, Z_3)$. Then there is $c_1 : V(G_1) \mapsto Z_3$ such that for every directed edge $xy \in E(G_1)$, $c_1(x) - c_1(y) \neq f(xy)$ and there is $c_2 : V(G_2) \mapsto Z_3$ such that for every directed edge $zw \in E(G_2)$, $c_2(z) - c_2(w) \neq f(zw)$. It follows that there is $a \in Z_3$ such that $c_1(u) = c_2(u) + a$. Define

$$c(z) = \begin{cases} c_1(z) & \text{if } z \in V(G_1), \\ c_2(z) + a & \text{if } z \in V(G_2). \end{cases}$$

Then c is a (Z_3, f)-coloring of G. □

Lemma 3.3 *Suppose that H_1 and H_2 are both planar graphs in the construction procedure in Theorem 3.1, that $\{u, v\}$ is a separating set of a 2-sum of H_1 and H_2 and that $uv \notin E(H_1) \cap E(H_2)$. Then $H_i + uv$ is planar or K_5 for $i = 1, 2$.*

Proof. By contradiction, we assume that $H_1 + uv$ is not planar. If $H_1 + uv$ has a $K_{3,3}$-minor N, then new edge $uv \in E(N)$ since H_1 does not have a $K_{3,3}$-minor. Since G is 2-connected and since $\{u, v\}$ is 2-vertex cut of G, H_2 must have a (u, v)-path. It then follows that G has a $K_{3,3}$-minor, contrary to the assumption of G. Therefore $H_1 + uv$ does not have a $K_{3,3}$-minor. By Theorem 3.1, $H_1 + uv \cong K_5$. Thus $H_1 + uv$ is planar or K_5. The proof for that case $H_2 + uv$ is similar. □

Theorem 3.4 *Suppose that G is a connected $K_{3,3}$-minor free graph with girth at least 5. For each edge $e = uv \in E(G)$, let $H = G[\{u, v\}]$. Then (G, H) is Z_3-extensible.*

Proof. By contradiction, suppose that G is an counterexample with $|V(G)|$ minimized. By Theorem 2.1, we may assume that G is not planar. By Theorem 3.1 and Lemma 3.2, we may assume that G is 2-connected and G can be constructed recursively by 2-sums of planar graphs since the girth of G is at least 5.

We construct a new graph Γ as follows. The vertices of Γ are the planar graphs in the construction procedure in Theorem 3.1. Two vertices are adjacent if and only if the corresponding planar graphs have common a separating set of two vertices. It follows that Γ is a tree. Thus Γ has at least two vertices of degree 1. Let G_1 and G_2 be two corresponding planar graphs to two vertices of degree 1 in Γ. Since G is of girth at least 5, we may assume that $|V(G_i)| \geq 5, i = 1, 2$. Let $c_0 : \{u, v\} \mapsto Z_3$ such that c_0 is a (Z_3, f)-coloring of H.

By Theorem 2.1, Γ contains at least 2 vertices. When $|V(\Gamma)| = 2$, G is a 2-sum of G_1 and G_2. If $\{u, v\}$ is a separating set of 2-sum of G_1 and G_2, by Corollary 2.18, c_0 can be extended to a (Z_3, f)-coloring of G_1 and c_0 can be also extended to a (Z_3, f)-coloring of G_2. Thus the theorem follows.

So we may assume that $|V(\Gamma))| \geq 3$ and $\{u, v\}$ is not a separating set of 2-sum of G_1 and G_2 if $|V(\Gamma)| = 2$. Therefore $e = uv$ does not belong to at least one of G_1 and G_2, say $e \notin E(G_1)$. Let $G = G_1 \cup G_1'$ such that $V(G_1) \cap V(G_1') = \{u', v'\}$. Then $e \in E(G_1')$. If $u'v' \in E(G)$, $e = uv \neq u'v'$ by assumption. By the minimality of G, c_0 can be extended to a (Z_3, f)-coloring c_1 of G_1'. By Corollary 2.18, $c_1 |_{\{u',v'\}}$ can be extended to a (Z_3, f)-coloring of G_1. So c_0 can be extended to a (Z_3, f)-coloring of G, a contradiction.

Thus we assume that $u'v' \notin E(G)$. Define G^* obtained from G by adding two new vertices v_1, v_2 and three new edges $u'v_1, v_1v_2, v_2v'$ such that G_i^* is obtained from G_1 and G_1' by adding these two vertices and these three edges ($1 \leq i \leq 2$), respectively. By Lemma 3.3, G_1^* is planar and each of G^*, G_1^* and G_2^* has of girth at least 5. Assume that the path $u'v_1v_2v'$ is oriented from u' to v_1, from v_1 to v_2, from v_2 to v'. Define $f_1 : E(G^*) \mapsto Z_3$ by

$$f_1(e) = \begin{cases} f(e) & \text{if } e \in E(G), \\ 0 & \text{if } e \in \{u'v_1, v_1v_2, v_2v'\}. \end{cases}$$

Note that $uv \in E(G_2^*) - E(G_1^*)$ and $|V(G_2^*)| = |V(G)| - |V(G_1)| + 2 + 2 < |V(G)|$. By the minimality of G, c_0 can be extended to a (Z_3, f)-coloring c_1 of G_2^*.

Let $G_1^{**} = G_1^* - \{v_1, v_2\} + u'v'$. Since G_1^* is planar, G_1^{**} is also planar. We can imbed G_1^{**} in a plane such that the edge $u'v'$ is in the out face. Now we replace the edge $u'v'$ by the path $u'v_1v_2v'$. The resulting plane graph is the same one that G_1^* is embedded in a plane such that the edges $u'v_1, v_1v_2, v_2v'$ are in the outer face of G_1^*. Let $W = \{u', v'\}$ and define $b_{u'} \in Z_3 - \{c_1(u'), c_1(v_1) + f_1(u'v_1)\}, b_{v'} \in Z_3 - \{c_1(v'), c_1(v_2) - f_1(v_2v')\}$. By Theorem 2.1, $c_1 |_{\{v_1,v_2\}} \mapsto Z_3$ can be extended to a (Z_3, f_1)-coloring c_2 of G_1^* such that $c_2 |_{\{v_1,v_2\}} = c_1 |_{\{v_1,v_2\}}$ and $c_2(u') \neq b_{u'}, c_2(v') \neq b_{v'}$. It follows that $c_2(u') = c_1(u')$ and $c_2(v') = c_1(v')$. Define

$$c(z) = \begin{cases} c_1(z) & \text{if } z \in V(G_1), \\ c_2(z) & \text{if } z \in V(G_1'). \end{cases}$$

Then c is a (Z_3, f)-coloring of G, a contradiction. □

4 Remark

The proof of the main results in this paper utilizes techniques in [12] developed for studying list-coloring. In [8], Lai and Zhang applied similar skills ([13]) for choosability to prove that every K_5-minor free graph G satisfies $\chi_g(G) \leq 5$. It is natural to conjecture that there might be a close relationship between the group chromatic number $\chi_g(G)$ and the choice number $\chi_l(G)$. In particular, examples ([7] and [9]) have let us to conjecture that $\chi_l(G) \leq \chi_g(G)$. Moreover, former studies of $\chi_g(G)$ (Theorem 3.1 and Corollary 4.2 of [9], Theorem 2 of [7] and Theorem 1.2 [8]) also let us to consider the following analogue of Hadwiger conjecture: if G does not have a K_k-minor, then $\chi_g(G) \leq k$.

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Proof for the other three cases of Lemma 2.12.

Suppose first that such 2-path $Q = x_{i+2}u'x_j$ exists, where $j \in \{i+4, \ldots, m\}, u' \in V(G)-V(C)$ and $x_j \in V(C)-W$. If $x_j = x_{i+4}$, by (F1) the cycle $x_{i+4}x_{i+3}x_{i+4}u'x_{i+2}$ is facial. Thus $d_G(x_{i+3}) = 2$, contrary to Lemma 2.4.

Thus we assume that $j \in \{i+5, \ldots, m\}$. Let C_3 be the cycle in $C \cup Q$ containing $u'x_{i+2}$ and xy and C_4 the cycle in $C \cup Q$ containing $u'x_{i+2}$ but not xy. Let $G_i = C_i \cup int(C_i), i = 3, 4$. By (3), we can extend c_1 to a (Z_3, f)-coloring c_2 of G_3. Recall that Q is oriented from $x_{i+2}$to u' and from x_j to u'.

Let $W''' = (W \cap V(G_4)) \cup \{x_{i+2}\}$. By (W2), $x_{i+3} \notin W$. By Lemma 2.5, $G_4[W''']$ is edgeless. Let $b_{x_{i+2}} \in Z_3 - \{c_2(x_{i+2}), c_2(u') + f(x_{i+2}u')\}$. Therefore both G_4 and W''' satisfy the hypotheses of Theorem 2.1. By (3) again, $c_2 \mid_{\{u', x_j\}}$ can be extended to a (Z_3, f)-coloring c_3 of G_4. Using c_2 and c_3, we can get a required (Z_3, f)-coloring of G, contrary to (2).

Suppose then that such 3-path $P = x_{i-1}u'v'x_j$ exists, where $u', v' \in V(G) - V(C)$, $x_j \in W$ and $j \in \{3, \ldots, i-4\}$. Let C_5 be the cycle in $C \cup P$ containing $u'v'$ and xy and Let C_6 be the cycle in $C \cup P$ containing $u'v'$ but not xy. Let $G_j = C_j \cup int(C_j), j = 5, 6$. By (3), c_1 can be extended to a (Z_3, f)-coloring c_2 of G_5. Assume that $u'v'$ is oriented from u' to v'.

Let $W^{(4)} = (W \cap V(G_6) \cup \{x_{i-1}\}$. By the assumption that xy is only edge of $G[W]$ and Lemma 2.11, $x_{i-1}x_{i-2}$ is only edge of $G_6[W^{(4)}]$. By Lemma 2.6, there is no 2-path from x_{i-1} to any vertex of $W \cap V(G_6)$. Then G_6 and $W_{(4)}$ satisfy the hypotheses of Theorem 2.1 with x_{i-1} playing the role of x of G. Define

$$b'''_z = \begin{cases} b_z & \text{if } z \in W^{(4)} - \{x_{i-1}, x_j\}, \\ b_{x_{i-1}} \in Z_3 - \{c_2(x_{i-1}), c_2(u') + f(x_{i-1}u')\} & \text{if } z = x_{i-1}, \\ b_{x_j} \in Z_3 - \{c_2(x_j), c(v') + f(x_jv')\} & \text{if } z = x_j. \end{cases}$$

By (3), $c_2 \mid_{\{u', v'\}}$ can be extended to a (Z_3, f)-coloring c_3 of G_6 such that $c_3(w^{(4)}) \neq b_{w^{(4)}}$ for every $w^{(4)} \in W^{(4)}$. It follows that $c_3(x_{i-2}) = c_2(x_{i-2})$ and $c_3(x_j) = c_2(x_j)$. Combining c_2 and c_3, we get a required (Z_3, f)-coloring of G extending c_1 such that $c(z) \neq b_z$ for each vertex $z \in W$, contrary to (2).

Finally we suppose then that such 2-path $Q = x_{i-1}u'x_j$ exists, where $u' \in V(G) - V(C), x_j \in V(C) - W$ and $j \in \{3, \ldots, i-4\}$. Let C_7 be the cycle in $C \cup Q$ containing $u'x_{i-1}$ and xy and C_8 the cycle in $C \cup Q$ containing $u'x_{i-1}$ but not xy. Let $G_i = C_i \cup int(C_i), i = 7, 8$. Let $G_i = C_i \cup int(C_i), i = 7, 8$. By (3), we can extend c_1 to a (Z_3, f)-coloring c_2 of G_7. Recall that Q is oriented from $x_{i-1}$to u' and from x_j to u'.

Let $W^{(5)} = (W \cap V(G_8)) \cup \{x_{i-1}\}$. By Lemma 2.11 and by the assumption that xy is only edge of $G[W]$, it follows that $x_{i-1}x_{i-2}$ is only edge of $G_8[W^{(5)}]$. By Lemma 2.6, there is no 2-path from x_{i-1} to any vertex of $W \cap V(G_8)$. Let $b_{x_{i-1}} \in Z_3 - \{c_2(x_{i-1}), c_2(u') + f(x_{i-1}u')\}$. Therefore both G_8 and $W^{(5)}$ satisfy the hypotheses of Theorem 2.1 with x_{i-1} playing the role of x of G. By (3) again, $c_2 \mid_{\{u', x_j\}}$ can be extended to a (Z_3, f)-coloring c_3 of G_8. Using c_2 and c_3, we can get a required (Z_3, f)-coloring of G, contrary to (2).□

Proof of Lemma 2.9 (ii).

Assume that such a 4-cycle exists. Define $G' = G - \{x_{i-1}, x_i, x_{i+1}, q\}$ and $W' = W \cup N(x_{i-1}) \cup N(x_i) \cup N(x_{i+1}) \cup N(q) - \{x_{i-1}, x_i, x_{i+1}, q\}$. If $x_{i-2} \in W$, define $a_{x_{i-1}} = b_{x_{i-2}} - f(x_{i-2}x_{i-1}), a_{x_i} \in Z_3 - \{b_{x_i}, a_{x_{i-1}} - f(x_{i-1}x_i)\}, a_{x_{i+1}} \in Z_3 - \{b_{x_{i+1}}, a_{x_i} - f(x_i x_{i+1}\}, a_q \in Z_3 - \{a_{i-1} - f(x_{i-1}q), a_{x_{i+1}} - f(x_{i+1}q)\}$. If $x_{i-2} \notin W$, define $a_{x_i} \in Z_3 - \{b_{x_i}\}, a_{x_{i-1}} \in Z_3 - \{a_{x_i} + f(x_i x_{i+1})\}, a_{x_{i+1}} \in Z_3 - \{b_{x_{i+1}}, a_{x_i} - f(x_i x_{i+1}\}, a_q \in Z_3 - \{a_{i-1} - f(x_{i-1}q), a_{x_{i+1}} - f(x_{i+1}q)\}$. Let $b_z = a_{x_j} - f(x_j z)$ for every $z \in N(x_j) - \{x_{i-1}, x_i, x_{i+1}, q\}, j \in \{i-1, i, i+1\}$ and $b_z = a_q - f(qz)$ for every $z \in N(q) - \{q, x_i, x_{i+2}$. By (F2) b_z is well-defined. If there is a 2-path $x_{i-1}qx_j$ where $x_j \in W$, by Lemma 2.6 $x_j \in \{x_{i+4}, x_{i+5}, \ldots, x_m\}$. By Lemma 2.7, there is no 3-path $x_{i_1}qq'x_j$ where $x_j \in W$. Thus G' may has some cut vertices. We will distinguish the following two cases.

Case 1. There is a 2-path $x_{i-1}qx_j$ where $x_j \in W, j \in \{i+3, i+4, \ldots, m\}$.

Then G' can be decomposed into blocks $B_1, B_2, \ldots, B_k$ such that
(1) B_1 contains the edge uv and B_k contains x_{i+2}, and
(2) $V(B_i) \cap V(B_{i+1})$ is a cut vertex.

We claim that c_1 can be extended to a (Z_3, f)-coloring of B_1. Suppose that $V(B_1) \cap V(B_2) = \{z\}$. Then $B_1[W']$ contains at most one edge zw where $w \in W$. By Lemma 2.6, there is no a 2-path from z to any vertex of $B_1[W']$. By (3), c_1 can be extended to a (Z_3, f)-coloring of B_1.

Now we claim that if c_1 can be extended to a (Z_3, f)-coloring of $B_1 \cup B_2 \cup \ldots \cup B_i$, then c_1 can be extended to a (Z_3, f)-coloring c_2 of $B_1 \cup B_2 \cup \ldots \cup B_{i+1}$. Observe the block B_{i+1}. $B_{i+1}[W']$ contains at most two edges one of those contains vertices of $V(B_i) \cap V(B_{i+1})$ and the other contains vertices of $V(B_{i+1}) \cap V(B_{i+2})$. Suppose that $V(B_i) \cap V(B_{i+1}) = \{z_1\}$. We define z_2 as follows. If $B_{i+1}[W']$ does not contain an edge one end of which is z_1, let $z_2 \in V(B_{i+1}) \cap V(C)$ such that $z_1z_2 \in E(G)$. If $B_{i+1}[W']$ contains an edge one end of which is z_1, let z_2 be the other end of such edge. Assume that z_1z_2 is oriented from z_1 to z_2. Define $c_1(z_2) \in Z_3 - \{b_{z_2}, c_1(z_1) - f(z_1z_2)\}$.

Now we consider the vertex, say z_1', of $V(B_{i+1}) \cap V(B_{i+2})$. Then $B_{i+1}[W']$ maybe contains an edge one end of which is z_1'. If $B_{i+1}[W']$ indeed contains such edge, let z_2' be the other end of that edge. By Lemma 2.6, there is no 2-path from z_1' to any vertex of $V(B_{i+1}) \cap W'$. Therefore z_1z_2 plays the role of uv in G and $z_1'z_2'$ plays the role of xy in G if such $z_1'z_2'$ exists. By (3), $c_2 |_{\{z_1, z_2\}}$ can be extended to a (Z_3, f)-coloring of $B_1 \cup B_2 \cup \ldots \cup B_{i+1}$.

Thus we assume that c_1 can be extended to a (Z_3, f)-coloring c' of G'. Define

$$c(z) = \begin{cases} c'(z) & \text{if } z \in V(G'), \\ a_{x_j} & \text{if } z = x_j, j \in \{i-1, i, i+1\}, \\ a_q & \text{if } z = q. \end{cases}$$

Then c is a required (Z_3, f)-coloring of G, contrary to (2).

Case 2. There is no a 2-path $x_{i-1}qx_j$ where $j \in \{i+3, i+4, \ldots, m\}$.

By Lemmas 2.6 and 2.7, $G'[W']$ contains no edges. By (3), c_1 can be extended to a (Z_3, f)-coloring c' of G'. Define

$$c(z) = \begin{cases} c'(z) & \text{if } z \in V(G'), \\ a_{x_j} & \text{if } z = x_j, j \in \{i-1, i, i+1\}, \\ a_q & \text{if } z = q. \end{cases}$$

Then c is a (Z_3, f)-coloring of G, contrary to (2). $\square$