Path-related vertex colorings of graphs

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Abstract

We investigate algorithms for a frequency assignment problem in cellular networks. The problem can be modeled as a special coloring problem for graphs. Base stations are the vertices, ranges are the paths in the graph, and colors (frequencies) must be assigned to vertices following the conflict-free property: In every path there is a color that occurs exactly once. We concentrate on the special case where the base stations lie on a chain and ranges are the non-empty subchains. We also consider other simple graphs, such as rings, trees, and grids. We discuss a whole hierarchy of related coloring problems.

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1. Introduction

A vertex coloring of a graph $G = (V, E)$ is an assignment C: $V \rightarrow N$ of colors to its vertices such that for every edge the colors of its two vertices are different. A hypergraph $H = (V, E)$ is a generalization of a graph for which hyperedges can be arbitrary-sized non-empty subsets of V. There are several ways to define vertex coloring in hypergraphs: On one extreme, it is required that for every hyperedge, not all colors are the same (there are at least two colors); on the other extreme, it is required that for every edge, no color is repeated (all the colors are different). In between these two extremes, there is another possible generalization: A vertex coloring C of hypergraph H is called conflict-free if in every hyperedge there is a vertex whose color is unique among all other colors in the hyperedge.

Conflict-free coloring models frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: base stations and mobile agents. Base stations have fixed positions and provide the backbone of the network; they are modeled by vertices in V. Mobile agents are the clients of the network and they are served by base stations. This is done as follows: Every base station has a fixed frequency; this is modeled by the coloring C, i.e., colors represent frequencies. If an agent wants to establish a link with a base station it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. The range of communication of every agent is modeled by a hyperedge, which is the set of base stations that can communicate with the agent. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not reused by some other base station in the range. This is modeled by the conflict-free property. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, where possible, is preferable.

The study of conflict-free colorings was originated in the work of Even et al. (2003) and Smorodinsky (2003). In addition to the practical motivation described above, this new coloring model has drawn much attention of researchers through its own theoretical interest and such colorings have been the focus of several recent papers (see references).

Fiat et al. (2005) considered the special case of the problem where the hypergraph is defined as follows: Vertices are identified by points that lie on a line and E consists of all subsets of V defined by intervals intersecting at least one vertex. A line with n points has $n(n+1)/2$ such subsets. We call these subsets intervals because for our problem, two intervals are equivalent if they contain the same vertices. We represent colorings by listing the colors of points from left to right in a string. For example, for n=5, 12312 is a conflict-free coloring, whereas 12123 is not.

Conflict-free coloring for intervals is important because it can model assignment of frequencies in networks where the agents' movement is approximately unidimensional, e.g., the cellular network that covers a single long road and has to serve agents that move along this road. Also, conflict-free coloring for intervals plays a role in conflict-free coloring for more complicated range spaces; Even et al. (2003).

The problem becomes more interesting when the vertices are given online by an adversary. Namely, at every given time step t, a new vertex v_t is given and the algorithm must assign v_t a color such that the coloring is a conflict-free coloring of the hypergraph that is induced by the vertices $V_t = \{v_1,\ldots, v_t\}$. Once v_t is assigned a color, that color cannot be changed in the future. This is an online setting, so the algorithm has no knowledge of how vertices will be given in the future. For this version of the problem, in the case of intervals, Fiat et al. (2005) provide several algorithms. Their randomized algorithm uses O(log$n$loglog$n$) colors with high probability. Their deterministic algorithm uses O(log^2n) colors in the worst case. Recently, randomized algorithms that use O(log$n$) colors have been found [Chen (2006), Bar-Noy et al. (2006c)], in the oblivious adversary online model.

For conflict-free coloring of n points with respect to (closed) disks, Pach and Toth prove a lower bound of Ω(log$n$) colors. They also generalize the result to homothetic copies of any convex body. Har-Peled and Smorodinsky (2005), for conflict-free coloring of points with respect to axis-parallel rectangles, give an efficient coloring. Pach and Toth (2003) improve on the above result.

Alon and Smorodinsky (2006) consider coloring a collection of n disks in which each disk intersects at most k others such that for each point p in the union of all disks there is at least one disk in the collection containing p whose color differs from that of all other member of the collection that contain p (this is the dual problem of coloring points with respect to ranges). The proof uses the probabilistic method, and especially the Lovasz Local Lemma.

Smorodinsky (2006) studies `traditional' coloring of hypergraphs that are induced by simple Jordan curves. He applies the above results to conflict-free coloring of regions with near linear union complexity (using a polylogarithmic number of colors), and axis-parallel rectangles (using $O(\log^2 n)$ colors).

Elbassioni and Mustafa (2006) consider an interesting variation of the problem of conflict-free coloring points with respect to axis-parallel rectangles: They prove that given any set of n points on the plane, one can add sublinearly many new points, so that all points can be conflict-free colored efficiently.

2. Conflict-free coloring

Given is a graph G, with vertex set $V(G)$ and edge set $E(G)$. The aim is to color the vertices of the graph such that for each path p in the graph, there is a vertex v in p whose color is not used by any other vertex in p. This is called a *conflict-free* coloring (CF coloring) of graph G with respect to paths. It is a minimization problem, i.e., the goal is to find such a coloring that uses as few colors as possible.

Definition 1. The *conflict-free chromatic number* of a graph G, denoted by $\chi_{cf}(G)$, is the minimum k for which has a k-CF-coloring (a coloring with k colors).

Since the above coloring involves sets of vertices included in a path, one can ask the same question in terms of hypergraphs. A hypergraph H is a generalization of a graph for which hyperedges can be arbitrary-sized non-empty subsets of V.

Definition 2. A vertex coloring C of hypergraph H is called *conflict-free* if in every hyperedge there is a vertex whose color is unique among all other colors in the hyperedge.

Proposition 1. Given a graph G with vertex set V, define the hypergraph H_G with vertex set V and a hyperedge for each path p of the graph containing all vertices of p. Then, a *conflict-free* coloring of graph G with respect to paths is a conflict-free coloring of H_G and vice versa.

3. Relation with other problems

3.1 Ordered coloring

A closely related problem is *ordered coloring* [Katschalski et al. (1995)] or *vertex ranking* [Iyer et al. (1988)]. Ordered coloring is like conflict-free coloring, but we have the following additional constraint: *the unique color in a path must also be the maximum color in the path*. Formally, we define:

Definition 3. A *unique maximum* CF coloring is a CF coloring in which the maximum color in every path *p* is also a unique color in path *p*.

We remark that the definition given above is not what is typical in the bibliography [Katchalski et al. (1995)]. Instead the following definition is more typical:

Definition 4. An *ordered k*-coloring of a graph is a function C: V→{1,…,*k*} such that for every pair of distinct vertices *v*, *v'*, and every path *p* from *v* to *v'*, if *C*(*v*)=*C*(*v'*), there is an internal vertex *v''* of *p* such that *C*(*v*)<*C*(*v''*). The *ordered chromatic number* of a graph *G*, denoted by $\chi_o(G)$, is the minimum *k* for which *G* has an ordered *k*-coloring..

We prove that the two definitions are equivalent:

Proposition 2. *C* is a unique maximum CF coloring if and only if *C* is an ordered coloring.

Proof. If *C* is a unique maximum CF-coloring, then for any two same color vertices *v*, *v'*, every (*v*,*v'*)-path has a unique maximum color, greater than *C*(*v*), which appears in some internal vertex of *p*. If *C* is an ordered coloring, then consider any path *p* in *G*. The maximum color in *p* has to occur exactly in one vertex. If it occurs in two vertices *v*, *v'* of *p* then there is a (*v*,*v'*)-path contained in *p* which has an internal vertex with a greater color; a contradiction to the maximality of *C*(*v*) in *p*.

Corollary 1. Every ordered coloring is also a CF-coloring and thus $\chi_{cf}(G) \leq \chi_o(G)$.

In ordered colorings an even stronger property is true:

Proposition 3. In any ordered coloring *C* of *G*, in every connected subset *S* of vertices of *G*, the maximum color appearing in *S* is unique in *S*.

Proof. By contradiction; if there are two different vertices *x*, *y* in *S* with the maximum color, then there is a (*x*,*y*)-path in *S*, for which there is no internal vertex with higher color.

Proposition 4. (Monotonicity under subgraphs) If $X \subseteq Y$, then $\chi_o(X) \leq \chi_o(Y)$.

Proof. Graph *X* contains a subset of the paths of *Y*, so the restriction of an optimal coloring of *V*(*Y*) to *V*(*X*) is a CF-coloring for *X*.

3.2 Squarefree colorings

Another related problem is obtained by looking at colorings of paths as strings. We impose the following restriction: Every coloring of a path, when viewed as a string, shall not contain a repetition. Formally, a string w of natural numbers (colors) is called *squarefree* if there is no substring of w of the form $x^2=xx$, where x is a nonempty string. Given a coloring C of the vertices of a graph, for every path $p=v_1\dots v_l$, we define the color string of p to be $C(v_1)\dots C(v_l)$.

Definition 5. A coloring C is a squarefree k-coloring if for every path in the graph its color string is squarefree.

Corollary 2. Every CF-coloring is squarefree and thus $\chi_{\mathrm{sf}}(G) \leq \chi_{\mathrm{cf}}(G)$.

We have the following relation between colorings:

C ← SF ← CF ← OC

where C is the class of `traditional' vertex coloring of graphs.

The above is a proper hierarchy as can be exhibited by the following colorings of the chain P_6:

121212	traditional but not squarefree
123132	squarefree but not conflict-free
313213	conflict-free but not ordered
121312	ordered

In terms of chromatic numbers:

Proposition 5. $\chi(G) \leq \chi_{\mathrm{sf}}(G) \leq \chi_{\mathrm{cf}}(G) \leq \chi_{\mathrm{o}}(G)$.

The problem of squarefree coloring looks a lot easier than CF coloring and ordered coloring: For example, a seminal result by Thue (1906) shows that 3 colors suffice to color any chain. More precisely:

$$\chi_{\mathrm{SF}}(P_n) = \begin{cases} n, & \text{if } n \leq 2 \\ 3, & \text{otherwise} \end{cases}$$

As we will see, for chains both ordered coloring and conflict-free coloring need $\Omega(\log n)$ colors.

Another, more recent result is by Currie (2002), on the squarefree chromatic number of rings is the following:

$$\chi_{\mathrm{SF}}(C_n) = \begin{cases} 4, & \text{if } n \in \{5,7,9,10,14,17\} \\ 3, & \text{otherwise} \end{cases}$$

As we will see, for rings both ordered coloring and conflict-free coloring need $\Omega(\log n)$ colors.

3.3 Cubefree colorings

Another related class of colorings consists of cubefree colorings, where color strings of paths can not contain an x^3 substring, for x nonempty. It is known from [Thue (1906)], but it is also implicit in [Prouhet (1851)], that 2 colors suffice to color any chain. Cubefree colorings can also be put in the above hierarchy over squarefree colorings but they are not comparable with traditional colorings.

4. Conflict-free coloring graphs

We study conflict-free coloring some graphs with respect to all paths.

4.1 Chain

Conflict-free coloring of a chain is better known as conflict-free coloring with respect to intervals [Even et al. (2003), Fiat et al. (2005)]. We will prove that it can be done with exactly $1+\lfloor \lg n \rfloor$ colors. This will also be an ordered coloring. Because of its symmetry, we call it a *recursively palindromic* coloring. A chain containing n points is given (points are linearly ordered in the chain). Colors are positive integers. A coloring is an assignment of colors to the points of the interval. The coloring is represented by an array of positive integers A[1..n], where for each point i, A[i] is the assigned color. A conflict-free coloring is an assignment C of colors to the points, such that for every subarray (sequence of consecutive elements in the array), there is a color that appears exactly once, i.e., for all i, j, with 1≤i≤j≤n, the subarray A[i..j] contains a color that appears exactly once in the subarray. Let $c(n)$ be the minimum number of colors for conflict-free coloring a chain of n points (vertices).

A lower bound. In order to find a lower bound for the number of colors needed for n points, first, we have to observe that in any conflict-free colored array A[1..n], there is a color that appears exactly once. If that color is assigned to point k, then the subarrays A[1..k-1] and A[k+1..n] use one less color than the whole interval. We can also assume that one of the two subintervals uses only colors that appear in the other interval, because any interval that spans points in both [1..k-1] and [k+1..n] will also span point k and thus have the color at the k-th entry as its uniquely appearing color. If we also consider the non-decreasing nature of the function c(n), we can concentrate on the conflict-free coloring of the interval of maximum length among [1..k-1] and [k+1..n]. The length of this interval is at least $\lfloor n/2 \rfloor$. Therefore, we have the recurrence:

$$c(n) \geq 1 + c(\lfloor n/2 \rfloor) \quad \text{and} \quad c(1) = 1 .$$

The solution of the above recurrence relation is

$$c(n) \geq 1 + \lfloor \lg n \rfloor.$$

A conflict-free coloring that achieves the bound. An offline coloring algorithm that achieves this lower bound is given below:

Starting at point 1, color with color 1 every 2 points

Starting at point 2, color with color 2 every 4 points

...

Starting at point 2^{i-1}, color with color i every 2^i points

...

and keep doing this until you have colored all points.

Color i is used only if $n \geq 2^{i-1}$, so in fact $1+\lfloor \lg n \rfloor$ colors are used by the algorithm. We can describe more easily the coloring produced by the algorithm by using a binary tree of n nodes. If $n=2^k-1$, then the tree is full, and it is as follows: The levels of the tree are numbered from the bottom, to the top (root). The lowest level is level 1. Nodes at level i are labeled with number (color) i. The root is at level $1+\lfloor \lg n \rfloor$, so it also has label $1+\lfloor \lg n \rfloor$. The conflict free coloring arises from an inorder traversal of the tree. If $n \neq 2^k-1$, the tree is missing its rightmost nodes and the description with levels is not that elegant in all cases.

Claim 1. For any color i that is repeated in some interval, there is a color i' in the coloring contained in that interval such that $i' > i$.

Proof. In the tree representation of the coloring, consider the node with the maximum color i' in the path (in the tree) connecting the two nodes colored with i.

Claim 2. The coloring produced by the algorithm is conflict-free.

Proof. In every interval, the maximum color occurs uniquely. Otherwise, if the maximum color is not unique, then (by the previous claim) an even greater color is contained in the interval; a contradiction.

4.2 Ring

To conflict-free color a ring, we use the above conflict-free coloring of a chain. We pick an arbitrary vertex v and color it with a unique color (not to be reused anywhere else in the coloring). The remaining vertices form a chain that we color with the method described above. This method colors a ring of n vertices with $2+\lfloor \lg(n-1) \rfloor$ colors. For example, if n=8, the coloring is 41213121, where `4' is the first unique color used for v. It is not difficult to see that the coloring is conflict-free: All paths that include v are conflict-free colored, and the remaining graph $G–v$ is a chain of $n–1$ nodes, so paths of $G–v$ are also conflict-free colored. This can be proved to be tight.

4.3 Tree

For a tree graph, we use the idea of a 1/2-separator [Jordan (1869), Lewis et al. (1965), Erlebach et al. (2003)]. A 1/2-separator is a vertex which, when removed, leaves connected components whose size is bounded by $n/2$. The method to color a tree is as follows: Find a 1/2-separator, color it with a unique color. Then color recursively the connected components, after the removal of the 1/2-separator. Thus, $\chi_{cf}(T) \leq 1+\lfloor \lg n \rfloor$ for a tree T with n vertices. See also [Katchalski et al. (1995)]. If a maximum color is used for every separator, the above is an ordered coloring. Moreover, one can find optimal ordered colorings of trees [Iyer et al. (1988)].

4.4 Grid

A grid of size $m\times m$, i.e, with $n=m^2$ vertices can be colored with an ordered coloring with at most $4m$ colors: The idea is to use unique maximum colors for the row closest to the middle and column closest to the middle (that is less than $2m$ colors), and then color recursively in the 4 subgrids with size at most $\lfloor m/2 \rfloor \times \lfloor m/2 \rfloor$ each. A slight variation gives a coloring with at most $3m$ colors: Use m unique maximum colors for the row closest to the middle, and then use about $m/2$ more unique colors for the part of the middle column over the middle row, and the same $m/2$ colors for the middle column under the middle row; then use recursion in the 4 subgrids with size at most $\lfloor m/2 \rfloor \times \lfloor m/2 \rfloor$ each. See also [Bar-Noy et al. (2006a), Cheilaris et al. (2006)].

5. Related and future research

One could study the conflict-free coloring problem in an online setting; for relevant results, see [Fiat et al. (2005), Bar-Noy et al. (2006b)]. The most important open problem in the online setting for chains is narrowing the gap between lower and upper bound in the deterministic online model: $\Omega(\log n)$, and $O(\log^2 n)$, respectively, which are a logarithmic factor apart. Better bounds can be obtained for the ordered chromatic number of the grid; see [Cheilaris et al. (2007)].

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