The Domination Parameters of Cubic Graphs

Igor E. Zverovich*†
RUTCOR – Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
Piscataway, NJ 08854-8003
USA

Vadim E. Zverovich*
Department of Mathematical Sciences
Brunel University, Uxbridge
Middlesex, UB8 3PH
UK
Vadim.Zverovich@brunel.ac.uk

Abstract

Let $ir(G)$, $\gamma(G)$, $i(G)$, $\beta_0(G)$, $\Gamma(G)$ and $IR(G)$ be the irredundance number, the domination number, the independent domination number, the independence number, the upper domination number and the upper irredundance number of a graph G, respectively. In this paper we show that for any integers k_1, k_2, k_3, k_4, k_5 there exists a cubic graph G satisfying the following conditions: $\gamma(G) - ir(G) \geq k_1$, $i(G) - \gamma(G) \geq k_2$, $\beta_0(G) - i(G) \geq k_3$, $\Gamma(G) - \beta_0(G) \geq k_4$, and $IR(G) - \Gamma(G) \geq k_5$. This result settles a problem posed in [8].

1 Introduction and Main Result

All graphs will be finite and undirected without multiple edges. If G is a graph, $V(G)$ denotes the set, and $|G|$ the number, of vertices in G. Let $N(x)$ denote the neighborhood of a vertex x, and let $\langle X \rangle$ denote the subgraph of G induced by $X \subseteq V(G)$. Also let $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$.

A set $I \subseteq V(G)$ is called *independent* if no two vertices of I are adjacent. A set X is called a *dominating set* if $N[X] = V(G)$. An *independent dominating set* is a vertex subset that is both independent and dominating, or equivalently, is maximal independent. The *independence number* $\beta_0(G)$ is the maximum cardinality of a (maximal) independent set of G, and the *independent domination number* $i(G)$ is the minimum cardinality taken

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over all maximal independent sets of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a (minimal) dominating set of G, and the *upper domination number* $\Gamma(G)$ is the maximum cardinality taken over all minimal dominating sets of G. For $x \in X$, the set

$$PN(x, X) = N[x] - N[X - \{x\}]$$

is called the *private neighborhood* of x. If $PN(x, X) = \emptyset$, then x is said to be *redundant* in X. A set X containing no redundant vertex is called *irredundant*. The *irredundance number* $ir(G)$ is the minimum cardinality taken over all maximal irredundant sets of G, and the *upper irredundance number* $IR(G)$ is the maximum cardinality of a (maximal) irredundant set of G. An *ir-set* X of G is a maximal irredundant set of cardinality $ir(G)$. A γ-set, an i-set, a β_0-set, a Γ-set and an IR-set are defined analogously.

The following relationship among the parameters under consideration is well-known [2, 3]:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

The above and related parameters for regular graphs were investigated by many authors [1],[4]–[16]. For example, Cockayne and Mynhardt [4] and independently Rautenbach [14] disproved the Henning-Slater conjecture [11] that $\Gamma(G) = IR(G)$ for any cubic graph G, while the Barefoot-Harary-Jones conjecture on the difference between the domination and independent domination numbers of cubic graphs was disproved independently in [12] and [16].

In this paper, we deal with the next problem:

Problem 1 **([8])** *Does there exist a cubic graph for which* $ir < \gamma < i < \beta_0 < \Gamma < IR$*?*

We define the graph W_k ($k \geq 0$) as follows. Take a disjunct union of the graphs

$$F_1, F_2, ..., F_{2k+8}, G_1, G_2, ..., G_{2k+6}, H_1, H_2, ..., H_{3k+6},$$

where F_i, G_i and H_i are shown in Figure 1, and add the edges

$$\{f'_i f_{i+1} : 1 \leq i \leq 2k+7\},\ f'_{2k+8} g_1,$$
$$\{g'_i g_{i+1} : 1 \leq i \leq 2k+1\},\ g'_{2k+6} h_1,$$
$$\{h'_i h_{i+1} : 1 \leq i \leq 3k+1\},\ h'_{3k+6} f_1.$$

Theorem 1 *For any integers* k_1, k_2, k_3, k_4, k_5 *there exists an integer* k *such that the cubic graph* W_k *satisfies the following conditions:* $\gamma(W_k) - ir(W_k) \geq k_1$*,* $i(W_k) - \gamma(W_k) \geq k_2$*,* $\beta_0(W_k) - i(W_k) \geq k_3$*,* $\Gamma(W_k) - \beta_0(W_k) \geq k_4$*, and* $IR(W_k) - \Gamma(W_k) \geq k_5$*.*

It follows from Lemmas 1–5 of Section 2 that the graph W_0 has the property

$$ir < \gamma < i < \beta_0 < \Gamma < IR,$$

thus solving Problem 1.

We conclude this section with the next conjecture.

Conjecture 1 *For any integers* k_1, k_2, k_3, k_4, k_5 *there exists a 3-connected cubic graph* G *satisfying the following conditions:* $\gamma(G) - ir(G) \geq k_1$*,* $i(G) - \gamma(G) \geq k_2$*,* $\beta_0(G) - i(G) \geq k_3$*,* $\Gamma(G) - \beta_0(G) \geq k_4$*, and* $IR(G) - \Gamma(G) \geq k_5$*.*

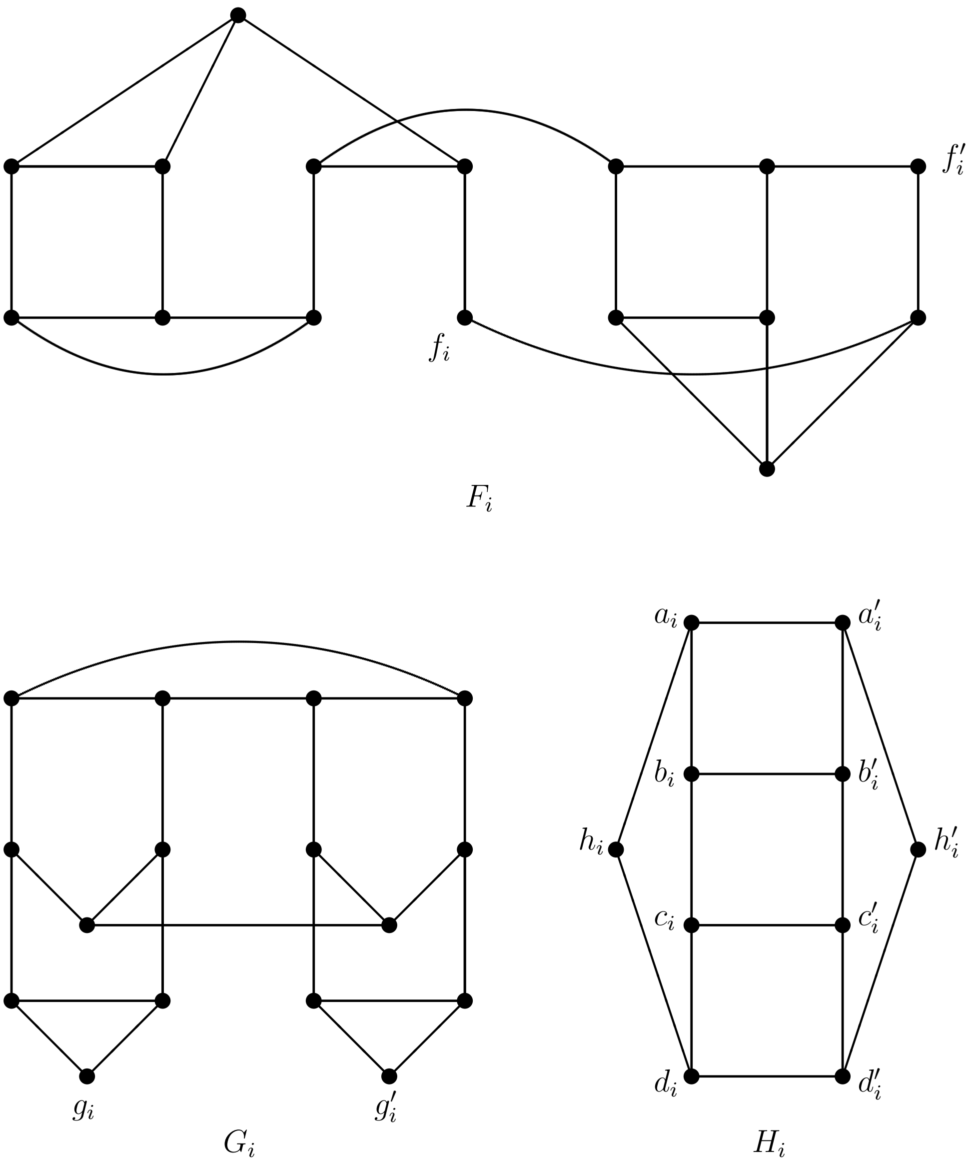

Figure 1. Graphs F_i, G_i, and H_i.

2 Proof of Theorem 1

The proof of Theorem 1 is based on 5 lemmas. Let us denote by F, G and H the graphs induced by the sets $\cup_{i=1}^{2k+8} V(F_i)$, $\cup_{i=1}^{2k+6} V(G_i)$, and $\cup_{i=1}^{3k+6} V(H_i)$, respectively.

Lemma 1 $\gamma(W_k) - ir(W_k) \geq k+1$.

Proof: Let D denote a γ-set of W_k. It is straightforward to check that $|D \cap V(G_i)| = 4$ whenever both g_i and g_i' are dominated by $D - V(G_i)$, and $|D \cap V(G_i)| = 5$ otherwise. Moreover, if $|D \cap V(G_i)| = 4$, then $g_i, g_i' \notin D$. Thus, the number of components G_i satisfying $|D \cap V(G_i)| = 4$ is at most $k+3$. We obtain

$$|D \cap V(G)| \geq 4(k+3) + 5(k+3) = 9k + 27.$$

Consider the set $J = (D - V(G)) \cup R$, where

$$R = \{N(g_i) \cap V(G_i), N(g_i') \cap V(G_i) : 1 \leq i \leq 2k+6\}.$$

We have

$$|R| = 8k + 24.$$

Let us construct a maximal irredundant set of W_k. We first put $J' = J$. Further, if $N[h_1] \cap J = \emptyset$, then we put $g'_{2k+6} \in J'$. If $N[f'_{2k+8}] \cap J = \emptyset$, then we put $g_1 \in J'$. If $h_1 \in D$ and $PN(h_1, D) = g'_{2k+6}$, then we put $h_1 \notin J'$. Finally, if $f'_{2k+8} \in D$ and $PN(f'_{2k+8}, D) = g_1$, then we put $f'_{2k+8} \notin J'$. It is easy to see that the set J' is a maximal irredundant set, and $|J'| \leq |J| + 2$. We obtain

$$\gamma(W_k) - ir(W_k) \geq |D| - |J'| \geq |D| - |J| - 2 = |D \cap V(G)| - |R| - 2 \geq k + 1.$$

∎

Lemma 2 $i(W_k) - \gamma(W_k) \geq k + 1.$

Proof: We denote by I an i-set of W_k.

Claim 1 *We have $|I \cap V(H_i)| = 3$ or 4 for any i, $1 \leq i \leq 3k+6$. Moreover, $|I \cap V(H_i)| = 3$ if and only if either h_i or h_i' is dominated by $I - V(H_i)$, and additionally $h_i, h_i' \notin I$.*

Proof: Assume that $h_i, h_i' \in I$ for some i, $1 \leq i \leq 3k+6$. We obtain $|I \cap V(H_i)| = 4$. Suppose now that exactly one vertex from h_i, h_i' belongs to I, say $h_i \in I$ and $h_i' \notin I$. If $b_i, c_i \notin I$, then these vertices cannot be dominated by an independent set, a contradiction. Therefore, without loss of generality, $b_i \in I$ and $c_i \notin I$. Hence $a_i' \in I$, and either $c_i' \in I$ or $d_i' \in I$. We have $|I \cap V(H_i)| = 4$. Consider the case $h_i, h_i' \notin I$. Since $I \cap \{b_i, b_i', c_i, c_i'\} \neq \emptyset$, we may assume without loss of generality that $b_i \in I$ and hence $a_i' \in I$. If $c_i' \in I$, then $d_i \in I$ and $|I \cap V(H_i)| = 4$. If $c_i' \notin I$, then $d_i' \in I$ and $|I \cap V(H_i)| = 3$. ∎

By Claim 1, the number of components H_i satisfying $|I \cap V(H_i)| = 3$ is at most $2k+4$. Therefore,

$$|I \cap V(H)| \geq 10k + 20.$$

Let us consider the set $D = \{h'_{3k+6}, h_i, b_i', c_i' : i = 1, 2, ..., 3k+6\}$. It is evident that the set $J = (I - V(H)) \cup D$ is a dominating set of W_k and

$$i(W_k) - \gamma(W_k) \geq |I| - |J| = |I \cap V(H)| - |D| \geq 10k + 20 - 9k - 19 = k + 1.$$

∎

Now we estimate the difference between the independence and independent domination numbers of W_k.

Lemma 3 $\beta_0(W_k) - i(W_k) \geq 2k + 4.$

Proof: It is easy to construct a maximal independent set I of W_k such that $|I \cap V(F_i)| = 6$, $|I \cap V(G_i)| = 6$, and $|I \cap V(H_i)| = 4$. We define the set $R \subset V(H)$ as follows. For each $i \in \{1, 2, ..., 3k+6\}$, we put $a_i, d_i, b'_i \in R$ if $i = 1 \pmod 3$, $h_i, h'_i, b'_i, c_i \in R$ if $i = 2 \pmod 3$, and $a'_i, b_i, d'_i \in R$ if $i = 0 \pmod 3$. Now, the set $J = (I - V(H)) \cup R$ is an independent dominating set and hence $i(W_k) \le |J|$. We obtain

$$\beta_0(W_k) - i(W_k) \ge |I| - |J| = |I \cap V(H)| - |R| = 12k + 24 - 10k - 20 = 2k + 4.$$

∎

Lemma 4 $\Gamma(W_k) - \beta_0(W_k) \ge 3k + 5.$

Proof: We can split $V(F_i)$ into three cycles C_3 and one C_7, $V(G_i)$ into two cycles C_5 and two cycles C_3, and $V(H_i)$ into two cycles C_5. Therefore,

$$\beta_0(W_k) \le 6(2k+8) + 6(2k+6) + 4(3k+6) = 36k + 108.$$

It is easy to construct a maximal independent set I of W_k such that $|I \cap V(F_i)| = 6$, $|I \cap V(G_i)| = 6$ and $g'_{2k+6} \in I$, and $|I \cap V(H_i)| = 4$. Thus, $|I| = 36k + 108$ and hence $\beta_0(W_k) = |I|$.

Consider the set $S = \{h'_i, a'_i, b'_i, c'_i, d'_i : \ 1 \le i \le 3k+6\} - h'_{3k+6}$. It is evident that $R = (I - V(H)) \cup S$ is a minimal dominating set. We have

$$\Gamma(W_k) - \beta_0(W_k) \ge |R| - |I| = |S| - |I \cap V(H)| = 15k + 29 - 12k - 24 = 3k + 5.$$

∎

Denote by D a Γ-set of W_k.

Proposition 1 $|D \cap V(F)| \le 13k + 53.$

Proof: Let us label the vertices of F_i as shown in Figure 2, and put $X = \{x, a, b, h, i, j\}$.

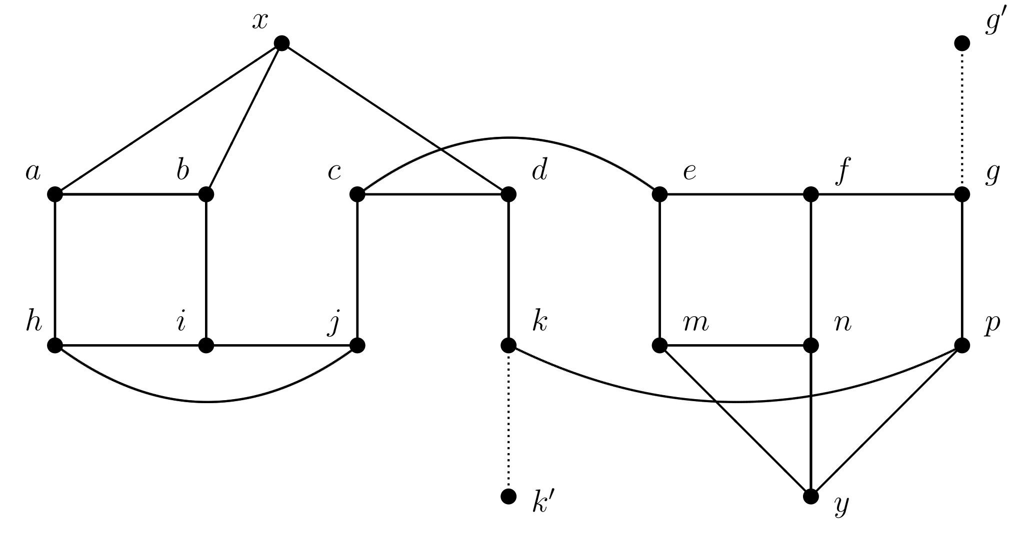

Figure 2.

Claim 2 *It holds* $|D \cap X| = 2$*. Moreover,* $f, e, m \notin D$ *if* $c, d \in D$ *and at least one of the vertices* k, k', p *belongs to* D*.*

Proof: We consider 4 cases.

Case 1. $c, d \notin D$. Since x must be dominated by D, it follows that one vertex from $\{a, b, x\}$ must belong to D. Analogously for the vertex j, one vertex from $\{h, i, j\}$ must belong to D. Now, the set X is dominated by those two vertices and hence $|D \cap X| = 2$.

Case 2. $c \notin D$, $d \in D$. Suppose that $k \notin D$. If $j \in D$, then $h, j \notin D$, for otherwise $PN(j) = \emptyset$. We see that exactly one vertex from $\{a, b, x\}$ belongs to D. Assume that $k \in D$. If $j \in D$, then $x = PN(d)$ and hence $a, b, x \notin D$. We have h or $i \in D$ and therefore $PN(j) = \emptyset$, a contradiction. Thus, $j \notin D$ and without loss of generality $h \in D$. It is easy to see that exactly one vertex from $\{a, b, x, i\}$ belongs to D and we are done.

Case 3. $c \in D$, $d \notin D$. If $x \in D$, then $a, b \notin D$, for otherwise $PN(x) = \emptyset$. We see that exactly one vertex from $\{h, i, j\}$ belongs to D and we are done. If $x \notin D$, then a or b belongs to D, say $a \in D$. Now, exactly one vertex from $\{b, h, i, j\}$ belongs to D and we are done.

Case 4. $c, d \in D$. Suppose that at least one of the vertices k, k', p belongs to D. We have $x = PN(d)$ and hence $a, b, x \notin D$. Therefore $h, i \in D$ and $j \notin D$. Moreover, $PN(c) = e$ and hence $m, e, f \notin D$. The case $k, k', p \notin D$ easily implies $|D \cap X| = 2$. ∎

We define 16 types for the component F_i as follows:

F_i has type A1 if $k', g' \in D$ and $k \in PN(k')$, $g \in PN(g')$;
F_i has type A2 if $k', g' \in D$ and $k \notin PN(k')$, $g \in PN(g')$;
F_i has type A3 if $k', g' \in D$ and $k \in PN(k')$, $g \notin PN(g')$;
F_i has type A4 if $k', g' \in D$ and $k \notin PN(k')$, $g \notin PN(g')$;
F_i has type B1 if $k' \in D, g' \notin D$ and $k \in PN(k')$, $g' \in N(D - V(F_i))$;
F_i has type B2 if $k' \in D, g' \notin D$ and $k \notin PN(k')$, $g' \in N(D - V(F_i))$;
F_i has type B3 if $k' \in D, g' \notin D$ and $k \in PN(k')$, $g' \in PN(g)$;
F_i has type B4 if $k' \in D, g' \notin D$ and $k \notin PN(k')$, $g' \in PN(g)$;
F_i has type C1 if $k' \notin D, g' \in D$ and $k' \in N(D - V(F_i))$, $g \in PN(g')$;
F_i has type C2 if $k' \notin D, g' \in D$ and $k' \in N(D - V(F_i))$, $g \notin PN(g')$;
F_i has type C3 if $k' \notin D, g' \in D$ and $k' \in PN(k)$, $g \notin PN(g')$;
F_i has type C4 if $k' \notin D, g' \in D$ and $k' \in PN(k)$, $g \in PN(g')$;
F_i has type D1 if $k', g' \notin D$ and $k' \in N(D - V(F_i))$, $g' \in N(D - V(F_i))$;
F_i has type D2 if $k', g' \notin D$ and $k' \in PN(k)$, $g' \in N(D - V(F_i))$;
F_i has type D3 if $k', g' \notin D$ and $k' \in N(D - V(F_i))$, $g' \in PN(g)$;
F_i has type D4 if $k', g' \notin D$ and $k' \in PN(k)$, $g' \in PN(g)$;

Let us denote $D_i = D \cap V(F_i)$.

Claim 3 *We have*
(a1) $|D_i| = 5$ *if* F_i *is of type A1;*
(a2) $|D_i| = 6$ *if* F_i *is of type A2;*
(a3) $|D_i| = 5$ *if* F_i *is of type A3;*
(a4) $|D_i| = 6$ *if* F_i *is of type A4;*
(b1) $|D_i| = 5$ *if* F_i *is of type B1;*
(b2) $|D_i| = 6$ *if* F_i *is of type B2;*

(b3) $|D_i| = 5$ *if* F_i *is of type B3;*
(b4) $|D_i| = 7$ *if* F_i *is of type B4;*
(c1) $|D_i| = 6$ *if* F_i *is of type C1;*
(c2) $|D_i| = 6$ *if* F_i *is of type C2;*
(c3) $|D_i| = 7$ *if* F_i *is of type C3;*
(c4) $|D_i| = 6$ *if* F_i *is of type C4;*
(d1) $|D_i| = 6$ *if* F_i *is of type D1;*
(d2) $|D_i| = 7$ *if* F_i *is of type D2;*
(d3) $|D_i| = 7$ *if* F_i *is of type D3;*
(d4) $|D_i| = 8$ *if* F_i *is of type D4;*

Proof: In what follows we will use the first part of Claim 2 without further reference.

(a1) Since $k \in PN(k')$ and $g \in PN(g')$, we have $d, k, p, g, f \notin D$. Also, $y \in D$, for otherwise p is not dominated. Suppose that $e \in D$. We have $m, n \notin D$ and $|D_i| = 5$. Assume that $e \notin D$. We obtain $n \in D$. It is easy to see that exactly one of the vertices c, m belongs to D and hence $|D_i| = 5$.

(a2) We have $p, g, f \notin D$. If $c \notin D$, then $|D_i - X| = 4$ and hence $|D_i| = 6$. Suppose that $c \in D$. If $d \in D$, then $m, e \notin D$ by Claim 2. Hence $n \in D$ and $|D_i| = 6$. If $d \notin D$, then again $|D_i| = 6$.

(a3) We have $d, k, p \notin D$. If $c \notin D$, then $|D_i - X| = 3$ and hence $|D_i| = 5$. Consider the case $c \in D$. If $y \in D$, then $|D_i| = 5$. If $y \notin D$, then $g \in D$, for otherwise p is not dominated. To dominate y we must take either m or n and hence $|D_i| = 5$.

(a4) Assume that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $p = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. The case $c \in D$ and $d \notin D$ is analogous. Finally, suppose that $c, d \in D$. By Claim 2, $e, f, m \notin D$. If $k \in D$, then $p = PN(k)$. Therefore, $y, p, g \notin D$, $n \in D$ and $|D_i| = 6$. If $k \notin D$, then exactly two vertices from $\{n, y, p, g\}$ belong to D and $|D_i| = 6$.

(b1) We have $d, k, p \notin D$. Suppose that $c \notin D$. If $y \in D$, then $|D_i - X| = 3$ and hence $|D_i| = 5$. If $y \notin D$, then $g \in D$ to dominate p. Again, $|D_i - X| = 3$ and $|D_i| = 5$. Consider the case $c \in D$. If $y \in D$, then $f \in D$ or $g \in D$, for otherwise g is not dominated. We have $|D_i| = 5$. If $y \notin D$, then $g \in D$, for otherwise p is not dominated. Also, one of the vertices m, n belongs to D to dominate y. We obtain $|D_i| = 5$.

(b2) Suppose that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $|D \cap \{c, d\}| = 1$. If $k \in D$, then $p = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$. If $k \in D$, then $PN(k) = p$ and hence $g, p, y \notin D$. Now g is not dominated, a contradiction. If $k \notin D$, then $|D_i| = 6$.

(b3) We have $d, k, p \notin D$ and $g \in D$. Suppose that $c \notin D$. If $y \in D$, then $f, m \notin D$, $e \in D$ and hence $|D_i| = 5$. If $y \notin D$, then again $|D_i| = 5$. Consider the case $c \in D$. To dominate y, exactly one of the vertices m, n, y belongs to D. Hence $|D_i| = 5$.

(b4) We have $g \in D$. Suppose that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 5$ and hence $|D_i| = 7$. Consider the case $|D \cap \{c, d\}| = 1$. If $k \in D$, then $PN(k) = \emptyset$, a contradiction. Therefore, $k \notin D$. It is easy to see that $|D_i| = 7$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$. If $k \in D$, then $PN(k) = \emptyset$, a contradiction. Therefore, $k \notin D$. We obtain $|D_i| = 6$. Since D is a maximum minimal dominating set, we conclude that $|D_i| = 7$.

(c1) We have $f, g, p \notin D$. Suppose that $k \notin D$. We obtain $y \in D$ to dominate p, and $d \in D$ to dominate k. Therefore, $|D_i| = 6$. Consider the case $k \in D$. If $c, d \notin D$, then $|D_i| = 5$. If exactly one vertex from $\{c, d\}$ is present in D, then it is checked directly that $|D_i| = 6$. Finally, suppose that $c, d \in D$. By Claim 2, $e, m \notin D$. We have $|D_i| = 6$. Since D is a maximum minimal dominating set, we conclude that $|D_i| = 6$.

(c2) Assume that $c, d \notin D$. It is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $p = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. Consider the case $c \in D$ and $d \notin D$. If $k \notin D$, then $p \in D$ to dominate k. We obtain $|D_i| = 6$. If $k \in D$, then $p \notin D$, for otherwise $PN(k) = \emptyset$. It is easy to see that $|D_i| = 6$. Finally, suppose that $c, d \in D$ and consider two cases.

Case 1. $k \in D$. By Claim 2, $e, f, m \notin D$. Further, $p = PN(k)$. Therefore, $g, p, y \notin D$, $n \in D$ and $|D_i| = 6$.

Case 2. $k \notin D$. Suppose that $p \in D$. By Claim 2, $e, f, m \notin D$. Also, $y \notin D$, for otherwise $PN(p) = \emptyset$. We obtain $n \in D$ and $|D_i| = 6$. Assume now that $p \notin D$. If $y \in D$, then $|D_i| = 6$. If $y \notin D$, then $g \in D$ to dominate p. Moreover, exactly one vertex from $\{m, n\}$ belongs to D. Thus, $|D_i| = 6$.

(c3) We have $k \in D$. Suppose that $c, d \in D$. By Claim 2, $f, e, m \notin D$. We see that $|D_i| = 7$. Consider the case $|D \cap \{c, d\}| = 1$. It is checked directly that $|D_i| = 7$. If $c, d \notin D$, then $|D_i| = 6$. Since D is a maximum minimal dominating set, we conclude that $|D_i| = 7$.

(c4) We have $f, g, p \notin D$. Suppose that $c \notin D$. It is checked directly that $|D_i| = 6$. Consider the case $c \in D$. If $d \notin D$, then $|D_i| = 6$. If $d \in D$, then $e, m \notin D$ by Claim 2. Again, $|D_i| = 6$.

(d1) Assume that $c, d \notin D$. If $k \notin D$, then $p \in D$ and $|D_i| = 5$. If $k \in D$, then it is not difficult to see that $|D_i - X| = 4$ and hence $|D_i| = 6$. Consider the case $c \notin D$ and $d \in D$. If $k \in D$, then $p = PN(k)$ and hence $g, p, y \notin D$. We have $|D_i| = 6$. If $k \notin D$, then one can easily check that again $|D_i| = 6$. Consider the case $c \in D$ and $d \notin D$. If $k \notin D$, then $p \in D$ and $|D_i| = 6$. If $k \in D$, then $p \notin D$, for otherwise $PN(k) = \emptyset$. It is easy to see that $|D_i| = 6$. Finally, suppose that $c, d \in D$. By Claim 2, $e, f, m \notin D$. If $k \in D$, then $p = PN(k)$. Therefore, $y, p, g \notin D$, $n \in D$ and $|D_i| = 6$. If $k \notin D$, then exactly two vertices from $\{n, y, p, g\}$ belong to D and $|D_i| = 6$. Since D is a maximum minimal dominating set, we conclude that $|D_i| = 6$.

(d2) The proof is analogous to the case (c3).

(d3) We have $g \in D$. The only difference between this case and the case (b4) is that the vertex k is dominated by k' in the latter case. Hence, if $d \in D$ or $k \in D$, then we use the corresponding reasoning of the case (b4) and obtain $|D_i| = 7$. Suppose now that $d, k \notin D$. We have $p \in D$, for otherwise k is not dominated. Obviously that $c, e, f \in D$ and $|D_i| = 7$.

(d4) We have $k, g \in D$. Suppose that $c, d \notin D$. It is not difficult to see that $D_i - X = \{k, e, f, g, p\}$. Hence $|D_i| = 7$. Consider the case $|D \cap \{c, d\}| = 1$. It is checked directly that $|D_i| = 8$. Finally, assume that $c, d \in D$. By Claim 2, $f, e, m \notin D$ and hence $|D_i| = 7$. Since D is a maximum minimal dominating set, we conclude that $|D_i| = 8$. ∎

Claim 4 *If F_i $(2 \leq i \leq 2k+7)$ has type D4, then both (i) and (ii) hold; if F_i has type D4 and $i = 2k+8$, then (i) holds. Furthermore, if F_i $(2 \leq i \leq 2k+7)$ is of type B4, C3, D2* *or D3, then at least one of the properties (i) and (ii) holds.*

(i) F_{i-1} has type A1, A2, C1 or C4 and $|D_{i-1}| \le 6$.

(ii) F_{i+1} has type A1, A3, B1 or B3 and $|D_{i+1}| = 5$.

Proof: This follows immediately from the definition and Claim 3. ∎

Let F_i be a component of type D4 for some $i \le 2k+7$. By Claim 3, $|D_i| = 8$. By Claim 4, F_{i+1} has type A1, A3, B1 or B3 and $|D_{i+1}| = 5$. We denote by m the number of such pairs. These components contain exactly $13m$ vertices of D, and any other component F_j with $j \le 2k+7$ has $|D_j| \le 7$. Suppose that there exist three sequential components F_i, F_{i+1}, F_{i+2} such that $|D_i| = |D_{i+1}| = |D_{i+2}| = 7$, i.e., they are of type B4, C3, D2 or D3 by Claim 3. Applying Claim 4 to F_{i+1} we arrive at a contradiction. Consider two components F_i, F_{i+1} of type B4, C3, D2 or D3 such that $i \le 2k+6$. We have $|D_i| = |D_{i+1}| = 7$. Applying Claim 4 to F_{i+1}, we obtain $|D_{i+2}| = 5$ for the component F_{i+2}. Denote by n the number of such triples. We see that these triples contain $19n$ vertices of D.

Suppose that the component F_{2k+8} belongs to one of the above pairs or triples, and consider a maximal sequence

$$F_{i+1}, F_{i+2}, ..., F_{i+r}$$

not containing the components from the above pairs and triples. It is obvious that either $|D_{i+r+1}| = 8$ or $|D_{i+r+1}| = |D_{i+r+2}| = 7$. In the first case we know that F_{i+r+1} is of type D4 and $|D_{i+r}| \le 6$ by Claim 4. For the latter case we know that F_{i+r+1} must have type B4, C3, D2 or D3. Hence, by Claim 4, $|D_{i+r}| \le 6$. Thus,

$$\sum_{j=1}^{r} |D_{i+j}| \le 6.5r.$$

Taking into account all such maximal sequences, we obtain

$$|D \cap V(F)| \le 13m + 19n + 6.5(2k+8-2m-3n) = 13k + 52 - 0.5n \le 13k + 52.$$

Assume now that the component F_{2k+8} does not belong to any of the above pairs or triples, and denote by L a maximal sequence

$$F_{l+1}, F_{l+2}, ..., F_{2k+8}$$

not containing the components from those pairs and triples. If $|D_{2k+8}| = 8$, then $|D_{2k+7}| = 6$ by Claim 4. We have

$$\sum_{j=1}^{2k+8-l} |D_{l+j}| \le 6.5(2k+8-l) + 1.5 = 6.5|L| + 1.5.$$

If $|D_{2k+8}| = 7$, then it is not difficult to see that

$$\sum_{j=1}^{2k+8-l} |D_{l+j}| \le 6.5(2k+8-l) + 1 = 6.5|L| + 1.$$

We already proved that if $F_{i+1}, F_{i+2}, ..., F_{i+r}$ $(i+r < 2k+8)$ is a maximal sequence not containing the components of the pairs and triples, then

$$\sum_{j=1}^{r} |D_{i+j}| \le 6.5r.$$

Taking into account all such maximal sequences and L, we obtain

$$|D \cap V(F)| \leq 13m + 19n + 6.5(2k + 8 - 2m - 3n - |L|) + 6.5|L| + 1.5 =$$

$$13k + 53.5 - 0.5n \leq 13k + 53.5.$$

Thus,

$$|D \cap V(F)| \leq 13k + 53,$$

as required. The proof of Proposition 1 is complete. ∎

Lemma 5 $IR(W_k) - \Gamma(W_k) \geq k + 1.$

Proof: Since D is a Γ-set, it follows that D is maximal irredundant. Adding to $D - V(F)$ some new vertices, we will construct a set D' which is maximal irredundant and

$$|D' \cap V(F)| \geq 14k + 54.$$

We first put $D' = D - V(F)$. Taking into account the definition of the 16 types of the component F_1, we consider 4 cases. Suppose that $k' \in D$ and $k \in PN(k', D)$. In this case, we put $a, b, x, m, n, y \in D'$. We do the same if $k' \in D$ and $k \notin PN(k', D)$. Assume that $k' \notin D$ and $k' \in N(D - V(F_1))$, say k' is adjacent to k''. Now, we put $a, b, x, m, n, y \in D'$ if $k' = PN(k'', D)$, and we put $h, i, j, k, m, n, p \in D'$ otherwise. Finally, suppose that $k' \notin D$ and $k' \in PN(k, D)$. We put $h, i, j, k, m, n, p \in D'$.

Let us consider the component F_{2k+8}. Suppose that $g' \in D$ and $g \in PN(g', D)$. We put $a, b, x, m, n, y \in D'$. Assume that $g' \in D$ but $g \notin PN(g', D)$. We put $a, b, x, m, n, y \in D'$. Consider now the case $g' \notin D$ and $g' \in N(D - V(F))$, say g' is adjacent to g''. We put $a, b, x, m, n, y \in D'$ if $g' = PN(g'', D)$, and we put $a, b, c, d, e, f, g \in D'$ otherwise. Finally, suppose that $g' \notin D$ and $g' \in PN(g, D)$. We put $a, b, c, d, e, f, g \in D'$.

For $2 \leq i \leq 2k + 7$, we put $a, b, c, d, e, f, g \in D'$ if i is even, and $h, i, j, k, m, n, p \in D'$ if i is odd. It is easy to see that the resulting set D' is a maximal irredundant set and $|D' \cap V(F)| \geq 14k + 54$. Applying Proposition 1, we obtain

$$IR(W_k) - \Gamma(W_k) \geq |D'| - |D| = |D' \cap V(F)| - |D \cap V(F)| \geq 14k + 54 - 13k - 53 = k + 1.$$

∎

Using Lemmas 1–5 we can easily choose the integer k such that the conditions of Theorem 1 are satisfied. The proof of Theorem 1 is complete.

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