Laplace, Turing and the "imitation game" impossible geometry: randomness, determinism and programs in Turing's test[1].

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Abstract: From the physico-mathematical view point, the imitation game between man and machine, proposed by Turing in 1950, is a game between a discrete and a continuous system. Turing stresses several times the laplacian nature of his discrete-state machine, yet he tries to show the undetectability of a functional imitation, by his machine, of a system (the brain) that, in his words, is not a discrete-state machine, as it is sensitive to limit conditions. We shortly compare this tentative imitation with Turing’s mathematical modeling of morphogenesis (his 1952 paper, focusing on continuous systems which are sensitive to initial conditions). On the grounds of recent knowledge about dynamical systems, we show the detectability of a Turing Machine from many dynamical processes. Turing’s hinted distinction between imitation and modeling is developed, jointly to a discussion on the repeatability of computational processes. The main references are of a physico-mathematical nature, but the analysis is purely conceptual.

Keywords: Turing Machine, classical determinism, dynamical systems, computational and dynamical hypotheses, fonctional analyses of cognition, obsessive iteration, Laplace today.

Introduction

In a famous 1950 article, Alan Turing proposes, in order to operate a functional comparison between brain and machine, a game he calls "imitation game". This text is, in many respects, as fundamental as his other writings, but in a completely different field since this time it consists of an article in philosophy and human cognition. These philosophical musings divide Turing's intellectual trajectory into two parts: the first moment of it being devoted to the modelling of the action executed by calculating thought, the "Human Computer" by means of the machine that tradition has endowed with Turing's own name (2, footnote); the second moment is devoted to the analysis, from 1950 on, of the morphogenetical potentialities of phenomena of chemical diffusion [Turing, 1952]. From as early as his first article of 1936, Turing had thus described his computing/deducting machine, a discrete-state machine, as he himself rightfully reminds: a record/playback head moves right or left, writes 1 or 0 on the tape, erases them. The fundamental idea: the machine consists of a software (the instructions) and a hardware (the material: the read/write head and the tape). This distinction, purely conceptual at the time, is the true beginning of modern Computer Science (you may recognize your Macintosh). This abstract machine can compute anything, there lies the extraordinary result of the years '36-37.

In fact, Turing himself, Kleene, and a few other pioneers demonstrate that all formalisms for

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computability, since the works of Herbrand and Gödel ('30-'31), are equivalent to Turing's machine: using lambda-calculus (Church '32; another fundamental formalism for computability, see [Barendregt, 1984] and §.3 below), they translate the various processes of arithmetic calculus the ones into the others. Consequently, all systems calculate the same class of functions on integers. That "we have an absolute" was clamored at the time (see the comment Gödel makes in 1963 on the re-edition of his 1931 article, reappearing in [Gödel & al., 1989]): this absolute is the class of calculable (partial) functions, of integers into integers, as locus of all which is effective, calculable, in fact thinkable ("... the laws of arithmetic govern all which is enumerable. This one is the vastest of all disciplines, since it contains not only the actual and the intuitive, but all which is thinkable." [Frege, 1884]). The lambda-calculus, its types, their semantic categories are extremely rich syntactical and mathematical structures (see [Hindley, Seldin, 1986], [Girard & al., 1990], [Krivine, 1990], [Asperti, Longo, 1991], [Amadio, Curien, 1998]): they are still at the heart of contemporary logic and theoretical Computer Science, although there are other problems today. These formalisms have indeed been the result of a remarkable conceptual and mathematical journey, the notion of logico-formal system and language, a pillar of the mathematics of the XXth century. In fact, a project of foundations of mathematics and of human knowledge.

Among the pioneers of this "formalist-linguistic turn" one must include the mathematicians Peano and Padoa: for them, mathematical certainty, in fact the certainty of thought and therefore thought itself, would situate itself among the "potentially mechanisable". So the first thing needing to be done was to reduce mathematics to a formal calculus, a numerical calculus that a machine should be capable of completely reproducing (hence the preliminary step: to encode mathematics in Peano's arithmetics). But which is this machine? One may also find a first intuition of it with Hilbert: he refers to "finite sequences of signs, constructed according to a finite number of rules", or to "laws of formal deduction" also written under the form of finite series of signs and, therefore, under the form of integers (and Hilbert knows what he's talking about, since he encodes, in his 1899 book, all the geometries, Euclidean and non-Euclidean, within Arithmetic by analytic means). Between 1930-1936, at last the intuition of these great pioneers will be formalized and, modulo a remarkable idea, gödelization (3), extended to an arithmetical encoding of all which is finite, Turing's machine replaces Vaucanson's and Diderot's automatons: potentially, it is able to simulate any human function, thought in particular (or primarily), [Gandy, 1988].

1. The game, the machine and the continuum

In 1950, Turing had the courage to submit Peano and Padoa's program to a sort of scientific-mental experiment: to demonstrate that a discrete-state machine, a DSM (his universal machine), is undistinguishable from a human brain, or, at least, that it is able to play and win what he calls the "imitation game", by playing against a man (or, rather, a woman?). In this text, we shall not discuss the specific question raised by this game between a man, a woman and a machine, but its general and dominant interpretation: as alleged proof of a "functional equivalence" between digital machine and human brain. And we shall address the issue within a purely physico-mathematical conceptual framework.

Turing's proof is cautious: it is based on mathematical hypotheses carefully made explicit, as shall be seen. Also to be noted is a capital difference from the modern claimants of "all is program", this "all" being replaced depending of the author by evolution, the genome, the brain, etc. (in fact, in this slogan, no hypothesis is formulated, its consists solely of a description of "reality", of the Universe, itself identified to a Discrete-State Machine). Turing is to the contrary aware of the strong hypotheses that are necessary to his reasoning. The conclusion, the success of the machine in the imitation game, is also very cautious. However, the central hypothesis as well as the conclusion is not corroborated. And, today, it can be proved for this great mathematician had well exhibited hypotheses and conclusions. There lies the interest of the article: explicit premises and rich arguments. We shall therefore play Turing's game from a mathematical viewpoint, with its hypotheses, without engaging into any discussion in Philosophy of Mind: it is not necessary in order to be certain of winning against any DSM.

In a DSM, Turing observes, "... it is always possible to predict all future states". And he continues "This is reminiscent of Laplace's view ... The prediction which we are considering is, however, rather nearer to practicability than that considered by Laplace" [Turing,1950; p. 47]. In fact, he explains, the Universe and its processes are "sensitive to initial conditions", should we say in modern terminology (Turing uses the following example: "The displacement of a single electron by a billionth of a centimeter at one moment might make the difference between a man being killed by an avalanche a year later, or escaping."). To the contrary, and there lies the greatest effectiveness of his approach, "It is an essential property of ...[DSMs] that this phenomenon does not occur. Even when we consider the actual physical machines instead of the idealized machines,...", prediction is possible, [Turing,1950; p. 47]. Thus Turing has no doubt: his machine is an ideal machine, indeed a logical one, as he called it, with a laplacian behaviour. And he is absolutely right: the notion of program and the mathematical structure of its implementation are deterministic in Laplace's sense, that is, the determination, by a finite number of rules (or equations, for laplacian mechanics), implies predictability. Of course, there may be some endowed indeterminacy (the machine can make steps which lead to an arbitrary element of a finite set of possible discrete states, instead of leading to a single one - we are then dealing with an indeterministic DSM), but it consists of probabilistic type of abstract indeterminacy, already well studied by Laplace, and which is not the same mathematical concept as the unpredictability of deterministic dynamical systems, in the modern sense which we shall discuss in length.

Though, as Turing understands well, "the nervous system is surely not a DSM" (ah, if only everyone would at least agree with that!). And he specifies: "a small error in the information about the size of the nervous impulse..." (p. 57). Once again, and in modern terminology, the brain rather is a dynamical system (Turing calls these systems "continuous"). Then how to compare a DSM with the brain? The comparison is functional and relative to the only possible access to the machine, during the imitation game: the finite sequences of a teleprinter's signs (your keyboard in front of your screen today, or mouse clicks, which start off a small program, a finite series of signs). Under these conditions, according to Turing, we would be unable to distinguish a continuous system, as the brain, or "… a more simple one, a differential analyzer...", from a DSM; if the continuous machine makes its response though a printer, it will be undistinguishable from a DSM's response, even if obtained by different means (continuous variations instead of discrete steps). So there is Turing's central hypothesis: if the interface with the dynamical system is given by a "discrete access grid", then it will be undistinguishable from a DSM.

In fact, today's physical DSM, our computers, simulate dynamical systems in a more than remarkable way! They develop finite approximations of the equations which model them with great efficiency: nowhere may we better see the "form" of an attractor than on the screen of a powerful enough machine. Their applications to aerodynamics (simulation of turbulence), for example, has considerably lowered the price of airplanes (almost no more need for wind tunnels). But... what are the conceptual, mathematical, physical differences?

Let's first evacuate any confusion between mathematical modelling and imitation, in Turing's sense. Take the discrete logistic equation $x_{n+1} = k\ x_n\ (1 - x_n)$, where $2 \leq k \leq 4$. Many physical systems (and even biological ones) are very well modelled by this function: typically in presence of an antagonist coupling, such as an x_n action coupled to a symmetric reaction $(1 - x_n)$. For some values of k, this obviously deterministic transformation from [0,1] to [0,1], has a chaotic behavior: a slightest variation of x_0, and the evolution will radically differ. Moreover, except for a countable subset of initial points x_0 (or a subset of "measure 0"), when $k = 4$ and n goes to infinity, the sequence $\{x_n\}$ is dense in [0,1] : its behavior is thus said to be ergodic (or quasi ergodic, to be précise, as it is so w.r.to a non-standard measure – not w.r.to Lebegues-measure). However, if you start your machine a second time on the same numerical value for x_0, you will obtain the same sequence, that is what a DSM is. Conversely, in a physical (classical) system, it makes no sense to say: "start with the exact same initial situation", for the physical measurement will always be an interval. And the dynamic is such that, as it happens, a perturbation beneath the possible measure, that is, within the interval, can shift the system towards very different evolutions.

In short, the trajectories, the portrait of the attractors (their geometrical structures), caused by variations beneath the finite grid measurement, can be very different. Now that is the complexity, from the Santa Fe' Institute to the CenECC of the ENS : it is in the possible bifurcations, in the richness of the attractors' geometrical structures, in their various forms of structural stability, up to the synchronization phenomena (in an epileptic's brain, for example) of which they might be the origin. The stakes are of geometrical nature.

So here we are with a first approximation of the winning strategy, if we endow "imitation", the word used by Turing, with a strong meaning, usually restricted to the notion of simulation: computational model or, more precisely, computational realization of the physico-mathematical modelling. In this case, a true physical dynamical system always wins the imitation game against a DSM, because it needs only to say:

"let's start over with the same initial conditions and then let's
compare the evolution of our phase portraits".

Measurement by interval and sensitivity to the initial conditions will mark the difference between the DSM and the physical system. If the system is a turbulent river, for example, it will win at its first turn and in few instants. A forced or double pendulum needs only a little more time. Start off, for example, your double pendulum (4) and the computer on, say, the values 3 and 7, twice in a row: the latter will use these exact values for the numerical simulation, each time. It will then obtain the same rounded values and, except in quite exceptional cases that shall be discussed, it will describe the same trajectory. However, it is not a matter of starting off the physical pendulum on 3 and 7, exactly: it can only be launched upon an interval, however small it may be, around those values. After a sufficiently long moment, the physical system shall follow a second different trajectory, very different indeed, from the first with regards to its phase space (the structure engendered by all the positions and speeds compatible with the system's data). Thus "more geometrico", a continuous system shows the unpredictability of its evolution in comparison to a DSM, even for an observer of the "linguistic turn", who swears but by a teleprinter, because no discrete reading grid, however fine it may be, allows to stabilize a system with an unstable dynamic.

For now, we have only applied Turing's statement concerning the sensitivity of dynamical systems to initial conditions, which is at the origin of the unpredictability, and his observation that "one of the essential properties of the ... DSM is that this phenomenon does not occur". Obviously, this game strategy is only a first mathematical response to what has been called, quite beyond Turing's thinking, "Turing's test", and to the myth of the machine as brain's model; it consists of a response within the framework Turing's mathematical hypotheses, which defines in several instances the brain as being "a continuous system" and his DSM as a "laplacian machine".

Before refining the game strategy and thoroughly discussing functional imitation, let’s briefly sum up the terms of this first confrontation between the machine and a physical system. We have thus supposed, as first approximation, that the machine attempts to simulate at best a dynamical system, by using a mathematical model designed on the basis of its deterministic nature (thus described by a finite number of equations, or formal rules of deduction for a logicist who wants to model thought (5)). At the first turn, it may be impossible to distinguish between the evolution of the DSM and that of the physical system, of which a teleprinter or a screen's pixels inform us of the numerical measurements: of course, the two evolutions are in general different, but neither is more realistic than the other (in physics, at least). However, the iteration of the simulation-modelling from the same initial conditions reveals the machine: if a DSM restarts upon the same numerical values, necessarily discrete, it will describe the exact same evolution in the phase space; however, the dynamical instability of a physical system, necessarely restarted within an approximating interval, will cause the second trajectory to differ from the first, after a sufficiently long time, and, moreover (see §.2 for more details), even the discrete reading of the physical measurements will display this difference. To conclude, we have shown that a DSM is surely not a model of the brain, which is, for Turing, a continuous system, as opposed to what is pleaded in the field of classical Artificial Intelligence and by many modern cognitivists. But can it imitate it? And what does this word mean, exactly, when referring to modelling? Turing's game allows to clarify these important concepts.

So let’s continue with our game. In order to thwart this first sketch of the iteration strategy that has just been proposed, the machine (the programmer) could in fact use the trick suggested by a comment by Turing on p. 58 ; he proposes to trick a continuous system's and a DSM's observer-comparator by having the latter produce a series of random numbers. This idea is at the center of a difference that demonstrates the philosophical and mathematical depth of the imitation game. In the concerned comment, Turing displays this radical difference which is of interest to us, and of which he is aware (see §.3 below), between his "imitation game" and the mathematical modelling of physical phenomena. Of course, by applying our strategy of iteration against ergodic simulation, we would find ourselves with 4 trajectories all differing from one another and, in some cases, being all as realistic as one another. But we had to renounce simulation as such, as modelling of the deterministic system by a system of equations or of formal rules of inference implemented on a computer, and we have gone towards a weaker notion, that of equivalence as indistinguishably modulo a finite interface, without engaging ourselves upon the identity of the laws of behavior (the machine's program is not supposed to implement the same laws which "determine" the physical system). In fact, that is what the imitation game is. And it brings us directly to the high stakes of the "simulation" of a deterministic system by ergodic method: a simulation which is in fact an imitation, to put it - like

Turing - in a quite appropriate but uncommon manner.

The precisions we shall add in the next section require somewhat more competence or mathematical attention: the humanist reader who has grasped this first difference between a DSM and a dynamical system may directly jump to §.3 (6).

2. Between randomness and deterministic chaos

Two questions are raised at this point. The first is quite general: from a computational viewpoint, may randomness be distinguished, in practice, from chaotic determinism? And if, during our game, in order to trick the observer of the strategy of iteration, we first accepted to simulate the dynamical system (to develop the computation of an equational model), but, at the second turn, the computer added small random perturbations to the initial data or to each step of the discrete evolution?

So we have two phases. During the first (single-turn game), we observe a physical system, of which we know the discrete measurements via a teleprinter (or by screen pixels), and a computer which generates a random trajectory. Now, there exists deterministic systems, maximally unstable, such that no known method allows us to distinguish between their evolutions, reproduced upon a screen, and the generation of a random sequence: these are the "Bernoulli systems" (7). For these systems, knowledge of the past does not allow to determine the future evolution; we then say that the flow is random. Draws at lottery or dice are typical examples of this: these systems are deterministic, yet perfectly chaotic. In the two cases, the number of parameters and of equations may be quite great, yet finite, and sensitivity to the initial conditions is such that it is absolutely not worth it to attempt to write these equations: it is preferable to analyze the phenomenon in terms of laws of probability ("limit laws", for "large numbers"). On the other hand, there exists very simple Bernoulli systems, described by one or two equations. It is thanks to these systems that we program a computer to generate random series: techniques based upon simple trigonometric properties and the multiplication of angles around 0, for example, will produce random series of + and - signs. Also the logistic equation of §.1, for k = 4, generates, and in a quite economic and deterministic fashion, series of which the "global geometry" is random (8).

INTERMEZZO 1 (determinism and knowledge)

The question to which Turing brings us becomes in fact quite delicate and interesting: we do not know of "proper random" systems, in classical physics. More precisely, in the discrete realm, we have an excellent concept, or even a mathematical definition, of random sequence (Kolmogorov, Matin-Löf, Chaitin: "the shortest program that generates it is the sequence itself" or ... "wait and see"), but all examples of natural or artificial sequences, that we know of, come from a physical deterministic system (chaotic) or from a deterministic computer program, in fact, laplacian. These programs, written in two lines, produce long "random" series: as generated by a DSM, Turing would soundly consider those sequences as being predictable (as a matter of fact, these sequences, called pseudo-random, are periodic, since they are generated by functions f as $x_{n+1} = f(x_n)$: on a concrete DSM, the finite decimal representation forces them to go back, soon or late, to the same number value, thus to the same sub-sequence. And, periodicity is the opposite of randomness, yet … the period may be *very very* long).

In a note, we have already observed that determinism is essential to the construction of scientific objectivity in classical physics (it is "objective"); we can now add that the classical randomness is epistemic (it is a matter of "perspective" and of knowledge, it is not inherent to theoretical construction; even a gas obeys deterministic laws of local interaction between particles). Shortly, the classical randomness which we know, is nothing but highly unstable determinism *or* of unstable appearance (the computer which calculates the logistic ergodic sequence, for a fixed x_0, remains, simply and permanently, upon a trajectory which is critical, but dense in the phase space - there is the purely epistemic chaos) *or* with a very great yet finite number of parameters (dice, a gas), these "or" not being exclusive. Once again, the sequences generated by the logistic function or by a game of dice, Bernoulli's fluxes, are deterministic and ergodic. However, there is a great difference between the number of laws and of degree of freedom which will determine them and, moreover, in the logistic equation, once x_n determined, we can compute and determine x_{n+1}, as opposed to dice where a draw in no manner determines the next (see preceding note). In this sense, their common ergocity is epistemic, for, on one hand, the observer writes the equations (the logistic equation) or knows the pertinent laws of evolution (dice) and, on the other hand, he observes a total lack of regularity in the two evolutions. It is the visible total irregularity, the geometry of the attractors if they exist, which are similar: the logistic series, just like the series of draws at dice, jumps from one end to the other of possible values, with no visible pattern. Through differing modalities, the objective determinism (or in principle) generates epistemic chaos and the phenomenal unpredictability associated to it.

But God, the perfect and infinite being who masters all laws of the Universe and who measures exactly, without approximation, without intervals, knows perfectly well the evolution of dice games and of the lottery - and of the Universe, as rightfully stated by Laplace, in a very famous and often misinterpreted page. By those words, Laplace merely lays the right absolute definition of deterministic system, outside of any construction of knowledge and of scientific objectivity, based upon strong and well-explicated hypotheses on God, and he is right. In classical physics, we write the same equations as God, as soon as we are capable of it, so had Gallileo already claimed. But we, men (and women), we have a few problems concerning physical measurement and a different on-look than His regarding the geometry of trajectories determined by these equations: and all this becomes very important for dynamical systems, as Poincaré proved, because they may be sensitive to initial conditions and, thus, to perturbations/ fluctuations below the possible measure interval. Laplace's erroneous conjecture lies elsewhere and consists within the central hypothesis at the origin of the "calculus of perturbations" to which it has greatly contributed: from *small* perturbations will follow *small* consequences. The determinism would therefore imply the predictability, modulo the inevitable approximation of the physical measurement, of which he is well aware. The invalidation of Laplace's conjecture by Poincaré will then make us understand classical randomness as particular case of deterministic chaos. And all this is very important to grasp Turing's attempt to imitate, and not to model, a continuous system by a laplacian DSM.

Now, if we want non-deterministic randomness, we can but recourse to quantum physics, thus beyond of our rather classical game: the indeterminism then, at least for the Heisenberg-type interpretation, is not epistemic, but becomes "essential" to the construction of scientific objectivity: the probabilities are "intrinsic" to the theory and... a needle, positioned with care upon its tip, falls, classically, upon a value or another of the green mat upon which it was, after an essentially random quantum fluctuation (God, himself, really knows to play dice, but only beneath Plank's h).

So there are the stakes which are the object of such debate: classical determinism does not know, in fact, proper randomness, but only the more or less chaotic evolutions, according to various modes of determination. On the other hand, for an important trend in physical thought, quantum indeterminism is inherent to the theory. Sometimes, the latter manifests itself to our classical observation, on the tip of a needle.

Let’s go back to the first phase of our game (single turn game): without God's help, we would be unable to distinguish a Bernoulli physical system from an ergodic imitation by the machine. However, there exists a continuum of classical dynamical systems which range from stable systems to Bernoulli’s fluxes: in intermediary situations, the future may be predicted for the more or less long term and, particularly, the past has a greater or lesser global influence upon future trajectories. Now there are measurements, of which some are based upon the notion of entropy (topological, see [Adler, 1979]), which allow to decide a deterministic system's degree of instability: on one hand, systems with nil entropy are predictable: on the other, in very high entropy systems, no observables are predictable. Between the two, numerous physical systems may be finely analyzed and, in certain cases, but there exists no general method, a partition of phase space (a topological covering by small cells), allows to conjecture the dynamic. That is, the experimental observation of a discrete trajectory allows the proposition of a deterministic law for the evolution; in these cases, different trajectories allow to guess different dynamics (in technical terms, the partitions have "generating series"). It therefore suffices to propose one of these moderately unstable systems for a good mathematician observer to be able to recognize the random imitation made by the computer. We shall further discuss this, below, to make sure that, in this case, the strategy is in fact a winning one.

Second phase. In order to thwart this latest strategy as well as that of iteration (the two-turn game of §.1) the computer implements an equational model of the physical system. However, at the second turn, in order to not fall into the trap of the genesis of an evolution identical to the first, it randomly introduces small perturbations, which may have huge consequences, of course. This second turn thus bases itself on the computation of a new deterministic system, that which adds the first to a random sequence's mechanical generator. The situation becomes delicate. If the system would admit generating series and if we were to fall upon, at the second turn, on two series which allow to guess out two differing dynamics, the distinction between the dynamical system and the DSM would be made: the series engendered by the computer would no longer be derived from the equations that modelled the physical system, but a variant due to the addition of a perturbation generator. And the mathematician who knows how to reconstruct equations from generating series, once again recognizes the formal machine. But, however ... even if we were to choose a system with the right level of entropy to play this game, it is not certain that we would fall upon generating series nor that we could use the rare applicable techniques to reconstruct the dynamics from these series: the machine, then, by this astute mix of modelling and ergodic imitation, would risk winning. We would then need to play the tough game of turbulence.

As of 1941, Kolmogorov and his school in fact proposed a stochastic approach to turbulence (see, with regards to this and more on turbulence, M. Farge's article in [Daham and al., 1992]). Kolmogorov's idea was that certain random systems could adequately model turbulent phenomena. This approach, still greatly studied today, bases itself upon a quite strong hypothesis, the ergodic hypothesis. It supposes, among others, the homogeneity, the isotropy and the self-similarity of the system's evolution. Lacking of something better, the ergodic methods represent an important tool for the analysis, but it is increasingly obvious that, in certain cases, the hypotheses upon which they base themselves are not corroborated and that, to the contrary, what is important, with turbulence, is exactly the complex mixture between relatively stable structures and strong instabilities (non homogeneity, non isotropy...). Generally speaking, one does not propose meteorological previsions using ergodic methods; likewise, these methods are strongly unrecommended for the modelling of turbulence generated by a plane's wing; it would be like to trust the lottery as for the conception and the security of flight structures. In mathematical physics and in Computer Science, normally and as early as possible, one would model, meaning that one would propose and program deterministic laws which reproduce at best the natural phenomenon in question. The turingian distinction between imitation and modelling then becomes crucial: stochastic imitation à la Kolmogorv vs. modelling, by the Navier-Stokes equations, in our case (see [Cannone, 2003] for these classical equations, today).

Now the ergodic hypothesis is invalidated by the presence of movement invariants, a sort of coherent structure, whirlpools for example, where rotation wins over deformation and who remain stable quite beyond what any statistical theory could predict. R. Thom in his work often considers these structures where, despite a highly unstable dynamic, there is a certain bearing of geometrical forms (structural stability); but that does not prevent - as Prigogine would state it - this game between locally stable structures and global system, of which the equations determine the range of possible regimes, from being based upon small fluctuations which, amplified, induce the choice of one of these regimes (9).

So, on one hand, thanks to the very specific geometry of the zones of stability and of fluctuations, we know today that pure ergodicity cannot trick the expert observer (according to M.Farge, Kolmogorov had understood already in 1949 the theoretical shortcomings of the ergodic hypothesis). On the other hand, we already observed that pure modelling is defeated, in the imitation game between a machine and a physical dynamical system (including a turbulent one), by iteration (§.1). Finally, if the programmer mixes both strategies (modelling + ergodicity) in order to play a second turn against a well-chosen turbulent system, the coherent structures, the movement invariants, can be broken in an unnatural way and allow to distinguish the machine: there lies our thesis, based upon an anterior experience of digital techniques, by finite elements methods, for the solution of differential equations. In fact, if we fix equations for turbulence (Navier-Stokes, typically, but others are beginning to be proposed) and we implement them in a machine, the addition of random perturbations during the computation will not allow to choose a priori (to program) the consequences of the perturbation. Meaning that the perturbation of a step of the digital computation might, in certain instants, not limit itself to the modification of incoherent residual flows (vorticity filaments, for example), nor to redirect the regime towards other possible ones, but may break structures which have all the macroscopic characteristics of coherence and of a long stability. In short, a pebble that is thrown in a whirlpool is visible, as foreign to the turbulence: it breaks it beyond the physically (geometrically) plausible, from an internal view point. And the physical world wins again against virtual reality.

By this, we hope to have answered to Turing's remark which proposes to imitate a continuous system, by a random system. In fact, we have taken it in a strong sense, of which he does not talk of explicitly: the possibility of a mix of strategies, modelling and ergodic imitation. Of course, we have not responded to the other great question that bothers Turing : which is the difference between a man and a woman? How to distinguish them if the man tries to imitate the woman? And if we replaced the man by a computer? Can we grasp the difference by the intermediary of a teleprinter, without seeing, without touching? (What a limitation of our material, visual and caressing humanity, but that's what the linguistic turn is (10).)

3. Logical, physical and biological machines

In our opinion, Turing is perfectly aware of the difference between imitation and mathematical modelling for a quite simple reason: he is already working upon a remarkable mathematical model of morphogenesis in a field of chemical diffusion (a fundamental article, one of the departing points, with the work of D'Arcy Thompson, of the modern analyses of morphogenesis). In fact, the most interesting property the equations to be found in [Turing, 1952], is that a very small variation of the boundary conditions, obviously in a continuous system, can radically change the evolution of the model. And this property is not the laplacian nondeterminism or randomness, but the sensitivity to the contour conditions and situates itself at the heart of the deterministic model of morphogenesis à la Turing. One thing is thus the "imitation game", another the mathematical modelling of physical and physico-chemical or biological phenomena: the turingian DSM does not claim to model the brain, in the physico-mathematical sense - the latter is a continuous system for Turing -, it can only attempt to trick an observer (for this reason, maybe and quite rightly so, some mark the beginning of classical Artificial Intelligence with this article by Turing). In the §.2 we have seen that even the imitation can be revealed: in general, imitation of a dynamical system cannot be accomplished in an indistinguishable, read satisfactory manner by ergodic means, in particular if it is somewhat turbulent, but not too much.

Second important precision to analyze in Turing's hypotheses. At page 47, he continues: "Even when we consider the actual physical machines instead of the idealized machines… " they are laplacian machines, as any DSM. True and false: true, the real (sequential) computer, as a DSM's realization, is by principle condemned to always make the same computation, from the same pool of discrete data and of programs, that is its logico-formal architecture (its logical gates and its programs, as formal languages). False, because it is also a physical machine, subject to variations below of its digital unpredictable approximations, due to the possible small defects of its electronic circuits, to the cosmic rays that would befall upon it... It’s extremely rare, but it happens. Evidently, these are sensitivities to limit conditions which have nothing to do with those, intrinsic, of continuous systems which happen to be simulated (and enormously more rare, therefore easy to detect by statistic means, by iterating the process a few times).

As a matter of fact, an abstract, mathematical DSM, such as Turing's machine, is not conceived as a physical machine, but as a logical machine, a human in "the minimal act of thought" - of formal thought (11). Consequently, its expressivity is mechanical yet purely logico-formal: typically, its expressive power is independent of spatial dimensions - of the tape, of the read/write head -, a property absolutely foreign to the physical processes, which all depend and strongly upon the dimensions of space. However, when we physically bring a DSM into being, it poses new physical problems - from cosmic radiation to the synchronicity, sometimes even relativistic, of modern systems, distributed in space. Let’s forget the comparison between formal DSMs and living machines, which are physical, obviously, but are moreover subject to phenomena of integration-regulation which keep them in an "extended critical state (12)"; this state is unknown by the non-living and its mathematics; mathematics which must therefore be extended and adapted to the new job (dynamical systems are "only" one of the best approximations we have, for the moment). It is exactly this integration of the brain within a body, their reciprocal regulation and by such a rich environment that confers it a quite peculiar structural and functional stability ; and when these regulative/integrative linkages by/of/in a body are weakened - in the course of a dream for example - the brain appears to be rather unstable (likewise in case of serious deprivation - artificial, for example - from sensation). A stability in the change (homeorhesis), anchored upon self-organization and being a feature of the living which appears extraordinarily apt to constitute invariants, from the invariants and stabilities of action to the cognitive, indeed conceptual invariants (at the heart of thought). In short, despite that we too never repeat the "same thing", in the sense of a DSM, we stabilize instabilities and critical states in a way still very ill understood, from the mathematical viewpoint. Some will then exchange the brain for a DSM: to the contrary, it is a dynamical system enormously more complex than any n-body physical system or turbulent stream (... think that the banks "regulate" a stream and, there the Navier-Stokes equations tell us very little of the turbulence close to the edges; and this is nothing compared to the complexity of a brain's friction with its environment, by way of its interactions with the different levels of organization of the body to which it belongs) (13).

INTERMEZZO II (machines and deductions)

Inter II.1 : The equivalence theorems of Turing-Kleene & al. of '36-37 (see introduction) should be considered as the second great negative result for logical formalisms, after Gödel's incompleteness theorem, 1931. That any formal deductive system, endowed with a notion of decidable proof (so any hilbertian system), can be completely simulated by a machine that goes "right, left, write/erase 0, 1", is a true catastrophy: what a conceptual misery these systems! (The difficulty is concealed within the monstrosity of the encoding). This philosophical shortcoming was already clear to Poincaré: "MM.Hilbert and Peano think that mathematics is like Chicago's sausage machine: porcs and axioms go in, theorems and sausages come out" (and there comes mathematics reduced to the "manipulations of concrete signs" of which some philosophers still talk today, logic conceived as "purely formal" and mathematics – an enormous logico-analytical tautology - ready to be entirely computer generated). In fact, DSMs are generalized sausage machines (and are absolutely tremendous, for their specific uses - but sausage machines too are quite useful!). Let's not forget, however, to appreciate the full half of the glass: what an idea that of Turing who, by inventing the notion of programmable machine, manages to compute all the partial recursive functions (an enormous class of functions on {0, 1}*, the integers) by a man/machine which goes "right, left, write/erase 0, 1". Quite obviously, this idea, with its notion of program, is the true beginning of Computer Science.

Inter II.2 : The typed lambda-calculus (Church '40) is the only system which allows to see with equilibrium the half-full glass: the formal deductions, with all their limits and their expressivity, directly become computations, without coding (this property is called "Curry-Howard isomorphism", see [Howard, 1980]). The "human computer" of Peano, Hilbert and Turing, this alienation of human rationality in a laplacian mechanism, instead of going "left, right, 0, 1", applies a little bit more complex basic formal rules - "implication-introduction", "implication-elimination" and a few others, by replacement of a sequence of signs by another and by sequence-matching (identification by mechanical superposition of signs without meaning). With recursion, the system is also a good digital computer. No miracle, only a very elegant constructive representation of formal proofs as programs, which placed this system at the center of the mathematics - Logic and Category Theory – for sequential calculi and languages (see [Girard & al., 1990], [Asperi, Longo, 1991]). Quite recently, it has been proposed to cognitivists to stop searching, in the brain, for a Turing Machine, but for a typed Lambda-machine (at last!): this DSM, at least, applies sequence-matching directly to rules for deduction. The lambda-calculus, "at last", because if, quite beyond of the Turing imitation game's objectives, one would obstinate oneself to seek the implementation of universal-formal rules of thought (the Laws of Thought) in the brain, one must know at least that the encoding of these laws is very important, just as under Unix or Mac-OS. In fact, the choice of the programming style (functional, logical, imperative, object oriented ..., for example) and the conception of a language with its own method for its specific coding-representation of the world and its actual expressivity, are at the heart of Computer Science, as a science, quite difficult and important, of DSMs. The computational equivalence proclaimed by the "Church thesis", is of no interest for Computer Science, since long (see the introduction at [Aceto & al., 2003]): a good share of the work happens to consist of the explicitation and use of the expressiveness of the language proposed or analyzed. Now, the terms-programs of the lambda-calculus, contrarily to the Turing Machines and to the other formalisms, encode a great part of "the architecture" of deduction in formal systems: and, in general, "a proof has an architecture", Poincaré had already exclaimed against Hilbert and his rather flat arithmetic encodings.

It should be clear, that the limits of lambda-calculus are those of any computational formalism: it proceeds by mechanical replacement of meaningless sequences of signs and by sequence-matching. To the contrary, we, when saying "if ... then ... else...", are not performing sequence-matching: we are displacing mountains of significations. That is the mathematical incompleteness of formalisms and the great, monist, cognitive stake for knowledge, well beyond the software/hardware/meaning distinction, quite convenient for machines and post-turigian functionalistic models of the mind, outside of this world (14).

Let's return a last time to our game, in order to reflect. How is it possible that a great mathematician such as Turing would believe that a discrete access grid, fixed once and for all (the letters of a teleprinter, the pixels of a screen), could conceal the geometrical difference between a dynamical system (very complex, the brain) and a laplacian mechanical machine? In fact, until the results by Kolmogorov-Arnold-Moser and Ruelle in the '60s and '70s, the complexity (geometrical!) of continuous systems was not entirely clear, particularly the idea that the "critical" points can be dense. But the possible philosophy existed. Let's explain ourselves.

Laplace already knew well that there are critical points: the summit of a mountain of potential, for example. It is Poincaré who, thanks to his work in celestial mechanics, will understand that the problem is "global", that it is proper to dynamical systems and to their geometry and not to a few isolated points. There is the meaning of his famous remark on sensitivity to the initial conditions: these critical points are "a bit everywhere", even though he did not exactly have the theorem which demonstrates it. It is also this attention to the physico-mathematical complexity that makes him also... conjecture the incompleteness of formal set theory, pretended universal sausage machine for mathematics (independence of the Continuum Hypothesis, in a letter to Zermelo: the theorems will come 34 and 60 years later). Just as Weyl conjectures the incompleteness of arithmetic in 1918, [Weyl, 1918]. Despite logicism, the philosophy of physics and that of mathematics must be profoundly linked, in order to better understand at least, as demonstrated by Poincaré and Weyl. In short, there are those who grasp the "secret darkness of milk" and its importance to knowledge and science and those who see the world through a laplacian DSM. Turing belongs to the first group, except that he pushes as far as possible, within the limits of the mathematical knowledge of his times, his genius idea, the modern DSM and its notion of program, last great invention of logico-formal mechanics. Others to the contrary will follow, claiming that a DSM is a model of the brain, or even that the brain is a DSM itself (even stronger). Their motivations are often based upon this article by Turing or upon the formal Set Theory and/or Type Theory: the first is a bad reading and the second is a mathematical error (that follows from the *mathematical*, concrete, incompleteness of formalisms).

4. Predictability and decidability

In a very brief text ("Laplace", downloadable, author's web) we argue the conceptual equivalence of Laplace's key hypothesis for the anlysis of perturbations (the predictability of deterministic systems – as decidablity of the evolution) and of the hypothesis of completeness (decidability of deducibility) of hilbertian systems, an analogy also hinted by Girard in his introduction to Turing's article. But with "Laplace" we also observed that the deterministic unpredictability à la Poincaré (the three bodies theorem, 1891) is the analog and the precursor of gödelian imcompleteness (undecidability) for any Hilbert-like formalism. One must however add a nuance to this analogy between the two great respective limitative results: unpredictability à la Poincaré and Gödel-like incompleteness (which corresponds to the undecidability of the halting problem, demonstrated by Turing in '36 for his logical machine, see Girard's introduction). The first appears "at a finite level", and very early (cf. Liapounov's divergence of coefficients in the Lindstedt-Fourier series), the latter is a problem "at infinity" (the halting problem or the non-termination of computations ... forever). So unpredictability is a "stronger" result, within the framework of an essential philosophical equivalence of the two approaches to knowledge (Laplacian in physics and formalist in logic) and of their limitative results (Poincaré and Gödel). The unpredictability of a physical dynamical system is related, in particular, to the impossibility in principle to travel the same path in the phase space, from the same initial conditions (measured by interval), whereas a DSM obstinates itself to do so. It must be observed that also Turing speaks of the unpredictability of a DSM with a large memory and very long programs (p. 59), a daily experience for any computer scientist, but he is clear in these regards: we are dealing with a practical unpredictability and not one of principle, mathematical (see [Turing, 1950; p. 47], already quoted above).

The analysis we are sketching here differs from many writings, in Theory of Mind and Artificial Intelligence, regarding the "Turing test" (15). In fact, our comparison develops itself between predictability and decidability and it is philosophical, in the sense of the theory of knowledge, but it must be reconstructed from mathematics. By this, we could understand why "imitation", such as defined by Turing, is detectable. Its mathematical (geometrical) limit finds itself exactly in the difference between the two unpredictability results. DSMs have properties of indecidability at infinity, but are predictable in the finite realm: by looking at the program and the discrete databases one can perfectly predict the next computation step and, above all, they are predictable with regards to the iteration of the process, as described in §.1. In a turingian DSM, all the laws of evolution/behavior of its own universe are explicitly and fully given (programmed) and measurement, as access to a digital database, is perfect; exactly as for God, who perfectly knows the laws and the exact measures in his universe, ours (first Intermezzo). The myth of formal machine and of absolute divinity meet and, both, their ways, detach the analysis of knowledge from its constitutive interface, between us and reality. Their counterparts in the foundations of mathematics have quartered the century between mechanistic formalism and ontologizing platonism.

Note that Turing is so firmly convinced that his DSM is laplacian that he makes a mistake: he explicitly claims that sensitivity to initial conditions does not apply to DSMs (he stresses "*discrete-state machines*", p. 47), even in the sense that «reasonably accurate knowledge of the state [of the machine] at one moment yields reasonably accurate knowledge any number of steps later » (p. 47). That is, DSMs would satisfy also Laplace's erroneous conjecture concerning approximations. Now, this happens to be false, since if the machine starts on very close but different values (reasonably accurate - but not exact - knowledge of the discrete state of the machine) for, say, x_0 in the computation of the logistic sequence, this may lead to very different evolutions and, thus, it suffices to make the trajectory eventually unpredictable for the observer. But digital data bases are exact and the machine is laplacian, since, as for Laplace's God, the access to and use of data base, which are *discrete* and *definite*, is meant to be exact: the machine computes over a precise x_0 , and not over an inevitably inexact physical measure. Morevover, the laws, organised as programs, are all given. This minor mistake by Turing is understandable, as there was little computational experience at the time on discrete sequences engendered by non-linear equations (a rare exception is [von Neumann, Ulam, 1947]; the topic came to the limelight only during the '70s). However, this is the same mistake that lays at the hearth of his attempted undetectable imitation: the idea that a discrete grid of access, would allow to control/predict also an unstable evolution. No, control and prediction, such as made explicit by perfect iteration, are due to the exact nature of digital data bases and of formally programmed dynamics, *within* a DSM.

It is modern mathematics then that makes us understand the extent to which logico-computational philosophy in cognition and foundations of mathematics stems from this newtonian-laplacian culture which has endured for too long in science, to the point of even inhibiting physico-mathematical work (and of stimulating the platonic response in philosophy of mathematics). In classical mechanics, after Poincaré, and with the exception of Hadamard and of one or two great russian mathematicians, we needed to wait for the '60s and '70s for his philosophies and his mathematics to be taken up. In philosophy, classical cognitivism, stuck in the "linguistic turn", suffered the consequences of it, since it has lost first of all, in the Boole and Frege mouvance and against the philosophy of Riemann and Poincaré, the "sense of space" and of geometrical complexity. Turing, in 1950, situates himself between the two cultures, as his article in philosophy proves: one must grasp the mathematical subtleties of his imitation game in order to appreciate it and to not proclaim, against Turing, that the brain is – or can be modelled by - a Turing machine, meaning a "programmable laplacian machine", all while adding ... "in the end", the fateful sentence of all simplistic reductions ever promised and never accomplished.

In fact, in cognition (but also in classical Artificial Intelligence and in - formalist - philosophy of mathematics, the loci of the discrete-arithmetic modelling of the world and of thought, along the lines of Hilbert's laplacian conjectures), we still await for a conscious reflection on paradigms comparable to the one explicitely made by Sir James Lighthill, during his chairman period at the International Association for Mechanics : «Here I have to pause and speak once again on the behalf of the broad global fraternity of practitioners of mechanics. We are deeply conscious today that the enthusiasm of the forebears for the marvelous achievements of Newtonian mechanics led them to make generalizations in this area of predictability which, indeed, we may have generally tended to believe before 1960, but which we now recognize to be false. We collectively whish to apologize for having mislead the general educated public by spreading ideas about the determinism of systems satisfying Newton's laws of motion that, after 1960, were to be proved incorrect» [Lightill, 1986].

In short, in Physics, Laplacian philosophy has played its part about two centuries ago; in logic, almost a century later, it suggested an elegant formalism which engendered the Computer

Science of sequentiality and its beautiful mathematics (but also a philosophy of knowledge anchored upon the physics of the XIXth century); yet, all this is over, even in Computer Science. Quite obviously, some of its great concepts remain pillars of the modern analyses of computer programming - the structures of types, polymorphism, for example - just as the notions of hamiltonian and of lagrangian in classical mechanics have diffused into the different branches of the physics of the XXth century, but the conceptual framework and its philosophy are radically changing. In fact, in Computer Science, the time has come for the computability of "data flows", of synchrony and of concurrency in (spatially) distributed systems, as opposed to that of "input-output" calculations, outside of the world - because beyond space and physical time (their time is secreted by the clock, see [Bailly, Longo, 2003]) - typical of Laplace-Turing sequential machines. These concurrent machines remain DSMs, so they are quite different from any dynamical system (continuous, said Turing), but they pose physical problems, as any real system, so also of spatio-temporal nature (synchronization, connectivity - as homotopy, for example, [Goubault, 2000]). Their mathematics are in the process of realization and are about to give us a novel theory of discrete computations which greatly enriches that of Turing, Church, and of the other greats of the '30s, because it responds to other questions than those of computability à la Turing (see [Aceto & al., 2003]).

Conclusion: irreversible vs. unrepeatable

We have briefly mentioned the essential, constitutive, role of determinism in the classical physical theories: a role confirmed by the great turning point of Poincaré, who has distinguished, mathematically, determinism from predictability. By this way, he has led us to understand randomness as epistemic, within the framework of deterministic theories (later, we even managed to say that a programmed sequence is random, if we do not know the laplacian program which generates it and if it has a behavior, a geometry, that is ergodic). On the other hand, an important trend in modern physics considers indeterminism as inherent to quantum theories and probabilities as intrinsic to this approach to microphysics.

Dynamical systems (thermodynamical and of critical type) have introduced, in modern fashion, "the arrow of time", following the essential irreversibility of their processes. But there is another concept which Computer Science places at the center of its own scientific construction: that of the *repeatability* of the process. In fact, it is inherent to the notion of program, the possibility of repeating the unfolding of the computation in time. That is, to start over from the same initial conditions and to follow the exact same evolution: the discrete nature of the system allows to avoid the consequences of a possibile sensitivity to initial conditions, even when they are implicit in the equations implemented. There lies an essential, constitutive component of the laplacian nature of DSMs, to which Turing so clearly refers: "It is an essential property of ...[DSMs] that this phenomenon does not occur". In summary, if a system is stable *or* if it is a DSM (discrete state machine!), its trajectories are repeatable, because it is not sensitive to the initial conditions *or* the eventual sensitivity does not manage to deploy its "destabilizing" effects, for re-initialization is perfect, and the unpredictability is "pushed to infinity" (the undecidability of the halting problem, Turing-style, see the beginning of §.4). As does a simple pendulum, as does a clock, the computer iterates without difficulty: in fact, iteration is their job. And iteration, in Computability Theory, begins by primitive recursion, characteristic of the functions of Herbrand and Gödel Arithmetic, goes through general recursion of this same formal system and of lambda-calculus, and arrives to a very important global property of programs: the portability of software (would you buy a piece of software if it was not transferable onto any machine and iterateble at will?). In short, the repeatability, along the discrete processes, is inherent to the Theory of Computability and to its remarkable practical development, Computer Science. Specifically, it tells us that one thing is the physico-mathematical modelling, by equations with their solutions, continuous or analytical for example and if possible, and another, an ulterior step, is the implementation of these on a DSM: the latter will give us an absolutely remarkable imitation (though detectable), which is indispensable to modern science, but essentially different from (our understanding of) the physical process, for it is a discrete realization of the continuous mathematical modelling. It is necessary to grasp this point in order to develop and apply at best this talent for imitation and iteration characteristic of DSMs. Galileo would have enormously envied our possibility to iterate without limit virtual physical experiences: he had to make do with throwing and throwing again his simple pendulum and its weight, in order to propose to us the first great laws of classical physics.

On the other hand, the dynamical processes, just slightly more complex – which interest us today, are not repeatable: a double pendulum or a turbulent river do not manage to follow again and exactly the same evolution. Moreover, for some dynamical systems, recurrence theorems confirm the difference: while a continuous system only goes very close to a previously explored state, its discrete implementation eventually forces identical iterations, when the recurrence interval is below the intended decimal approximation. Thus, sequences which are recurrent or ergodic, thus dense in the phase space, become ... periodic and start repeating themselves over and over again. More generally, any sequence generated by an iterated function system ($x_{n+1} = f(x_n)$) is periodic on a concrete DSM, as much as any pseudo-random generator, since they can take only a finite number of values. And, as already observed, periodicity is the opposite of density and ergodicity (but the period may be *very* long).

Unrepeatability is a concept to add to irreversibly: it does not coincide with the latter, because one can iterate the irreversible evolution of a gas, for example, as a *global*, statistic, evolution of the system. It is the *local* behaviour of a particle or the series of couplings (fluctuation, bifurcation) which are unrepeatable. Similarly, it is easy to describe a reversible process, which is unrepeatable. Conjointly with determination, the (fluctuation, bifurcation) couple is constitutive of classical dynamics and even more of biological processes: with structural stability, it participates in morphogenesis à la Turing and in the variability which is at the heart of evolution, phylogenetic and ontogenetic; it contributes to the dynamics of cognitive phenomena.

There are the stakes proposed by our response to Turing, based upon the irrepeatability of certain "continuous" processes, within the physical framework that he suggests himself for his game. A framework which constitutes a displacement of scientific attention from his behalf: his first works and his formal machine are part of the great ideas in Logic and in the foundations of the mathematics of the '30s; his reflections, in the 1950 article, enrich themselves with an on-look upon contemporary mathematical physics. He thus goes beyond the limits of laplacian philosophy that had characterized the first years of work in Logic. But how is it possible that a whole branch of scientific reflection, so important technically, Mathematical Logic, could have taken such a backlog, in philosophy of nature and of knowledge, in comparison with other disciplines, Physics particularly?

The weighty, historical, responsibility of the philosophies attached to logicism and to formalism was first to isolate the problem of the mathematical foundations of our relationship to phenomenal space (we discuss this in [Longo, 2002a and 2003]). This choice originally had good motivations, very well explicated by the two great founders, who were soundly worried for the upheaval of non-Euclidean geometries: it was urgent to abandon any reference to physical space and to base the foundational analysis upon pure logic and/or formal coherence ([Frege, 1884] and [Hilbert, 1899]) (16). This theoretical breakage gave us remarkable logico-formal machine, as perfect as out of this world (at least, until the arrival of todays networks and of concurrency). But, at the same time, it separated the analysis of the foundations of mathematics and, worse, of cognition, from that of Physics, because exactly at that time new theories emerged, between the XIXth and XXth centuries, strictly related to the problem of the mathematical intelligibility of space and time (geometry of dynamical systems and of relativistic spaces). Consequently, it separated them from our efforts in the construction of modern scientific knowledge, so strongly correlated to the constitution of mathematical concepts and structures, as well as from the major change in the philosophie of Nature proposed by the new physical theories. For example, symmetries and symmetry-breaking, at the heart of modern Physics, appear only in [Weyl,1952] as a component of the foundation (as genesis) of mathematical structures, and, more recently, in Proof Theory, by the work of Girard.

By consequence, the platonism/formalism scholastic dominant in the philosophy of mathematics (do triangles and real numbers really exist? … « the Scylla of ontologism, … the Charybdi of nominalism … from both sides I see the emergence of the ghost of a new scholastic» [Enriques, 1935]) missed out on the great foundational debates in Physics, about the structure of space, about determism, "non-locality" etc. (relativistic, dynamic, quantum systems), which marked the century. And it left us with formalisms, technically marvelous to invent and work on DSMs, but laplacian in their conception of the world - or in the organization of their own universe; a universe subdivided into small discrete boxes, well localized and perfectly stable, such as the bits of computer’s memory. Turing was in the process of grasping this point, as pointed out by his imitation game between deterministic systems with differing spatio-temporal evolution ("morphogeneses"), a game between the discrete and the continuum; but he died, at age 42.

Let’s try to not reach the same stalemate with Biology, of which cognitive sciences cannot do without, because the living makes even less sense without its space, its action within an ecosystem, its dynamic of forms. A dialogue with these rapidly growing sciences, within which mathematics cannot pretend to any hegemony, nor to ontological priority, and which would be at the same time technical and foundational, is essential to mathematics and to their foundation, because there cannot be a philosophy of mathematics without a philosophy of nature. There lies one of the great teachings of this article by Turing, and, long before, also of Poincaré and of H.Weyl, [Weyl, 1918 and 1927]; another "lone wolf" - according to his own definition - at a time when it was still being tried to demonstrate the laplacian completude of logico-formal potentially mechanizable systems. Deductive systems of which some seek, even today, the implementation in the brain and, sometimes, claiming to speak in Turing's name; and they go from imitation to model, up to the discreet seduction of the metaphor (17).

The distinction hinted by Turing, and at the hart of our analysis, betwen modeling (as mathematical proposal of constituive principles for a physical process) and imitation (fonctional imitation, with no commitment on the “nature” of phenomena) is a fundamental idea. It should be taken up today, both from a foundational and practical view point, as discrete-state machines are essential to modern science by their extraodinary modeling/imitation abilities.

A recent project, see the team "Morphological Complexity and Information" (18), attempts to propose a foundational dialogue with the natural sciences (see [Longo, 2003], [Bailly, Longo, 2003], [Bailly, Longo, 2003a]) as well as a few alternatives, modest and specific, to the stalemate of the arithmetic encoding of the world - a coding which is changing this very world by the descendants of Turing's DSM and their extraordinary networks, but which, transformed into a philosophy of knowledge, may prevent us of grasping its complexity and... to start thinking to the next machine.

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FOOTNOTES

2 The term "Turing machine" is traceable to A. Church, review of [Turing, 1936] in Journal of Symbolic Logic, 2, 42-43, 1937. The expression employed by Turing to designate his machine is "logical computing machine" (LCM).

3 Crucial technical aspect of Gödel's proof, 1931: it allows the encoding of the formal-deductive metatheory of Arithmetic in Arithmetic itself (see [Gödel et al., 1989]).

4 A mathematical description of a forced pendulum can be found in [Lighthill, 1986].

5 A system is deterministic, if we know to (or think we can) write a finite number of equations or rules of inference that will determine its evolution. In classical physics, determinism is inherent to the construction of scientific objectivity: the possibility to "determine" a system by a finite number of equations or of rules is intrinsic to its theoretical approach. Within this classical framework, Poincaré has demonstrated that equational determinism does not imply the predictability of the physical system. But we will come back to this, during an intermission.

6 This reader, while the others read the §.2, could consult the following page http://www.cse.ucsc.edu/~charlie/3body/
for about ten extraordinary examples of mechanical iteration of perfectly regular orbits, for 3, 6, ... 19, 99 bodies (crossed 8s, fantastical flowers ... absolutely no chaos). Once found, the exact initial conditions that generate these periodical orbits, thanks to very difficult mathematics, the machine, at each click of the observer, starts over with the exact same trajectories, as perfect as unreal. Unreal, because these orbits are critical: the gravitational field of a small comet at 10 billion kilometers would topple these "planets" far away from their periodical trajectories. Some of these images give rise to laughter (and the admiration for the mathematicians who worked on them), so much are they physically absurd: even in physics, some sense of humor can help us

distinguish between real world and virtual reality.

7 For an introduction to the determinism of chaotic systems, see [Dahan et al., 1992]. For an increasing technicity, see [Lighthill, 1986], [Devaney, 1989].

8 In these two last cases of programmable ergodicity, it is the global knowledge of the past which says nothing about the future (the series are globally random - they can concentrate for a long time near certain values, change suddenly of attraction zone, topple a group of values very far), but, locally, we perfectly know the next step - we have explicitly described (programmed) the laws of determination, conversely to dice and Lottery. It is the similar geometry of trajectories that allow to call ergodic all these series, physical or programmable.

9 Thom's and Prigogine's points of view have enormously enriched our knowledge and, despite important differences, they are mathematically and physically compatible: the analysis in [Petitot, 1990] shows it quite well. Unfortunately, the trap of ontologizing platonism gives rise to inescapable quarrels, because it leads to confound the mathematical construction of scientific objectivity that constitutes itself between us and the world, with preexisting ontologies. An objectivity constituted between us and this reality which canalizes and causes friction upon our organisative propositions, propositions that are in no way arbitrary because they are the result of our action in this world and they are embedded in our cognitive practices and structures ([Longo, 2002a and b]). In effect, the mathematical concepts require a conceptor who draws them on the phenomenal veil starting upon regularities that impose themselves upon his/her cognitive structure (those he/she "manages to see"); the mathematical explicitation of these regularities are part of the very process of the construction of mathematical knowledge and objectivity. To put it in husserlian terms, platonism reduces and confounds transcendental constitution and transcendence. How much damage has this understandable reaction, in foundational reflections, of numerous great mathematicians (Gödel, Thom, Connes ...) caused by the dominating formalist philosophies, which are technically difficult, but conceptually poor (those of foundations in meaningless logico-formal calculations, see next intermission). For example, in the quarrel about determinism, we even get to a dualistic separation that gives a different ontological status to fluctuation, a *material cause*, than to the global mathematical structure (the equations of a dynamic), *efficient or formal cause*, in the aristotelian terminology so dear to Thom. This latter would be the 'in-itself' or the platonic idea and would precede the phenomenal appearance [Petitot, 1990]. The revitalization of Aristotle's fine causal analysis is very interesting (but one must not forget the *'final cause'*, see [Stewart, 2002]); there is, however, no need of an ontological (platonician) distinction among these four different causes. To the contrary, their unity and temporal and conceptual simultaneity, within physical and biological phenomena, with their 'teleonomy', is the scientific challenge of today.

10 "[The game] is played with three people, a man (A), a woman (B), and an interrogator (C) who may be of either sex. The interrogator stays in a room apart from the other two. The object of the game for the interrogator is to determine which of the other two is the man and which is the woman. [...] We now ask the question, 'What will happen when a machine takes the part of A in this game?' Will the interrogator decide wrongly as often when the game is played like this as he does when the game is played between a man and a woman? These questions replace our original, 'Can machines think?' " [Turing, 1950].

11 "A man provided with paper, pencil, and rubber, and subject to a strict discipline, is in effect a universal machine!" ... " LCMs (logical computing machines, see note 2) can do anything that could be described as "rule of thumb" or "purely mechanical"!" [Turing, 1948]. And Wittgenstein continues: "Turing's 'Machines'. These machines are humans who calculate." [Wittgenstein, 1980; 1096]. "No insight or ingenuity on the part of the human being carrying out the computation": the LCM is the breaking down of formal thought into the simplest mechanical gesture, but as a human abstraction, upon a finite sequence of meaningless signs, outside of the world.

12 Turing refers to the brain as, at least, a dynamical physical system. To stay within his image, take a turbulent system that is at the same time very stable and very unstable, very ordinate and very inordinate; insert it sandwich-style between different levels of organization that regulate it and that it integrates. You will then have a very pale physical image of a biological object. Among these objects, quite material, soulless and without software distinct from the hardware (the modern dualism of the cognitivism of the formal rule and of the program), you will also find bodies with nervous systems that integrate and regulate them (as networks of exchange and communication), within which they integrate themselves (as organs) and by which they are regulated (by hormonal cascades. for example). These systems organize the action of the body by keeping it in a state that is physically critical, yet extended (it subsists in time and following relatively spaced out rails, see [Bailly, 1991]); within the limits of this state, we can find both stability and instability, variance and invariance, integration and differentiation, see [Bailly, Longo, 2003b]. And all this in a dynamic ecosystem and in the changing history of a community of bodies-brains that interact by gestures and language (ulterior levels of organization, external to, but generated by the biological objects, this time).

13 May it be said between us that the winning strategy proposed above for a dynamical system also applies to a man (or a woman): ask a thousand questions that require a few lines of answers each, to the human and to the machine, via a teleprinter as Turing would want. Ask the same questions the next day: you will not obtain the same responses from the human, only a continuity of meaning. In this case, the random mechanical genesis of variants is more of an attempt to trick than a mathematical counter-strategy, like those of which we speak above, because there is the vexed question of meaning as well as the dynamic stability of the biological object's identity, which would show the difference. But that goes beyond the modest ambitions of this article: here we are only talking about digital machines and Physics.

14 The mathematical incompleteness of formalisms is a theme strongly related to what we discuss here, see [Longo, 1999 and 2002; Bailly, Longo, 2003a] for analyses based upon recent results.

15 But why change the name given by Turing to the imitation game between a machine and a man/woman? The slip of scientific vision, implicit in this change of name, is very well underlined by [Lassègue, 1998]. But would have these authors failed to grasp the profound and dramatic irony of this improbable game in which to make a computer participate: to play the difference between man and woman? Would have they ignored the evolution and the mathematical stakes of Turing's scientific project, at the same time as the tragedy of the "game"

lived by this man of genius who first *projected himself* into a machine (human computer), then condemned for his homosexuality and soon to commit suicide; would they have so badly understood his mathematics as much as ignored his suffering between being and imitation: man/woman/machine?

16 This issue of well explicating the hypotheses must be a feature of the Greats (Laplace, Frege, Hilbert, Turing ...): probably because they understand the novelty of the original conceptual framework they are proposing. If not, one may find, even quite recently, people who say they have "demonstrated" Church Thesis; small implicit hypothesis: the Universe, with all of its sub-systems, is an enormous laplacian machine. But, Church Thesis is an implication, which goes from an informal definition, that of potentially mechanizable deductive calculus à la Hilbert, to specific formal systems (Church, Turing ...). As an implication, today one could say that it is certainly within the limits of truth, in Thom's sense: "the limit of the true is not the false, but the insignificant" (see for a modern appreciation [Aceto et al., 2003]). Quite obviously the ultimate goal of these "proofs" is to talk of the brain, finite sub-systems of the Universe (for a brief history of Church's Thesis -Church-Turing's, more specifically - and of its physical and cognitive caricatures, see [Copeland, 2002], in
http://plato.stanford.edu/entries/church-turing/#Bloopers).

17 "The model simplifies, the metaphor complicates" [Nouvel, 2002]; it adds information, it refers to a (another) impregnating conceptual framework, a universe of methods and of knowledge that we transfer onto the first one. "When a model functions as metaphor, the model becomes an object of seduction for thought. If we then use it as a suggestion for the solution of a philosophical question, we will manage, abetted by this confusion, to make this metaphor appear as a 'philosophical consequence' " of mathematical modelling [Nouvel, 2002].

18 Web page: http://www.di.ens.fr/users/longo/CIM/projet.html .

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(Preliminary or revised versions of Longo's articles are downloadable from: http://www.di.ens.fr/users/longo , including the introduction to [Aceto et al., 2003]).

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