

ON PARTITIONING THE ORBITALS OF A TRANSITIVE PERMUTATION GROUP

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ABSTRACT. Let G be a permutation group on a set Ω with a transitive normal subgroup M . Then G acts on the set $\text{Orbl}(M, \Omega)$ of nontrivial M -orbitals in the natural way, and here we are interested in the case where $\text{Orbl}(M, \Omega)$ has a partition \mathcal{P} such that G acts transitively on \mathcal{P} . The problem of characterising such tuples $(M, G, \Omega, \mathcal{P})$, called TODs, arises naturally in permutation group theory, and also occurs in number theory and combinatorics. The case where $|\mathcal{P}|$ is a prime-power is important in algebraic number theory in the study of arithmetically exceptional rational polynomials. The case where $|\mathcal{P}| = 2$ exactly corresponds to self-complementary vertex-transitive graphs, while the general case corresponds to a type of isomorphic factorisation of complete graphs, called a homogeneous factorisation. Characterising homogeneous factorisations is an important problem in graph theory with applications to Ramsey theory. This paper develops a framework for the study of TODs, establishes some numerical relations between the parameters involved in TODs, gives some reduction results with respect to the G -actions on Ω and on \mathcal{P} , and gives some construction methods for TODs.

1. INTRODUCTION

Each transitive permutation group M on a set Ω has a natural induced action on the set

$$\Omega^{(2)} = (\Omega \times \Omega) \setminus \{(\alpha, \alpha) \mid \alpha \in \Omega\} = \{(\alpha, \beta) \mid \alpha \neq \beta \in \Omega\},$$

given by $(\alpha, \beta)^x = (\alpha^x, \beta^x)$ for $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$ and $x \in M$. The M -orbits in $\Omega^{(2)}$ are called M -orbitals on Ω , and a partition \mathcal{P} of $\Omega^{(2)}$ is called an M -orbital decomposition if each class $P \in \mathcal{P}$ is a union of one or more M -orbitals. Let $\text{Orbl}(M, \Omega)$ denote the set of M -orbitals in $\Omega^{(2)}$. In this paper we investigate the situation where $M < G \leq \text{Sym}(\Omega)$, and G induces a transitive action on an M -orbital decomposition \mathcal{P} of $\Omega^{(2)}$; that is to say, we assume

- (i) for all $P \in \mathcal{P}$ and $g \in G$, $P^g \in \mathcal{P}$,
- (ii) for $P, P' \in \mathcal{P}$, there exists $g \in G$ such that $P^g = P'$, and
- (iii) \mathcal{P} is refined by $\text{Orbl}(M, \Omega)$.

If these conditions hold, then we call the tuple $(M, G, \Omega, \mathcal{P})$ a *transitive orbital decomposition*, or a *TOD* for short. The cardinalities $|\Omega|$ and $|\mathcal{P}|$ are called the *degree* and the *index* of the TOD, and sometimes we refer to $(M, G, \Omega, \mathcal{P})$ as a k -TOD if $k = |\mathcal{P}|$.

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Interest in transitive orbital decompositions has arisen from several different areas. The case $k = 2$ corresponds to self-complementary graphs, which are discussed in more detail below. The case where k is a prime-power arose also in algebraic number theory in the study of arithmetically exceptional rational polynomials [11]. (See [7] for the more general context of these investigations.)

Assume that $k = 2$. Then the two classes P, P' in \mathcal{P} can be regarded as the edge sets of directed graphs (digraphs) $\Gamma = (\Omega, P)$ and $\Gamma' = (\Omega, P')$ such that the group M is a vertex-transitive automorphism group of both Γ and Γ' , and each $g \in G$ that interchanges P and P' induces a graph isomorphism between Γ and Γ' . Thus Γ, Γ' is a pair of self-complementary vertex-transitive digraphs. Moreover, if \mathcal{P} is *symmetric*, in the sense that each $P \in \mathcal{P}$ is equal to its reverse $P^* = \{(\beta, \alpha) \mid (\alpha, \beta) \in P\}$, and if \mathcal{P} has index $k = 2$, then the two graphs Γ, Γ' may be considered as undirected self-complementary graphs, namely, $\Gamma = (\Omega, E)$ and $\Gamma' = (\Omega, E')$, where the edge sets E, E' are the sets of unordered pairs $\{\alpha, \beta\}$ from Ω with (α, β) in P, P' , respectively.

The study of vertex-transitive self-complementary graphs began with a construction of a family of self-complementary circulant graphs given by H. Sachs [24]. Vertex-transitive self-complementary graphs have received considerable attention in the literature, see for example [14, 17, 18, 20, 25, 27], and they have been used to investigate Ramsey numbers [3, 4, 5]. Most of the known vertex-transitive self-complementary graphs are Cayley graphs, see for example [14, 18, 25, 19, 23]; the first infinite family of vertex-transitive self-complementary graphs that are not Cayley graphs was constructed recently in [17]. With regard to TODs of arbitrary index k , we give necessary and sufficient conditions for the existence of a k -TOD in Proposition 3.3, and the proof of this result includes a general construction for them. Explicit constructions are given in Section 6.

A lot of effort was expended in determining the positive integers n such that there exist vertex-transitive self-complementary graphs with n vertices, see [1, 8, 15, 20, 27], and recently, Muzychuk [20] completely determined such positive integers. One of the major results of this paper, Theorem 1.1, is a generalisation of Muzychuk's result. A k -TOD $(M, G, \Omega, \mathcal{P})$ is called *cyclic* if the transitive permutation group $G^{\mathcal{P}}$ induced by G on \mathcal{P} is cyclic. Thus a pair of vertex-transitive self-complementary digraphs corresponds to a cyclic 2-TOD. Our result classifies the possibilities for the degree of a cyclic k -TOD, and the case $k = 2$ is the result of Muzychuk [20].

Theorem 1.1. *Let k be an integer such that $k > 1$, and let $n = r_1^{d_1} r_2^{d_2} \dots r_m^{d_m}$, where the r_i are distinct primes, $d_i \geq 1$, and $m \geq 1$. Then*

- (i) *there exists a cyclic k -TOD of degree n if and only if, for all $i = 1, \dots, m$,*

$$r_i^{d_i} \equiv 1 \pmod{k};$$

- (ii) *there exists a cyclic symmetric k -TOD of degree n if and only if, for all $i = 1, 2, \dots, m$,*

$$\begin{aligned} r_i^{d_i} &\equiv 1 \pmod{2k} \text{ if } r_i \text{ is odd, or} \\ r_i^{d_i} &\equiv 1 \pmod{k} \text{ if } r_i = 2. \end{aligned}$$

For a transitive group $M \leq \text{Sym}(\Omega)$, let $M^{(2)}$ denote the 2-closure of M in the sense of Wielandt; that is, $M^{(2)}$ is the largest subgroup of $\text{Sym}(\Omega)$ with the same orbits as M in $\Omega^{(2)}$. One way that a subgroup $G \leq \text{Sym}(\Omega)$ may leave invariant a partition refined by $\text{Orbl}(M, \Omega)$ is if G permutes the M -orbitals setwise. We show

(Proposition 2.1) that a subgroup $G \leq \text{Sym}(\Omega)$ leaves the set $\text{Orbl}(M, \Omega)$ invariant (that is, $\Delta^g \in \text{Orbl}(M, \Omega)$ for all $\Delta \in \text{Orbl}(M, \Omega)$ and all $g \in G$) if and only if G normalises $M^{(2)}$. However, whether or not G leaves $\text{Orbl}(M, \Omega)$ invariant in a TOD $(M, G, \Omega, \mathcal{P})$, we may always replace M by the kernel \hat{M} of the action of G on \mathcal{P} , and thereby obtain a TOD $(\hat{M}, G, \Omega, \mathcal{P})$ with \hat{M} normal in G . From Subsection 2.2 onwards we shall always assume that M is normalised by G . We begin by presenting in Section 2 several elementary properties of TODs and in Section 3 constructions of new TODs from a given TOD. Some of these constructions yield examples of cyclic TODs.

If $(M, G, \Omega, \mathcal{P})$ is a k -TOD and \mathcal{B} is a nontrivial block system for G in Ω , then the induced structure $(M_B^B, G_B^B, B, \mathcal{P}_B)$ on a block $B \in \mathcal{B}$ is also a k -TOD (Lemma 4.1), but there seems to be no natural induced TOD corresponding to the actions of M and G on \mathcal{B} (see Subsection 4.2). However, for cyclic k -TODs, we are able to induce a cyclic k -TOD for the quotient action on \mathcal{B} .

Theorem 1.2. *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic k -TOD. Then for any G -invariant partition \mathcal{B} of Ω , there exists a partition \mathcal{Q} of $\mathcal{B}^{(2)}$ such that $(M^B, G^B, \mathcal{B}, \mathcal{Q})$ is a cyclic k -TOD.*

This theorem will be proved in Section 3. It raises the question of classifying cyclic k -TODs $(M, G, \Omega, \mathcal{P})$ with G primitive on Ω . For the case where k is a prime-power, a classification is given in [10]. In Section 4 we give several constructions of cyclic TODs proving the existence assertions of Theorem 1.1, and we complete the proof of Theorem 1.1 in Section 5.

1.1. Further graph-theoretic links. From another viewpoint, a symmetric TOD of degree n is a special type of isomorphic factorisation of the complete graph K_n on n vertices. An *isomorphic factorisation* of K_n with vertex set V is a decomposition $\{E_1, \dots, E_k\}$ of the unordered pairs of vertices such that the k graphs $(V, E_1), \dots, (V, E_k)$ are pairwise isomorphic; the graphs (V, E_i) are called the *factors* of the factorisation. Isomorphic factorisations of complete graphs have been investigated for a long time, see for instance [12, 13]. If $(M, G, \Omega, \mathcal{P})$ is a symmetric k -TOD of degree n and if $E_i = \{\{x, y\} \mid (x, y) \in P_i\}$ where $\mathcal{P} = \{P_1, \dots, P_k\}$, then $\{E_1, \dots, E_n\}$ is an isomorphic factorisation of K_n with vertex set Ω with the additional property that the group G permutes $\{E_1, \dots, E_k\}$ transitively. Thus, isomorphisms between each pair (V, E_i) and (V, E_j) can be induced by elements of G . Isomorphic factorisations of K_n with this property are called *homogeneous factorisations*.

We end this section with a discussion of two special classes of TODs. Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD such that $\mathcal{P} = \text{Orbl}(M, \Omega)$, that is, \mathcal{P} is a trivial partition of $\text{Orbl}(M, \Omega)$. It then follows that G is a 2-transitive permutation group on Ω , and M is a permutation group of rank $k + 1$. By Proposition 2.1, we may assume that G contains $M^{(2)}$, and then $M^{(2)}$ is a normal subgroup of G of rank $k + 1$. Inspecting the classification of 2-transitive permutation groups, see [2], we see that either G is affine, or $M \cong \text{PSL}_2(8)$ and $G \cong \text{Aut}(\text{PSL}_2(8))$. The latter indeed gives rise to a 3-TOD $(M, G, \Omega, \mathcal{P})$ such that $\mathcal{P} = \text{Orbl}(M, \Omega)$, which is symmetric and has degree 28. This, in particular, shows that the complete graph K_{28} may be factorised into three isomorphic arc-transitive graphs of valency 9. Further, this, together with Theorem 1.1, also shows that a complete graph K_n having a non-trivial homogeneous factorisation with M -arc-transitive factors implies that n is a

prime-power or $n = 28$. The special case where $k = 2$ corresponds to arc-transitive self-complementary graphs, which are classified in [21, 28].

A more general interesting class of TODs is the class of homogeneous factorisations of complete graphs with edge-transitive factors. This corresponds to the class of symmetric k -TODs $(M, G, \Omega, \mathcal{P})$ such that each $P_i \in \mathcal{P}$ is an orbital, or a union of two paired orbitals of M^Ω . In this case, it follows that we can take G to be a 2-homogeneous permutation group on Ω with a transitive normal subgroup $M^{(2)}$ of rank $k + 1$. The TOD arising from $\text{PSL}_2(8)$ is also an example for this case.

In subsequent work [16], a complete description will be given of homogeneous factorisations of complete graphs with arc-transitive or edge-transitive factors.

2. GENERAL PROPERTIES OF TODS

2.1. On the normality of M . Some TODs $(M, G, \Omega, \mathcal{P})$ arise with the group G having an induced action on the set $\text{Orbl}(M, \Omega)$ of M -orbitals. This condition is not part of the definition of a TOD. However, we do not have any examples where it does not hold. We first prove the result mentioned in the introduction, which relates this property to the 2-closure of M .

Proposition 2.1. *Let M be a transitive permutation group on a set Ω . Then a subgroup $G \leq \text{Sym}(\Omega)$ leaves $\text{Orbl}(M, \Omega)$ invariant if and only if G normalises $M^{(2)}$.*

Proof. Suppose that G normalises $M^{(2)}$. Let $g \in G$ and $\Delta \in \text{Orbl}(M, \Omega)$. We need to prove that $\Delta^g \in \text{Orbl}(M, \Omega)$. For $x \in M$, we have $x^{g^{-1}} \in M^{(2)}$, and hence $\Delta^{gxg^{-1}} = \Delta^{x^{g^{-1}}} = \Delta$. So $\Delta^{gx} = \Delta^g$, that is, Δ^g is M -invariant. For any $(\alpha, \beta), (\gamma, \delta) \in \Delta$, there exists $x \in M$ such that $(\alpha, \beta)^x = (\gamma, \delta)$. Now $g^{-1}xg \in M^{(2)}$, $(\alpha^g, \beta^g), (\gamma^g, \delta^g) \in \Delta^g$, and $(\alpha^g, \beta^g)^{g^{-1}xg} = (\gamma^g, \delta^g)$. It follows that $M^{(2)}$ is transitive on Δ^g , and so Δ^g is an $M^{(2)}$ -orbital. Thus by the definition of $M^{(2)}$, $\Delta^g \in \text{Orbl}(M, \Omega)$, and hence G leaves $\text{Orbl}(M, \Omega)$ invariant.

Conversely, suppose that G leaves $\text{Orbl}(M, \Omega)$ invariant. Let $x \in M^{(2)}$ and $g \in G$. We claim that $x^g \in M^{(2)}$. Take an arbitrary element $\Delta \in \text{Orbl}(M, \Omega)$. Then $\Delta^{g^{-1}} \in \text{Orbl}(M, \Omega)$, and hence $(\Delta^{g^{-1}})^x = \Delta^{g^{-1}}$. Therefore, $\Delta^{xg} = \Delta^{g^{-1}xg} = \Delta$, and so x^g fixes every element of $\text{Orbl}(M, \Omega)$. Thus $x^g \in M^{(2)}$, and so G normalises $M^{(2)}$. \square

If $(M, G, \Omega, \mathcal{P})$ is a k -TOD and G normalises $M^{(2)}$, then $(M^{(2)} \cap G, G, \Omega, \mathcal{P})$ is also a k -TOD, and by Lemma 2.1, $M^{(2)} \cap G$ is a normal subgroup of G . Whether or not this is the case, if \hat{M} is the kernel of the action of G on \mathcal{P} , then $(\hat{M}, G, \Omega, \mathcal{P})$ is a k -TOD. We will assume from now on that, for a k -TOD $(M, G, \Omega, \mathcal{P})$, the group M is a normal subgroup of G . Moreover, we note the following.

Lemma 2.2. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD, and let K be the kernel of the G -action on \mathcal{P} . Assume that N is a normal subgroup of G such that $N \leq K$ and N is transitive on Ω . Then $(N, G, \Omega, \mathcal{P})$ is a k -TOD.*

Proof. This is clear from the definition of TODs. \square

2.2. TODs and partitions of Ω . Choose a point $\omega \in \Omega$. Now we give a relation between a partition of $\Omega^{(2)}$ and a partition of $\Omega \setminus \{\omega\}$. For a partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of $\Omega^{(2)}$, let $P_i(\omega) = \{\omega' \in \Omega \mid (\omega, \omega') \in P_i\}$, and let

$\mathcal{P}(\omega) = \{P_1(\omega), P_2(\omega), \dots, P_k(\omega)\}$. Let M be a transitive permutation group on Ω , and let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a partition of $\Omega^{(2)}$ refined by $\text{Orbl}(M, \Omega)$. Then $\mathcal{P}(\omega)$ is a partition of $\Omega \setminus \{\omega\}$ refined by the set of M_ω -orbits in $\Omega \setminus \{\omega\}$. Conversely, if $\mathcal{P}(\omega)$ is a partition of $\Omega \setminus \{\omega\}$ refined by the set of M_ω -orbits in $\Omega \setminus \{\omega\}$, then we obtain a partition \mathcal{P} of $\Omega^{(2)}$ refined by $\text{Orbl}(M, \Omega)$, by defining $\mathcal{P} = \{P_1, \dots, P_k\}$ with $P_i = \{(\omega, \omega')^g \in \Omega^{(2)} \mid \omega' \in P_i(\omega), g \in M\}$. The next lemma shows that for an overgroup G of M that leaves \mathcal{P} invariant, the G -action on \mathcal{P} is equivalent to the G_ω -action on $\mathcal{P}(\omega)$. Whenever a group G has an action on a set \mathcal{P} we shall denote by $G^{\mathcal{P}}$ the permutation group on \mathcal{P} induced by G .

Lemma 2.3. *Let M be a transitive permutation group on Ω , and suppose that \mathcal{P} is a partition of $\Omega^{(2)}$ refined by $\text{Orbl}(M, \Omega)$. Let G be such that $M \triangleleft G \leq \text{Sym}(\Omega)$, and let $\omega \in \Omega$. Then*

- (1) \mathcal{P} is G -invariant if and only if $\mathcal{P}(\omega)$ is G_ω -invariant;
- (2) in the case where \mathcal{P} is G -invariant, $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$ and the G_ω -actions on \mathcal{P} and $\mathcal{P}(\omega)$ are equivalent; in particular, G is transitive on \mathcal{P} if and only if G_ω is transitive on $\mathcal{P}(\omega)$, and thus $(M, G, \Omega, \mathcal{P})$ is a TOD if and only if G_ω is transitive on $\mathcal{P}(\omega)$.

Proof. Since M is transitive on Ω , $G = MG_\omega$, and thus each element $x \in G$ may be written as $x = gy$ for some $g \in M$ and some $y \in G_\omega$. Then for $P_i, P_j \in \mathcal{P}$, $P_i^x = P_i^y = P_j$ if and only if $P_i(\omega)^y = P_j(\omega)$. Thus \mathcal{P} is G -invariant if and only if $\mathcal{P}(\omega)$ is G_ω -invariant. Moreover, since $G = MG_\omega$, we have $G^{\mathcal{P}} = G_\omega^{\mathcal{P}}$ and the G_ω -action on \mathcal{P} is equivalent to the G_ω -action on $\mathcal{P}(\omega)$. In particular, G is transitive on \mathcal{P} if and only if G_ω is transitive on $\mathcal{P}(\omega)$. □

2.3. A congruence involving n and k . Lemma 2.3 implies a congruence relation that must be satisfied by the parameters n and k for k -TODs of degree n .

Lemma 2.4. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD, where $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. Then for each G_ω -orbit Δ in $\Omega \setminus \{\omega\}$, $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$ is a G_ω -invariant partition of Δ ; in particular, k divides $|\Delta|$.*

Proof. By definition, \mathcal{P} is a G -invariant partition of $\Omega^{(2)}$. Thus by Lemma 2.3, $\mathcal{P}(\omega)$ is a G_ω -invariant partition of $\Omega \setminus \{\omega\}$. It follows, since G_ω fixes Δ setwise, that $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$ is a G_ω -invariant partition of Δ . Also, by Lemma 2.3, G_ω is transitive on $\mathcal{P}(\omega)$ and hence also on $\{\Delta \cap P_i(\omega) \mid 1 \leq i \leq k\}$. Thus the sets $\Delta \cap P_i(\omega)$ all have the same size; so $|\Delta|$ is divisible by k . □

Lemma 2.5. *For positive integers n and k , if there exists a k -TOD of degree n , then $n \equiv 1 \pmod{k}$, and in particular, k is coprime to n . Moreover, if the k -TOD is symmetric and n is odd, then $n \equiv 1 \pmod{2k}$.*

Proof. Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD of degree n , and let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. By Lemma 2.4, it follows that k divides the length of each G_ω -orbit in $\Omega \setminus \{\omega\}$, and hence k divides $n - 1$.

Now assume that \mathcal{P} is symmetric. Then each P_i is a symmetric relation on Ω , and so there are $n(n - 1)/(2k)$ unordered pairs $\{\alpha, \beta\}$ such that P_i contains (α, β) and (β, α) . If n is odd, then $2k$ is coprime to n since k is coprime to n , and thus $(n - 1)/(2k)$ is an integer, that is, $n \equiv 1 \pmod{2k}$. □

3. SOME CONSTRUCTION METHODS FOR TODS

3.1. TODs and partitions of $\text{Orbl}(M, \Omega)$. Suppose that $(M, G, \Omega, \mathcal{P})$ is a k -TOD with $M \triangleleft G$. Then, for each G -orbit Q in $\text{Orbl}(M, \Omega)$, the partition \mathcal{P} of $\Omega^{(2)}$ determines a partition $\mathcal{B}(Q) = \{B(P) \mid P \in \mathcal{P}\}$ of Q , where $B(P)$ is the set of M -orbitals $\Delta \in Q$ such that $\Delta \subseteq P$. The next result explores the connection between the existence of a k -TOD and the existence of k -part partitions $\mathcal{B}(Q)$ for the G -orbits Q in $\text{Orbl}(M, \Omega)$. This result underlies our later constructions of cyclic TODs in Section 6.

Actions of a group G on Ω and Ω' are said to be *permutationally equivalent* if there is a bijection $f: \Omega \rightarrow \Omega'$ such that, for all $g \in G$ and $\omega \in \Omega$, $(\omega^g)f = (\omega f)^g$. For a transitive subgroup $M \leq \text{Sym}(\Omega)$ and a subset $X \subseteq \text{Orbl}(M, \Omega)$, define $X^* = \{\Delta^* \mid \Delta \in X\}$. We say that X is *symmetric* if $X = X^*$.

Construction 3.1. Let M be a transitive permutation group on Ω , and let $M \trianglelefteq G \leq \text{Sym}(\Omega)$. Suppose that for each G -orbit Q in $\text{Orbl}(M, \Omega)$, there exists a G -invariant partition $\mathcal{B}(Q)$ of Q with k parts. Choose a particular G -orbit Q and let $B \in \mathcal{B}(Q)$. Suppose that the actions induced by G on the $\mathcal{B}(Q)$ are permutationally equivalent, that is, for any other G -orbit Q' in $\text{Orbl}(M, \Omega)$, there exists a bijection $f_{Q'}: \mathcal{B}(Q) \rightarrow \mathcal{B}(Q')$ such that, for all $C \in \mathcal{B}(Q)$ and all $g \in G$, $(C^g)f_{Q'} = (Cf_{Q'})^g$. Let $J = \{g_1, \dots, g_k\}$ be a set of coset representatives for the setwise stabiliser G_B in G , and note that $G_B = G_{(B)f_{Q'}}$ for each Q' . Define P_1 to be the union, over all G -orbits Q' in $\text{Orbl}(M, \Omega)$, of all ordered pairs of points contained in M -orbitals in $(B)f_{Q'}$; that is,

$$P_1 = \bigcup_{Q'} \left(\bigcup_{\Delta \in (B)f_{Q'}} \Delta \right).$$

Let $P_i = P_1^{g_i}$ for $i = 1, \dots, k$, and let $\mathcal{P} = \{P_1, \dots, P_k\}$.

That the 4-tuple $(M, G, \Omega, \mathcal{P})$ produced by this construction is a TOD is proved below.

Lemma 3.2. *For M, G, \mathcal{P} as in Construction 3.1, $(M, G, \Omega, \mathcal{P})$ is a k -TOD, and moreover it is symmetric if and only if*

- (a) *for each symmetric orbit R , each part of the partition $\mathcal{B}(R)$ is symmetric, and*
- (b) *for each nonsymmetric orbit R , and each $B \in \mathcal{B}(Q)$, we have $(Bf_R)^* = (B)f_{R^*}$.*

Proof. Suppose that $(M, G, \Omega, \mathcal{P})$ is a tuple produced in Construction 3.1. Since $G_B = G_{(B)f_{Q'}}$, it follows that $\mathcal{B}(Q') = \{(Bf_{Q'})^{g_i} \mid i = 1, \dots, k\}$ for each Q' , and therefore \mathcal{P} is a partition of $\Omega^{(2)}$. Moreover,

$$P_i = \bigcup_{Q'} \left(\bigcup_{\Delta \in Bf_{Q'}} \Delta \right)^{g_i} = \bigcup_{Q'} \left(\bigcup_{\Delta' \in (Bf_{Q'})^{g_i}} \Delta' \right) = \bigcup_{Q'} \left(\bigcup_{\Delta' \in (B^{g_i})f_{Q'}} \Delta' \right).$$

Let $g \in G$. Then a similar argument gives $P_i^g = \bigcup_{Q'} (\bigcup_{\Delta \in (B^{g_i g})f_{Q'}} \Delta) = P_j$, where $B^{g_i g} = B^{g_j}$. Thus \mathcal{P} is G -invariant, and $G^{\mathcal{P}}$ is permutationally equivalent to $G^{\mathcal{B}(Q)}$ and, in particular, is transitive. Thus $(M, G, \Omega, \mathcal{P})$ is a k -TOD.

Suppose that $(M, G, \Omega, \mathcal{P})$ is a symmetric k -TOD, and let R be a symmetric orbit and $B \in \mathcal{B}(R)$. Let $\Delta \in B$, and suppose that $\Delta \subset P$ with $P \in \mathcal{P}$. Since R is symmetric we have $\Delta^* \in R$, and since $(M, G, \Omega, \mathcal{P})$ is symmetric we have $\Delta^* \subseteq P$.

By Construction 3.1, B consists of all orbitals Δ' such that $\Delta' \in R$ and $\Delta' \subseteq P$. Therefore $\Delta^* \in B$. It follows that $B = B^*$ is symmetric. Let R be a nonsymmetric orbit. Let $\Delta \in R$ lie in the block Bf_R of $\mathcal{B}(R)$. Then $\Delta^* \in (Bf_R)^*$ and $\Delta^* \in R^*$. Suppose that $\Delta \in P$, $P \in \mathcal{P}$. Since \mathcal{P} is symmetric, we have $\Delta^* \in P$. By the definition of f_{R^*} , $(B)f_{R^*}$ is the set of orbitals $\Delta^* \in R^*$ that are contained in P . Hence $\Delta^* \in (B)f_{R^*}$, and so $(Bf_R)^* = (B)f_{R^*}$.

Conversely, suppose that, for each symmetric orbit R , each part of the partition $\mathcal{B}(R)$ is symmetric and for each nonsymmetric orbit R and $C \in \mathcal{B}(Q)$, $(C)f_{R^*} = (Cf_R)^*$. We claim that $(M, G, \Omega, \mathcal{P})$ is symmetric. Since $G^{\mathcal{P}}$ is transitive, it is sufficient to prove that $P_1 = P_1^*$. Let $\Delta \subseteq P_1$ lie in an orbit R . If R is symmetric, then Bf_R is the set of orbitals $\Delta' \in R$ such that $\Delta' \in P_1$. Hence $\Delta \in Bf_R$. Now $\Delta^* \in R$ since R is symmetric, and $\Delta^* \in Bf_R$ since the part Bf_R of $\mathcal{B}(R)$ is symmetric, and hence $\Delta^* \subseteq P_1$. Now let R be not symmetric. Again $\Delta \in Bf_R$; so $\Delta^* \in (Bf_R)^* = (B)f_{R^*}$ and, by Construction 3.1, $\Delta^* \subseteq P_1$. Thus $P_1 = P_1^*$; so $(M, G, \Omega, \mathcal{P})$ is symmetric. \square

Now we obtain a set of necessary and sufficient conditions for the existence of TODs based on the action on $\text{Orbl}(M, \Omega)$.

- Proposition 3.3.** (i) *Let M be a transitive permutation group on Ω , and let $M \trianglelefteq G \leq \text{Sym}(\Omega)$. Then there exists a partition \mathcal{P} of $\Omega^{(2)}$ such that $(M, G, \Omega, \mathcal{P})$ is a k -TOD if and only if for each G -orbit Q in $\text{Orbl}(M, \Omega)$, there exists a G -invariant partition $\mathcal{B}(Q)$ of Q with k parts, and the actions of G on $\mathcal{B}(Q)$, for all G -orbits Q in $\text{Orbl}(M, \Omega)$, are pairwise permutationally equivalent.*
- (ii) *Moreover, there exists a symmetric k -TOD $(M, G, \Omega, \mathcal{P})$ if and only if, in addition, for each symmetric G -orbit Q in $\text{Orbl}(M, \Omega)$ (if any such exists), there exists a partition $\mathcal{B}(Q)$ as in part (i), each part of which is symmetric.*

Proof. Suppose that there exists $\mathcal{P} = \{P_1, \dots, P_k\}$ such that $(M, G, \Omega, \mathcal{P})$ is a k -TOD. Let $J = \{g_1, \dots, g_k\} \subset G$ be such that $P_1^{g_i} = P_i$ for each i . Since $G^{\mathcal{P}}$ is transitive, each G -orbit Q in $\text{Orbl}(M, \Omega)$ contains (at least one) M -orbital $\Delta \subseteq P_1$. Let B_1 be the set of M -orbitals Δ such that $\Delta \in Q$ and $\Delta \subseteq P_1$. For each i , set $B_i = B_1^{g_i}$. Then B_i is the set of M -orbitals Δ such that $\Delta \in Q$ and $\Delta \subseteq P_i$. Thus $\mathcal{B}(Q) = \{B_1, \dots, B_k\}$ is a G -invariant partition of Q with k parts, and $G^{\mathcal{B}(Q)}$ is permutationally isomorphic to $G^{\mathcal{P}}$. Conversely, if suitable G -invariant partitions $\mathcal{B}(Q)$ exist for each Q , then Construction 3.1 gives the required k -TOD by Lemma 3.2. Thus part (i) is proved.

Suppose further that $(M, G, \Omega, \mathcal{P})$ is symmetric, and suppose that Q is a symmetric G -orbit in $\text{Orbl}(M, \Omega)$. Then, since $P_1 = P_1^*$, the set B_1 of M -orbitals $\Delta \in Q$ that are contained in P_1 satisfies $B_1^* = B_1$. By the definition of $\mathcal{B}(Q)$ above, each part of $\mathcal{B}(Q)$ is symmetric.

Conversely, suppose that for each symmetric orbit Q , there is a $\mathcal{B}(Q)$ with all parts symmetric. For any such Q , we choose a partition $\mathcal{B}(Q)$ with this extra property. Choose a particular G -orbit Q in $\text{Orbl}(M, \Omega)$, and for each G -orbit R let $f_R: \mathcal{B}(Q) \rightarrow \mathcal{B}(R)$ be the bijection defining the permutational equivalence of the G -actions (taking f_Q to be the identity map). Suppose that $R \neq R^*$. Then R^* is also a G -orbit in $\text{Orbl}(M, \Omega)$; so, in particular, $R \cap R^* = \emptyset$. If necessary we replace $\mathcal{B}(R^*)$ by $\mathcal{B}^*(R) := \{B^* \mid B \in \mathcal{B}(R)\}$, and we replace f_{R^*} by $f_R^*: \mathcal{B}^*(Q) \rightarrow \mathcal{B}^*(R)$ defined by $(B)f_R^* = (Bf_R)^*$, for $B \in \mathcal{B}(Q)$. Let $(M, G, \Omega, \mathcal{P})$ be as

in Construction 3.1 using these partitions $\mathcal{B}(R)$. Then by Lemma 3.2, $(G, M, \Omega, \mathcal{P})$ is a symmetric TOD. \square

3.2. Some TODs derived from a given one. Our first construction varies the partition \mathcal{P}' but involves the same subgroup M .

Lemma 3.4. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD.*

- (1) *If \mathcal{P}' is a nontrivial G -invariant partition of $\text{Orbl}(M, \Omega)$ refined by \mathcal{P} , then $k' = |\mathcal{P}'| \geq 2$, k' divides k , and $(M, G, \Omega, \mathcal{P}')$ is a k' -TOD.*
- (2) *If $H < G$ is such that $H^{\mathcal{P}}$ is semiregular and nontrivial with orbits of length k' , and H normalises M , then $k'|k$, $k' \geq 2$, and $(M, \langle M, H \rangle, \Omega, \mathcal{P}')$ is a k' -TOD for some H -invariant partition \mathcal{P}' refined by \mathcal{P} .*

Proof. (1). Since G is transitive on \mathcal{P} and \mathcal{P}' is refined by \mathcal{P} , G is transitive on \mathcal{P}' , and hence $(M, G, \Omega, \mathcal{P}')$ is a TOD of index k' dividing k .

(2). Choose a representative from each H -orbit in \mathcal{P} , and let P'_1 be the union of these representatives. Set $\mathcal{P}' = \{(P'_1)^h \mid h \in H\}$. Since H is semiregular on \mathcal{P} with orbits of length k' , it follows that \mathcal{P}' is an H -invariant partition of $\Omega^{(2)}$ with k' parts and refined by \mathcal{P} . \square

The next construction is the key both to a reduction to consideration of TODs $(M, G, \Omega, \mathcal{P})$ with G primitive on Ω , and also to the proof of Theorem 1.1. For a partition \mathcal{P} of $\Omega^{(2)}$ and a subset $\Delta \subset \Omega$, by the *restriction* of \mathcal{P} to $\Delta^{(2)}$ we mean the partition $\mathcal{Q} = \{P_i \cap (\Delta \times \Delta) \mid 1 \leq i \leq k\}$ of $\Delta^{(2)}$. Let $M \leq G \leq \text{Sym}(\Omega)$ with M transitive. For $\Delta \in \text{Orbl}(M, \Omega)$, the paired orbital $\Delta^* = \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$ also lies in $\text{Orbl}(M, \Omega)$. If G leaves $\text{Orbl}(M, \Omega)$ invariant, then, for each $g \in G$, $(\Delta^*)^g = (\Delta^g)^*$. For a subgroup $N \leq \text{Sym}(\Omega)$ and $\omega \in \Omega$, we denote by ω^N the N -orbit containing ω .

Lemma 3.5. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD with M normal in G , let $\omega \in \Omega$, and suppose that N is a subgroup of M with no fixed points in Ω . Assume that $E \leq G_\omega$ is such that $E^{\mathcal{P}}$ is transitive and $E \leq \mathbf{N}_G(N)$. Set $F = NE$ and $\Delta = \omega^N$. Then the restriction \mathcal{Q} of \mathcal{P} to $\Delta^{(2)}$ is such that $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is a k -TOD, and $E^{\mathcal{P}}$ is permutationally isomorphic to $F^{\mathcal{Q}}$. Further, if in addition $(M, G, \Omega, \mathcal{P})$ is symmetric, then also $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is symmetric.*

Proof. Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$; so $\mathcal{P}(\omega) = \{P_1(\omega), P_2(\omega), \dots, P_k(\omega)\}$. Since \mathcal{P} is refined by $\text{Orbl}(M, \Omega)$, each of the $P_i(\omega)$ is M_ω -invariant and hence also N_ω -invariant. Thus each $Q_i(\omega) := \Delta \cap P_i(\omega)$ is N_ω -invariant. Define $\mathcal{Q}(\omega) = \{Q_i(\omega) \mid 1 \leq i \leq k\}$. Observe that $\bigcup_i Q_i(\omega) = \Delta \setminus \{\omega\}$, and if $i \neq j$, then $Q_i(\omega) \cap Q_j(\omega) = \emptyset$; so $\mathcal{Q}(\omega)$ is a partition of $\Delta \setminus \{\omega\}$. Also, $\mathcal{Q}(\omega)$ is invariant under N_ω .

By assumption $E^{\mathcal{P}}$ is transitive, and since $E \leq G_\omega$, E also acts transitively on $\mathcal{P}(\omega)$. Further, since E normalises N and fixes ω , it follows that E fixes Δ setwise. Thus, E has an induced action on $\mathcal{Q}(\omega)$ given by $Q_i(\omega)^g = \Delta \cap P_i(\omega)^g$ ($g \in E$, $i \leq k$), which is permutationally isomorphic to its actions on $\mathcal{P}(\omega)$ and \mathcal{P} . In particular, E acts transitively on $\mathcal{Q}(\omega)$, and all the $Q_i(\omega)$ are nonempty. Note that $F_\omega = N_\omega E$ and the induced action on $\mathcal{Q}(\omega)$ satisfies $F_\omega^{\mathcal{Q}(\omega)} = (N_\omega E)^{\mathcal{Q}(\omega)} = E^{\mathcal{Q}(\omega)}$. By Lemma 2.3, the corresponding partition \mathcal{Q} of $\Omega^{(2)}$ is F -invariant, and the F -action on \mathcal{Q} is equivalent to the F_ω -action on $\mathcal{Q}(\omega)$. Thus $F^{\mathcal{Q}}$ is permutationally isomorphic to $E^{\mathcal{P}}$, and $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is a k -TOD. From the definition of \mathcal{Q} (preceding Lemma 2.3) it is clear that $Q_i \subseteq P_i$ for each i , and therefore $Q_i = P_i \cap (\Delta \times \Delta)$, for $1 \leq i \leq k$.

Assume in addition that $(M, G, \Omega, \mathcal{P})$ is symmetric, that is, each P_i is symmetric. Let $O \in \text{Orbl}(M, \Omega)$, and let $\hat{O} \in \text{Orbl}(N, \Delta)$ be such that $\hat{O}(\omega) \subseteq O(\omega)$. Then $\hat{O}^*(\omega) \subseteq O^*(\omega)$. Assume that $O(\omega) \subseteq P_i(\omega)$. Then, since P_i is symmetric, $O(\omega) \cup O^*(\omega) \subseteq P_i(\omega)$. By the definition of $Q_i(\omega)$, $\hat{O}(\omega) \cup \hat{O}^*(\omega) \subseteq Q_i(\omega)$. It follows that Q_i is symmetric, and so $(N^\Delta, F^\Delta, \Delta, \mathcal{Q})$ is symmetric. \square

Our final construction is not elementary, since it relies on an application of the result of Fein, Kantor and Schacher [6] that a transitive permutation group contains a fixed-point-free element of prime-power order. This result relies on the finite simple group classification. Recall that we may always assume, for a TOD $(M, G, \Omega, \mathcal{P})$, that M is normal in G and hence that G leaves $\text{Orbl}(M, \Omega)$ invariant.

Theorem 3.6. *If $(M, G, \Omega, \mathcal{P})$ is a k -TOD with M normal in G , then there exists a p -TOD $(M, H, \Omega, \mathcal{Q})$ for some prime divisor p of k and some partition \mathcal{Q} of $\Omega^{(2)}$ refined by \mathcal{P} , where $H = \langle M, \tau \rangle$ for some $\tau \in G_\omega \setminus M_\omega$ where $\omega \in \Omega$. In particular, τ fixes no element of $\text{Orbl}(M, \Omega)$.*

Proof. Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD with M normal in G . Let $\omega \in \Omega$. Since M is transitive, $G = MG_\omega$; so $G^\mathcal{P} = G_\omega^\mathcal{P}$. Thus $G_\omega^\mathcal{P}$ is transitive, and it follows from [6] that, for some prime p , G_ω contains an element τ of p -power order such that $\tau^\mathcal{P}$ has no fixed points in \mathcal{P} . Label the parts of \mathcal{P} as P_{ij} , so that the i^{th} -orbit of $\langle \tau^\mathcal{P} \rangle$ in \mathcal{P} is $\{P_{i,j} \mid 1 \leq j \leq p^{a_i}\}$, $a_i \geq 1$, and $P_{ij}^\tau = P_{i,j+1}$ (reading the second subscript modulo p^{a_i}). For $l = 1, \dots, p$, define $Q_l = \bigcup_i (P_{i,l} \cup P_{i,l+p} \cup \dots \cup P_{i,l-p+p^{a_i}})$. Then $\mathcal{Q} = \{Q_1, \dots, Q_p\}$ is permuted cyclically by τ , and \mathcal{Q} is a partition of $\Omega^{(2)}$ refined by \mathcal{P} . Thus $(M, \langle M, \tau \rangle, \Omega, \mathcal{Q})$ is a p -TOD. Since \mathcal{Q} is refined by \mathcal{P} , it is also refined by $\text{Orbl}(M, \Omega)$, and it follows that τ acts on $\text{Orbl}(M, \Omega)$ with no fixed points. \square

This result has an immediate consequence:

Lemma 3.7. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD such that G has a regular normal subgroup N contained in M . Then N is soluble.*

Proof. By Lemma 2.2, $(N, G, \Omega, \mathcal{P})$ is a k -TOD, and then, by Theorem 3.6, there exists an element $\tau \in G \setminus N$ that fixes no element of $\text{Orbl}(N, \Omega)$. Since N is regular on Ω , it follows that τ fixes no non-identity element of N . Thus the automorphism of N induced by conjugation by τ is fixed-point-free, and hence N is soluble, see [9, Thm. 1.48]. \square

4. TODS AND IMPRIMITIVE GROUP ACTIONS

4.1. TODs on blocks of imprimitivity. We show that the induced configuration of a k -TOD $(M, G, \Omega, \mathcal{P})$ on a block B of imprimitivity for G on Ω is also a k -TOD. Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD, and let \mathcal{B} be a G -invariant partition of Ω . Let $\mathcal{P}_B := \{P_1^B, P_2^B, \dots, P_k^B\}$, where $P_i^B := P_i \cap (B \times B)$. Then each P_i^B is a union of M_B -orbitals on B , and \mathcal{P}_B is a partition of $\mathcal{B}^{(2)}$. We denote the setwise stabilisers of B in M, G by M_B and G_B , respectively.

Lemma 4.1. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD with M normal in G . Then for a non-trivial block B of imprimitivity for G^Ω , $(M_B^B, G_B^B, B, \mathcal{P}_B)$ is a k -TOD; further, $G^\mathcal{P}$ is permutationally isomorphic to $G_B^{\mathcal{P}_B}$.*

Proof. The setwise stabiliser M_B has no fixed points in Ω and is normalised by G_ω . Also, since $M^\mathcal{P} = 1$, we have that $G^\mathcal{P} = G_\omega^\mathcal{P}$ is transitive. The result now follows from Lemma 3.5 applied with $N = M_B, E = G_\omega$, and $\Delta = B$. \square

By choosing B to be a minimal block of imprimitivity, we may assume that the group G_B^B is primitive. This suggests studying k -TODs $(M, G, \Omega, \mathcal{P})$ with G primitive on Ω .

Proposition 4.2. *Let $(M, G, \Omega, \mathcal{P})$ be a k -TOD such that G is primitive on Ω . Then G^Ω is of O’Nan-Scott type HA, AS, SD, CD or PA (as defined in [22]).*

Proof. This is an immediate consequence of Lemma 3.7. □

Such TODs are investigated further in [10] and, in particular, a classification is obtained of cyclic p^a -TODs where p is a prime. It is shown there, in particular, that there exist TODs corresponding to each of the five O’Nan-Scott types specified in the proposition.

4.2. Quotients of TODs. Let $(M, G, \Omega, \mathcal{P})$ be a TOD and let \mathcal{B} be a G -invariant partition of Ω . There is a natural map from $\Omega^{(2)}$ to $\mathcal{B} \times \mathcal{B}$ given by $(\omega, \omega') \rightarrow (B, B')$, where $\omega \in B \in \mathcal{B}$ and $\omega' \in B' \in \mathcal{B}$. This induces a map from subsets of $\Omega^{(2)}$ to subsets of $\mathcal{B} \times \mathcal{B}$. However, there are at least two reasons why, in general, a partition \mathcal{P} of $\Omega^{(2)}$ is not mapped to a partition of $\mathcal{B}^{(2)}$. First there is the problem that distinct points ω, ω' may lie in the same block of \mathcal{B} . One might hope still to achieve a partition of $\mathcal{B}^{(2)}$ simply by ignoring such pairs. However, the second problem is that disjoint subsets of $\Omega^{(2)}$ may correspond to non-disjoint subsets of $\mathcal{B}^{(2)}$. This second problem makes it impossible, in general, to find a natural partition of $\mathcal{B}^{(2)}$ corresponding to a given partition of $\Omega^{(2)}$. In particular, there seems to be no natural construction of a TOD for the actions of M and G on \mathcal{B} from a given TOD $(M, G, \Omega, \mathcal{P})$. This is demonstrated by the following simple example.

Example 4.3. Let $M = \langle (123456789) \rangle \cong \mathbb{Z}_9$, $G = D_{18}$, and $\Omega = \{1, 2, \dots, 9\}$. Then $\text{Orbl}(M, \Omega) = \{\Delta_i = (1, i)^M \mid 2 \leq i \leq 9\}$. Let $P_1 = \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5$ and $P_2 = \Delta_6 \cup \Delta_7 \cup \Delta_8 \cup \Delta_9$. Then $\mathcal{P} = \{P_1, P_2\}$ is a partition of $\Omega^{(2)}$ and $(M, G, \Omega, \mathcal{P})$ is a 2-TOD. Now $B_1 = \{1, 4, 7\}$, $B_2 = \{2, 5, 8\}$ and $B_3 = \{3, 6, 9\}$ form a G -invariant partition \mathcal{B} of Ω , but the images of both P_1 and P_2 , under the map $\Omega^{(2)} \rightarrow \mathcal{B} \times \mathcal{B}$ defined above, are $\mathcal{B} \times \mathcal{B}$.

However, we show in the next section that, for a cyclic TOD $(M, G, \Omega, \mathcal{P})$ with G -invariant partition \mathcal{B} of Ω , it is possible to construct an induced quotient TOD on \mathcal{B} of the same index. (This is Theorem 1.2, stated in the introduction.)

5. CYCLIC TODS

To prove Theorem 1.2, we first prove a very useful result about cyclic TODs. A similar result can be found in [11].

Proposition 5.1. *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic k -TOD and let K be the kernel of the action of G on \mathcal{P} . Then each element $\tau \in G \setminus K$ has exactly one fixed point in Ω .*

Proof. Let $\omega \in \Omega$. We have $G = KG_\omega$, and so $G = \langle K, \sigma \rangle$ for some $\sigma \in G_\omega$. Let $k = \prod_{i=1}^r p_i^{e_i}$ for distinct primes p_i , $e_i \geq 1$, and $r \geq 1$. Let $\tau \in G \setminus K$. Then there exists i such that the order of τ modulo K is divisible by p_i , and hence $\tau \notin \langle K, \sigma^{p_i^{e_i}} \rangle$. Let $\hat{\mathcal{P}}$ be the partition of $\Omega^{(2)}$ such that each part is the union of the parts of \mathcal{P} contained in some orbit of $\langle (\sigma^{p_i^{e_i}})^{\mathcal{P}} \rangle$. Then by Lemma 3.4(1), $(M, G, \Omega, \hat{\mathcal{P}})$ is a cyclic $p_i^{e_i}$ -TOD. Set $\hat{M} = \langle K, \sigma^{p_i^{e_i}} \rangle$. Then by Lemma 2.2, $(\hat{M}, G, \Omega, \hat{\mathcal{P}})$ is a cyclic

$p_i^{e_i}$ -TOD, and \hat{M} is the kernel of the action of G on $\hat{\mathcal{P}}$. There is a p_i -element σ' such that $G = \langle \hat{M}, \sigma' \rangle = \langle \hat{M}, \sigma \rangle$ (taking σ' to be the “ p_i -part” of σ).

Then $\tau = x(\sigma')^l$ for some $x \in \hat{M}$ and some integer l not divisible by $p_i^{e_i}$. Since x fixes $\hat{\mathcal{P}}$ pointwise, the $\langle \tau \rangle$ -action on $\hat{\mathcal{P}}$ is equivalent to the action of $\langle \sigma' \rangle$ on $\hat{\mathcal{P}}$. Thus $\langle \tau \rangle$ is nontrivial and half-transitive on $\hat{\mathcal{P}}$. In particular, τ fixes no element of $\hat{\mathcal{P}}$, and so τ fixes no $(\omega, \omega') \in \Omega^{(2)}$ for any distinct $\omega, \omega' \in \Omega$. Hence τ fixes at most one point of Ω .

If τ has p_i -power order, then, since $p_i \nmid |\Omega|$ by Lemma 2.5, τ fixes at least one point of Ω , so that it fixes exactly one point of Ω .

Suppose now that the order $o(\tau) = n_1 n_2$ is such that n_1 is a p_i -power, $\gcd(n_1, n_2) = 1$, and $n_2 > 1$, and write $\tau = \tau_1 \tau_2$ such that $o(\tau_i) = n_i$ and $\tau_1 \tau_2 = \tau_2 \tau_1$. Since $|G : \hat{M}| = p_i^{e_i}$, it follows that $\tau_2 \in \hat{M}$, and therefore τ_1 is of p_i -power order and lies in $G \setminus \hat{M}$. By the argument of the previous paragraph, τ_1 has exactly one fixed point in Ω . Let $\Delta_1, \Delta_2, \dots, \Delta_t$ be the $\langle \tau \rangle$ -orbits in Ω . Since $\langle \tau_1 \rangle$ is a normal subgroup of $\langle \tau \rangle$, we have that $\langle \tau_1 \rangle$ acts on each Δ_j half-transitively. Thus either $\langle \tau_1 \rangle$ acts on Δ_j trivially, or $|\Delta_j|$ is divisible by p_i . Since τ_1 has exactly one fixed point in Ω , it follows that exactly one of the Δ_j has size 1 and all the others have size a multiple of p_i . Therefore, τ fixes exactly one point in Ω . \square

Now we deduce a corollary for imprimitive cyclic TODs.

Lemma 5.2. *Let $(M, G, \Omega, \mathcal{P})$ be a cyclic k -TOD. Let K be the kernel of the G -action on \mathcal{P} , and let \mathcal{B} be a nontrivial G -invariant partition of Ω . Then each element of $G \setminus K$ fixes exactly one block of \mathcal{B} .*

Proof. Here $G = \langle K, \sigma \rangle$ for some $\sigma \in G$ such that $\sigma^k \in K$ and $G/K \cong \mathbb{Z}_k$. Then, by Lemma 2.2, $(K, G, \Omega, \mathcal{P})$ is a cyclic k -TOD. Write $\mathcal{B} = \{B_0, B_1, \dots, B_t\}$ for some $t \geq 2$. Let $\tau \in G \setminus K$. By Proposition 5.1, τ fixes a point of Ω and hence fixes setwise a block of \mathcal{B} , say B_0 . Now $\tau = f\sigma^r$ for some integer r and $f \in K$, and τ has order k_0 modulo K for some $k_0|k$ with $k_0 > 1$. Since K acts trivially on \mathcal{P} , $\langle \tau^P \rangle = \langle (\sigma^r)^P \rangle$ has k/k_0 orbits of length k_0 in \mathcal{P} . Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$, and observe that, for any $i \neq 0$,

$$\begin{aligned} (P_1 \cap (B_0 \times B_i)) \cup \dots \cup (P_k \cap (B_0 \times B_i)) &= B_0 \times B_i, \text{ and} \\ (P_j \cap (B_0 \times B_i)) \cap (P_{j'} \cap (B_0 \times B_i)) &= \emptyset \quad \text{if } j \neq j', \end{aligned}$$

where $B_0 \times B_i = \{(\omega, \omega') \mid \omega \in B_0, \omega' \in B_i\}$. Suppose that τ fixes B_i setwise for some $i \in \{1, 2, \dots, t\}$. Then $(B_0 \times B_i)^\tau = B_0 \times B_i$, and for any $j \in \{1, 2, \dots, k\}$, there exists $j' \in \{1, 2, \dots, k\}$ such that

$$(P_j \cap (B_0 \times B_i))^\tau = P_{j'} \cap (B_0 \times B_i).$$

If $P_j \cap (B_0 \times B_i) \neq \emptyset$, then for each of the k_0 parts $P_{j'}$ in the $\langle \tau^P \rangle$ -orbit containing P_j , we have $|P_j \cap (B_0 \times B_i)| = |P_{j'} \cap (B_0 \times B_i)|$. Since this is true for all $\langle \tau^P \rangle$ -orbits, it follows that k_0 divides $|B_0 \times B_i|$. However, by Lemma 4.1, $(K_{B_0}, G_{B_0}^{B_0}, B_0, \mathcal{P}_{B_0})$ is a k -TOD, and hence by Lemma 2.5, $|B_0| \equiv 1 \pmod{k}$. Therefore, $|B_0 \times B_i| = |B_0|^2 \equiv 1 \pmod{k}$, and so $|B_0 \times B_i| \equiv 1 \pmod{k_0}$, which is a contradiction since $k_0 > 1$. Therefore, B_0 is the unique fixed block of τ . \square

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let K be the kernel of G acting on \mathcal{P} . Then $G/K \cong \mathbb{Z}_k$, and $G = \langle K, \sigma \rangle$ for some $\sigma \in G$ such that σ normalises K and $\sigma^k \in K$. Since K

is transitive on Ω , we have that $G = KG_\omega$, where $\omega \in \Omega$. Hence $\sigma = f\sigma'$ where $f \in K$ and $\sigma' \in G_\omega$. Since f fixes every element of \mathcal{P} , $\langle \sigma' \rangle$ induces a transitive action on \mathcal{P} . Thus, without loss of generality, we may assume that $\sigma \in G_\omega$. Let $\mathcal{B} = \{B_0, B_1, \dots, B_t\}$ be a G -invariant partition of Ω such that $\omega \in B_0$. Then in particular $B_0^\sigma = B_0$, and hence σ normalises K_{B_0} . Let \mathcal{D} be the set of K_{B_0} -orbits in $\Omega \setminus B_0$. Then \mathcal{D} is $\langle \sigma \rangle$ -invariant.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be such that $P_i^\sigma = P_{i+1}$ for each $i < k$ and $P_k^\sigma = P_1$, and let $\mathcal{P}(B_0) = \{P_1(B_0), P_2(B_0), \dots, P_k(B_0)\}$, where

$$P_i(B_0) = \{\alpha \in \Omega \setminus B_0 \mid (\beta, \alpha) \in P_i, \text{ for some } \beta \in B_0\}.$$

Then

$$P_i(B_0)^\sigma = \{\alpha^\sigma \in \Omega \setminus B_0 \mid (\beta^\sigma, \alpha^\sigma) \in P_i^\sigma, \text{ for some } \beta^\sigma \in B_0\} = P_{i+1}(B_0),$$

reading the subscripts modulo k . Since K_{B_0} acts trivially on \mathcal{P} , it follows that K_{B_0} fixes each $P_i(B_0)$ setwise, and hence each $P_i(B)$ is a union of some subset of \mathcal{D} .

Suppose that $\Delta \in \mathcal{D}$ is contained in $P_1(B_0)$, and suppose that $B \in \mathcal{B} \setminus \{B_0\}$ is such that $B \cap \Delta$ contains a point α . Suppose further that $1 \leq i \leq k$, and $B \cap \Delta^{\sigma^i}$ also contains a point, say β . Then $\beta = \delta^{\sigma^i}$ for some $\delta \in \Delta$, and since K_{B_0} is transitive on Δ , $\delta = \alpha^g$ for some $g \in K_{B_0}$. Thus $\beta = \alpha^{g\sigma^i} \in B \cap B^{g\sigma^i}$, and hence $g\sigma^i$ fixes B . However, $g\sigma^i \in G_{B_0}$ and $B \neq B_0$. It follows from Lemma 5.2 that $g\sigma^i \in K$. Hence $\sigma^i \in K$, and so $i = k$. Thus the k sets $\Delta, \Delta^\sigma, \dots, \Delta^{\sigma^{k-1}} \in \mathcal{D}$ meet disjoint subsets of $\mathcal{B} \setminus \{B_0\}$. Moreover, $\Delta^{\sigma^i} \subseteq (P_1(B_0))^{\sigma^i} = P_{i+1}(B_0)$ for each $i < k$.

For a K_{B_0} -orbit Δ , let $\mathcal{B}(\Delta)$ denote the subset of blocks B of $\mathcal{B} \setminus \{B_0\}$ such that $B \cap \Delta \neq \emptyset$. Suppose that $\mathcal{B}(\Delta) \cap \mathcal{B}(\Delta')$ (where $\Delta, \Delta' \in \mathcal{D}$) contains a block B , and let B' be an arbitrary block in $\mathcal{B}(\Delta)$. Then, since $B \cap \Delta \neq \emptyset$, $B' \cap \Delta \neq \emptyset$, and Δ is a K_{B_0} -orbit, some element $x \in K_{B_0}$ maps a point of $B \cap \Delta$ to a point of $B' \cap \Delta$, and hence $B^x = B'$. Since Δ' is a K_{B_0} -orbit, we have $(B \cap \Delta')^x = B' \cap \Delta'$, and therefore $B' \in \mathcal{B}(\Delta')$. Thus $\mathcal{B}(\Delta) \subseteq \mathcal{B}(\Delta')$, and a similar argument proves that $\mathcal{B}(\Delta') \subseteq \mathcal{B}(\Delta)$. Thus, for $\Delta, \Delta' \in \mathcal{D}$, $\mathcal{B}(\Delta)$ and $\mathcal{B}(\Delta')$ are either equal or disjoint. It may happen that $\mathcal{B}(\Delta) = \mathcal{B}(\Delta')$ for distinct K_{B_0} -orbits Δ, Δ' , but we have just proved that, in this case, Δ and Δ' lie in different $\langle \sigma \rangle$ -orbits. Thus $\langle \sigma \rangle$ permutes the set $\{\mathcal{B}(\Delta) \mid \Delta \in \mathcal{D}\}$ with all orbits of length k . Suppose that $\langle \sigma \rangle$ has m orbits in this set, and suppose, without loss of generality, that $\mathcal{B}(\Delta_1), \dots, \mathcal{B}(\Delta_m)$ are representatives of these m orbits. Define $Q_1(B_0) := \mathcal{B}(\Delta_1) \cup \dots \cup \mathcal{B}(\Delta_m)$ and, for $2 \leq i \leq k$, set $Q_i(B_0) := (Q_1(B_0))^{\sigma^{i-1}}$, and set $\mathcal{Q}(B_0) := \{Q_i(B_0) \mid 1 \leq i \leq k\}$. Then $\langle \sigma \rangle$ is transitive on $\mathcal{Q}(B_0)$, and it follows from Lemma 2.3 that $(M^{\mathcal{B}}, G^{\mathcal{B}}, \mathcal{B}, \mathcal{Q})$ is a cyclic k -TOD, where $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ with $Q_i = \{(B_0, B)^g \mid B \in Q_i(B_0), g \in G\}$ (as defined before Lemma 2.3). \square

6. EXPLICIT CONSTRUCTION FOR CYCLIC TODS

For a cyclic k -TOD $(M, G, \Omega, \mathcal{P})$, we may assume by Lemma 2.2 that M is normal in G and therefore that there exists $\sigma \in G \setminus M$ such that σ normalises M and $\langle \sigma \rangle$ acts transitively on \mathcal{P} . The following is a consequence of Proposition 3.3, and gives a criterion for a transitive permutation group M to give rise to a cyclic TOD.

Lemma 6.1. *Let M be a transitive permutation group on Ω , and let*

$$\sigma \in \mathbf{N}_{\text{Sym}(\Omega)}(M) \quad \text{and} \quad G = \langle M, \sigma \rangle < \text{Sym}(\Omega).$$

Then there exists a partition \mathcal{P} of $\Omega^{(2)}$ such that

- (i) *$(M, G, \Omega, \mathcal{P})$ is a k -TOD if and only if k divides the size of each $\langle \sigma \rangle$ -orbit on $\text{Orbl}(M, \Omega)$;*
- (ii) *$(M, G, \Omega, \mathcal{P})$ is a symmetric k -TOD if and only if k divides the size of each $\langle \sigma \rangle$ -orbit on $\text{Orbl}(M, \Omega)$, and for each $\Delta \in \text{Orbl}(M, \Omega)$ and each $\tau \in \langle \sigma \rangle \setminus \langle \sigma^k \rangle$, $\Delta^\tau \neq \Delta^*$, where Δ^* is the paired orbital of Δ .*

Proof. Part (i) follows immediately from Proposition 3.3. For part (ii), the extra condition is that, for each symmetric $\langle \sigma \rangle$ -orbit Q in $\text{Orbl}(M, \Omega)$, there exists a symmetric k -part partition $\mathcal{B}(Q)$ with appropriate $\langle \sigma \rangle$ -action. Now Q is a symmetric $\langle \sigma \rangle$ -orbit if and only if $Q = Q^*$, that is, $\Delta^* \in Q$ whenever $\Delta \in Q$. We require that each part $B \in \mathcal{B}(Q)$ should be symmetric, that is, $\Delta^* \in B$ whenever $\Delta \in B$. Since $\mathcal{B}(Q) = \{B^{\sigma^i} \mid 0 \leq i < k\}$, this condition implies that, for each $\Delta \in \text{Orbl}(M, \Omega)$, $\Delta^{\sigma^i} \neq \Delta^*$ for $i = 1, \dots, k - 1$ (either because Δ and Δ^* lie in different $\langle \sigma \rangle$ -orbits, or because they lie in the same block B of $\mathcal{B}(Q)$, where $\Delta, \Delta^* \in Q$). Conversely, the condition $\Delta^{\sigma^i} \neq \Delta^*$ for all Δ and for $i = 1, \dots, k - 1$ enables us to define a G -invariant partition with all parts symmetric, for each symmetric G -orbit Q . (Take $B = \{\Delta^{\sigma^{ki}} \mid i \geq 0\}$.) □

If an element $\sigma \in \mathbf{N}_{\text{Sym}(\Omega)}(M)$ has p -power order with p prime, and σ fixes no element of $\text{Orbl}(M, \Omega)$, then p divides the size of every $\langle \sigma \rangle$ -orbit on $\text{Orbl}(M, \Omega)$. Suppose that τ is an automorphism of M that normalises a point stabiliser in M . Then τ is induced by some element $\tilde{\tau} \in \text{Sym}(\Omega)$ such that $\tilde{\tau} \in \mathbf{N}_{\text{Sym}(\Omega)}(M)$. Thus $\tilde{\tau}$ acts on $\text{Orbl}(M, \Omega)$. By Lemma 6.1, we have a criterion for a transitive permutation group to have a TOD of prime index in terms of certain special automorphisms of the group.

Corollary 6.2. *A transitive group M acting on Ω has a TOD of prime index p if and only if M has an automorphism τ of p -power order such that τ normalises some point stabiliser in M , and $\tilde{\tau}$ fixes no element of $\text{Orbl}(M, \Omega)$, where $\tilde{\tau}$ is as above.*

To conclude this section, we construct explicit examples of cyclic TODs for all values of n and k occurring in Theorem 1.1.

Construction 6.3. Let $n = r_1^{d_1} \dots r_m^{d_m}$, where the r_i are distinct primes, $d_i \geq 1$ and $m \geq 1$. Let $M = \mathbb{Z}_{r_1}^{d_1} \times \dots \times \mathbb{Z}_{r_m}^{d_m}$. Then $\text{Aut}(M) = \prod_{i=1}^m \text{GL}(d_i, r_i) \geq \prod_{i=1}^m \text{GL}(1, r_i^{d_i})$.

(a) Suppose that $r_i^{d_i} \equiv 1 \pmod{k}$ for each i . Choose $\sigma_i \in \text{GL}(1, r_i^{d_i})$ such that σ_i is of order k . Let $\sigma = \sigma_1 \dots \sigma_m \in \text{Aut}(M)$. Take $P_1(1)$ to consist of one representative of each of the $\langle \sigma \rangle$ -orbits in $M \setminus \{1\}$, and set

$$P_1 = \{(x, y) \mid x, y \in M, xy^{-1} \in P_1(1)\}, \text{ and}$$

$$\mathcal{P} = \{P_1^{\sigma^i} \mid 0 \leq i < k\}.$$

(b) Suppose that $r_i^{d_i} \equiv 1 \pmod{2k}$ whenever r_i is odd, and $r_i^{d_i} \equiv 1 \pmod{k}$ if $r_i = 2$. For each i , if r_i is odd, then let $\sigma_i \in \text{GL}(1, r_i^{d_i})$ have order $2k$, and if $r_i = 2$,

let $\sigma_i \in \text{GL}(1, r_i^{d_i})$ have order k . Let $\sigma = \sigma_1\sigma_2 \dots \sigma_r \in \text{Aut}(M)$. Take $P_1(1)$ to consist of one representative of each of the $\langle \sigma \rangle$ -orbits in $M \setminus \{1\}$, and set

$$P_1 = \{(x, y) \mid x, y \in M, xy^{-1} \in P_1(1)\}, \text{ and}$$

$$\mathcal{P} = \{P_1^{\sigma^i} \mid 0 \leq i < k\}.$$

The next lemma shows that these constructions produce cyclic k -TODs.

- Lemma 6.4.** (i) For $M, \sigma, \mathcal{P}, n, k$ as in Construction 6.3 (a), $(M, \langle M, \sigma \rangle, M, \mathcal{P})$ is a cyclic k -TOD of degree n ; if in addition k is odd, then $(M, \langle M, \sigma \rangle, M, \mathcal{P})$ is a symmetric cyclic k -TOD of degree n .
- (ii) For $M, \sigma, \mathcal{P}, n, k$ as in Construction 6.3 (b), $(M, \langle M, \sigma \rangle, M, \mathcal{P})$ is a symmetric cyclic k -TOD of degree n .

Proof. The cyclic group $\text{GL}(1, r_i^{d_i})$ is regular on $\mathbb{Z}_{r_i}^{d_i} \setminus \{1\}$, and hence $\prod_{i=1}^m \text{GL}(1, r_i^{d_i})$ is semiregular on $M \setminus \{1\}$. Thus in both constructions (given in Construction 6.3) $\langle \sigma \rangle$ acts semiregularly on $M \setminus \{1\}$, and so $\langle \sigma \rangle$ acts on $\text{Orbl}(M, M)$ with all orbits of length k in part (i) and length $2k$ or k in part (ii). It follows from Lemma 6.1 that in both cases, $(M, \langle M, \sigma \rangle, M, \mathcal{P})$ is a cyclic k -TOD.

Now suppose that $M, \sigma, \mathcal{P}, n, k$ are as in Construction 6.3 (a) with k odd, or as in Construction 6.3 (b). We show that the condition of Lemma 6.1 (ii) holds. Suppose to the contrary that $\Delta^{\sigma^i} = \Delta^*$, where $\Delta \in \text{Orbl}(M, M)$ and $\sigma^i \notin \langle \sigma^k \rangle$. Then in particular $k \nmid i$. Now $\Delta(1) = \{x\}$ for some $x \in M \setminus \{1\}$, and $\Delta^*(1) = \{x^{-1}\}$. Since σ^i fixes 1, $x^{\sigma^i} = x^{-1}$, and so $x^{\sigma^{2i}} = x$. Now all $\langle \sigma \rangle$ -orbits in $M \setminus \{1\}$ have length $2k$ or k . Since $x^{\sigma^{2i}} = x$, we have that $k|2i$. Since $k \nmid i$, k must be even. Thus the proof of part (i) is complete. Continuing with the proof of part (ii), by Lemma 2.5, n is odd. It then follows from the definition of σ that all $\langle \sigma \rangle$ -orbits in $M \setminus \{1\}$ have length $2k$. Hence $2k|2i$ and $k|i$, which is a contradiction. So $\Delta^{\sigma^i} \neq \Delta^*$, satisfying Lemma 6.1 (ii). □

7. DEGREES AND INDICES

In this section, we prove a relation between index k and degree n for a cyclic k -TOD of degree n , and complete the proof of Theorem 1.1. First, we show that if $(M, G, \Omega, \mathcal{P})$ is a cyclic TOD, then some Sylow subgroups induce cyclic TODs. For a prime r , by $r^a \parallel n$ we mean that r^a is the highest power of r dividing n .

Lemma 7.1. Let $(M, G, \Omega, \mathcal{P})$ be a cyclic p^e -TOD of degree n with M normal in G , where p is a prime and $G = \langle M, \sigma \rangle$ for some element σ of p -power order. Let r be a prime such that $r^d \parallel n$ with $d > 0$, and let R be a Sylow r -subgroup of M . Then there exist an element $\sigma_0 \in G$, an orbit Σ of R in Ω , and a partition \mathcal{Q} of $\Sigma^{(2)}$ such that $(R, \langle R, \sigma_0 \rangle, \Sigma, \mathcal{Q})$ is a cyclic p^e -TOD of degree r^d . If in addition $(M, G, \Omega, \mathcal{P})$ is symmetric, then $(R, \langle R, \sigma_0 \rangle, \Sigma, \mathcal{Q})$ is also symmetric.

Proof. There exists $l \geq d$ such that $r^l = |R|$, so that r^{l-d} is the order of a Sylow r -subgroup of M_ω , where $\omega \in \Omega$. Let τ be an arbitrary element of $G \setminus \langle M, \sigma^p \rangle$. Then $\langle M, \sigma^p, \tau \rangle = G$. By Lemma 5.1, τ fixes a unique point in Ω , and we denote the point by ω_τ . Thus $\tau \in G_{\omega_\tau}$, and it follows that τ normalises the point-stabilizer M_{ω_τ} .

Let $\rho \in G \setminus \langle M, \sigma^p \rangle$, and let S be a Sylow r -subgroup of M_{ω_ρ} . Then $|S| = r^{l-d}$, and S^ρ is also a Sylow r -subgroup of M_{ω_ρ} . By Sylow's theorem, $S^\rho = S^g$ for some $g \in M_{\omega_\rho}$. Thus $S^{\rho'} = S$, where $\rho' := \rho g^{-1} \in G \setminus \langle M, \sigma^p \rangle$. Since both ρ and g fix ω_ρ ,

we have that ρ' fixes ω_ρ . Hence by Lemma 5.1, $\omega_{\rho'} = \omega_\rho$. Thus S is an r -subgroup of M for which there exists an element $\rho' \in G \setminus \langle M, \sigma^p \rangle$ such that $S^{\rho'} = S$ and $|S_{\omega_{\rho'}}| = r^{l-d}$.

Assume now that X is maximal by inclusion among r -subgroups of M such that there exists $\tau \in G \setminus \langle M, \sigma^p \rangle$ satisfying

$$X^\tau = X \quad \text{and} \quad |X_{\omega_\tau}| = r^{l-d}.$$

Let $N = \mathbf{N}_M(X)$, and let Y be a Sylow r -subgroup of N . Note that N has no fixed points in Ω , for if S is a Sylow r -subgroup of M containing X , then $\mathbf{N}_S(X)$ properly contains X , and hence has no fixed points in Ω . Now τ normalises N , $X \trianglelefteq Y$, and Y^τ is a Sylow r -subgroup of N . Thus $Y^\tau = Y^x$ for some $x \in N$, and so $Y^{\hat{\tau}} = Y$, where $\hat{\tau} := \tau x^{-1}$. Let $\Delta = \omega_\tau^N$, the orbit of N containing ω_τ . Since τ fixes ω_τ and normalises N , we have that τ fixes Δ setwise. Thus, in particular, $\Delta^{\hat{\tau}} = \Delta^{\tau x^{-1}} = \Delta$. By Lemma 3.5, $(N, \langle N, \tau \rangle, \Delta, \mathcal{P}')$ is a cyclic p^e -TOD for some partition \mathcal{P}' of $\Delta^{(2)}$. Thus by Lemma 5.1, $\hat{\tau}$ fixes a point of Δ , and so $\omega_{\hat{\tau}} \in \Delta$. Since N is transitive on Δ , we have $\omega_{\hat{\tau}}^y = \omega_{\hat{\tau}}$ for some $y \in N$, and thus $X_{\omega_\tau}^y \leq N_{\omega_\tau}^y = N_{\omega_{\hat{\tau}}}$. So $X_{\omega_\tau}^y \leq X^y \cap N_{\omega_{\hat{\tau}}} = X \cap N_{\omega_{\hat{\tau}}} = X_{\omega_{\hat{\tau}}}$. Hence $r^{l-d} = |X_{\omega_\tau}| = |X_{\omega_\tau}^y| \leq |X_{\omega_{\hat{\tau}}}| \leq |Y_{\omega_{\hat{\tau}}}|$. However, since $r^{l-d} \parallel |M_{\omega_{\hat{\tau}}}|$, we have that $|Y_{\omega_{\hat{\tau}}}| = r^{l-d}$. Therefore, Y is an r -subgroup of M such that $Y^{\hat{\tau}} = Y$ and $|Y_{\omega_{\hat{\tau}}}| = r^{l-d}$. By the maximality of X , we have $Y = X$. Thus X is a Sylow r -subgroup of $\mathbf{N}_M(X)$, and hence X is a Sylow r -subgroup of M . In particular, $|X| = r^l$ and X has no fixed points in Ω .

By Sylow's theorem, $R = X^g$ for some $g \in M$. Let $\tau_0 = \tau^g$. Then by Lemma 3.5 applied to R and $K = \langle R, \tau_0 \rangle$, there exists a p^e -TOD $(R, \langle R, \tau_0 \rangle, \Sigma, \mathcal{Q})$, where $\Sigma = \omega_{\tau_0}^R$ and $|\Sigma| = |R : R_{\omega_{\tau_0}}| = r^d$. If in addition $(M, \langle M, \sigma \rangle, \Omega, \mathcal{P})$ is symmetric, then by Lemma 3.5, $(R, \langle R, \tau_0 \rangle, \Sigma, \mathcal{Q})$ is symmetric. \square

We now give a relation between the degrees and indices of cyclic k -TODs in the case where k is a prime-power.

Lemma 7.2. *Let p be a prime, and let $n = r_1^{d_1} r_2^{d_2} \dots r_m^{d_m}$ where the r_i are distinct primes.*

- (i) *If there exists a p^e -TOD of degree n , then $r_i^{d_i} \equiv 1 \pmod{p^e}$, for all r_i .*
- (ii) *If there exists a symmetric p^e -TOD of degree n , then $r_i^{d_i} \equiv 1 \pmod{2p^e}$ for all odd r_i , and $r_i^{d_i} \equiv 1 \pmod{p^e}$ if $r_i = 2$.*

Proof. Let $(M, G, \Omega, \mathcal{P})$ be a cyclic p^e -TOD of degree n , where $G = \langle M, \sigma \rangle$ for some element $\sigma \in G$ of p -power order. Let r be a prime such that $r^d \parallel n$ with $d > 0$. Let R be a Sylow r -subgroup of M . By Lemma 7.1, there exists a p^e -TOD $(R, \langle R, \tau \rangle, \Sigma, \mathcal{Q})$ of degree r^d . Thus, by Lemma 2.5, $r^d \equiv 1 \pmod{p^e}$. If in addition $(M, \langle M, \sigma \rangle, \Omega, \mathcal{P})$ is symmetric, then by Lemma 7.1, $(R, \langle R, \tau \rangle, \Sigma, \mathcal{Q})$ is symmetric. Thus by Lemma 2.5, either $r^d \equiv 1 \pmod{2p^e}$, or $r = 2$ and $2^d \equiv 1 \pmod{p^e}$. \square

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. If k is a prime-power, then Theorem 1.1 follows from Lemmas 6.4 and 7.2. Thus we may assume that k is not a prime-power. Let $(M, G, \Omega, \mathcal{P})$ be a cyclic k -TOD of degree n . Write $k = p_1^{e_1} p_2^{e_2} \dots p_l^{e_l}$, where p_i are distinct primes, $e_i \geq 1$, and $l \geq 2$. Let $\sigma \in G \setminus M$ be such that $\langle \sigma \rangle$ is transitive on \mathcal{P} . Recall that we may take $\sigma \in G_\omega$. Let $k_i = k/p_i^{e_i}$, and let $\sigma_i = \sigma^{k_i}$. Then $\langle \sigma_i \rangle$ acts half-transitively

on \mathcal{P} , and each $\langle \sigma_i \rangle$ -orbit on \mathcal{P} has size $p_i^{e_i}$. Hence $\langle \sigma_i \rangle$ acts on the set of M -orbitals such that each orbit has size divisible by $p_i^{e_i}$. By Lemma 6.1, there exists a cyclic $p_i^{e_i}$ -TOD $(M, \langle M, \sigma_i \rangle, \Omega, \mathcal{P}_i)$ for some partition \mathcal{P}_i of $\Omega^{(2)}$. By Lemma 7.2, if r is a prime and $r^d \parallel |\Omega|$, then $r^d \equiv 1 \pmod{p_i^{e_i}}$. It then follows that $r^d \equiv 1 \pmod{k}$.

Assume further that $(M, G, \Omega, \mathcal{P})$ is symmetric. By Lemma 3.5 (taking $N = M$ and $E = \langle \sigma \rangle \leq G_\omega$), $(M, \langle M, \sigma_i \rangle, \Omega, \mathcal{P}_i)$ is symmetric. By Lemma 7.2, if r is odd and $r^d \parallel |\Omega|$, then $r^d \equiv 1 \pmod{2p_i^{e_i}}$. It then follows that $r^d \equiv 1 \pmod{2k}$ if r is odd.

The converse assertion of Theorem 1.1 follows from Lemma 6.4. \square

REFERENCES

- [1] B. Alspach, J. Morris and V. Vilfred, Self-complementary circulant graphs, *Ars Combinatoria* **53** (1999), 187–191. MR **2001b**:05184
- [2] P. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc* **13** (1981), 1–22. MR **83m**:20008
- [3] N. J. Calkin, P. Erdős and C. A. Tovey, New Ramsey bounds from cyclic graphs of prime order, *Siam J. Discrete Math.* **10** (1997), 381–387. MR **98e**:05078
- [4] V. Chvátal, P. Erdős and Z. Hedrlín, Ramsey’s theorem and self-complementary graphs, *Disc. Math.* **3**(1972), 301–304. MR **47**:1674
- [5] C. R. J. Clapham, A class of self-complementary graphs and lower bounds of some Ramsey numbers, *J. Graph Theory* **3** (1979), 287–289. MR **81d**:05054
- [6] B. Fein, W. M. Kantor and M. Schacher, Relative Brauer groups II. *J. Reine Angew. Math.* **328** (1981), 39–57. MR **83a**:12018
- [7] M. Fried, R. Guralnick and J. Saxl, Schur covers and Carlitz’s conjecture, *Israel J. Math.* **82** (1993), 157–225. MR **94j**:12007
- [8] D. Fronček, A. Rosa and J. Širan, The existence of self-complementary circulant graphs, *European J. Combin.* **17** (1996), 625–628. MR **97d**:05235
- [9] D. Gorenstein, *Finite Simple Groups*, 1982, Plenum Press, New York. MR **84j**:20002
- [10] R. M. Guralnick, C. H. Li, C. E. Praeger and J. Saxl, On orbital partitions and exceptionality of primitive permutation groups, preprint.
- [11] R. M. Guralnick, P. Müller and J. Saxl, The rational function analogue of a question of Schur and exceptionality of permutation representations, Mem. Amer. Math. Soc. (to appear)
- [12] F. Harary, R. W. Robinson and N. C. Wormald, Isomorphic factorisations I: Complete graphs. *Trans. Amer. Math. Soc.* **242** (1978), 243–260. MR **58**:27646a
- [13] F. Harary and R. W. Robinson, Isomorphic factorisations X: unsolved problems, *J. Graph Theory* **9** (1985), 67–86. MR **87a**:05111
- [14] R. Jajcay and C. H. Li, Constructions of self-complementary circulants with no multiplicative isomorphisms, *European J. Combin.* **22** (2001), 1093–1100. MR **2002g**:05098
- [15] C. H. Li, On self-complementary vertex-transitive graphs, *Comm. Algebra* **25** (1997), 3903–3908. MR **98k**:05119
- [16] C. H. Li, T. K. Lim and C. E. Praeger, Homogeneous factorisations of complete graphs with edge-transitive factors, in preparation.
- [17] C. H. Li and C. E. Praeger, Self-complementary vertex-transitive graphs need not be Cayley graphs, *Bull. London Math. Soc.* **33** (2001), 653–661. MR **2002h**:05085
- [18] V. Liskovets and R. Pöschel, Non-Cayley-isomorphic self-complementary circulant graphs, *J. Graph Theory* **34** (2000), 128–141. MR **2001e**:05056
- [19] R. Mathon, On selfcomplementary strongly regular graphs, *Disc. Math.* **69** (1988), 263–281. MR **89d**:05150
- [20] M. Muzychuk, On Sylow subgraphs of vertex-transitive self-complementary graphs, *Bull. London Math. Soc.* **31** (1999), 531–533. MR **2001i**:05093
- [21] W. Peisert, All self-complementary symmetric graphs, *J. Algebra* **240** (2001), 209–229. MR **2002e**:05074
- [22] C. E. Praeger, Finite quasiprimitive graphs, *Surveys in Combinatorics 1997 (London)*, 65–85, London Math. Soc. Lecture Notes series 241, (Cambridge Univ. Press, Cambridge, 1997). MR **99b**:05076

- [23] S. B. Rao, On regular and strongly regular selfcomplementary graphs, *Disc. Math.* **54** (1985), 73-82. MR **86g**:05089
- [24] H. Sachs, Über selbstkomplementäre Graphen, *Publ. Math. Debrecen* **9** (1962), 270-288. MR **27**:1934
- [25] D. A. Suprunenko, Selfcomplementary graphs, *Cybernetics* **21** (1985), 559-567. MR **87g**:05197
- [26] M. Suzuki, *Group Theory I*, (Springer, New York 1982). MR **82k**:20001c
- [27] B. Zelinka, Self-complementary vertex-transitive undirected graphs, *Math. Slovaca* **29** (1979), 91-95. MR **81h**:05116
- [28] H. Zhang, Self-complementary symmetric graphs, *J. Graph Theory* **16** (1992), 1-5. MR **92k**:05061

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