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O.R. Applications

Viability of infeasible portfolio selection problems: A fuzzy approach ☆

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Abstract

This paper deals with fuzzy optimization schemes for managing a portfolio in the framework of risk–return trade-off. Different models coexist to select the best portfolio according to their respective objective functions and many of them are linearly constrained. We are concerned with the infeasible instances of such models. This infeasibility, usually provoked by the conflict between the desired return and the diversification requirements proposed by the investor, can be satisfactorily avoided by using fuzzy linear programming techniques. We propose an algorithm to repair infeasibility and we illustrate its performance on a numerical example. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Fuzzy mathematical programming is developed for treating uncertainty in the setting of optimization problems. Mathematical programming models require much of well-defined and precise data which involve high information costs. However, the use of fuzzy models not only avoids unrealistic modeling but also offers a chance for reducing information costs. Fuzzy sets are used in fuzzy mathematical programming both to define the objective and constraints and also to reflect the aspiration levels given by the decision makers. Numerous books and papers are devoted to introducing and developing fuzzy mathematical programming techniques (see, for instance, [8,3,21]).

In Watada [20], the fuzzy portfolio selection problem has been used to introduce vague goals for the expected return rate and risk. Tanaka and Guo (see [18,19]) use possibility distributions to model uncer-

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tainty in the returns. They identify two possibility distributions – upper and lower – from given possibility degrees to security data. Their approach permits the incorporation of expert knowledge by means of a possibility grade, to reflect the degree of similarity between the future state of stock markets and the state of previous periods. In Inuiguchi and Ramik [5], the portfolio selection problem exemplifies the advantages and disadvantages of different fuzzy mathematical programming approaches.

On the other hand, infeasible instances arise quite frequently in the modeling process. Sometimes the model contains what the modeler intends to reflect, but individually reasonable constraints are globally inconsistent. There are a number of algorithmic methods which assist in the analysis of infeasible mathematical programs. Most of them are designed to identify the portion(s) of the linear program containing the cause of infeasibility [1]. In the context of the portfolio selection models, this kind of method is not specially appealing because for these problems, where the investing rate seeks the minimum risk under a given expected return, infeasibility is provoked by the conflict between the desired return and the diversification requirements given by the decision maker. Hence, the viability of the instance (in the investor's opinion) will depend on the severity of the perturbations that should be made to the model to make it feasible.

An early work by Negoita and Sularia [14] used fuzzy logic to solve the problem of turning a system of linear inequalities without a solution into one with a solution by observing some tolerances given at free terms. In [9] a fuzzy method is proposed to make some linear programs feasible.

In this paper some fuzzy optimization schemes for managing portfolio selection problems efficiently are discussed in the framework of risk–return trade-off. Section 2 is devoted to describing the formulations of some portfolio selection problems. The basic concepts necessary to present our approach for repairing infeasibility are given in Section 3. Then, in Section 4 we develop an interactive fuzzy algorithm to manage infeasible instances of portfolio selection that respects the original will of the investor as much as possible.

Finally, in Section 5 we use some data in a paper by Markowitz (see [18]) to illustrate our method. We show how slight perturbations in the bound-type constraints (portfolio diversification) and in the expected return make the instance feasible.

2. Formulations of portfolio selection problem

The classical portfolio selection problem was formulated by Markowitz in the 1950s as a quadratic programming problem (MV) in which the risk variance is minimized and the investment diversification is treated in computational terms (see [12]):

$$
\text{(MV)} \qquad \begin{aligned} \min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ \text{s.t.} \quad & \sum_{i=1}^{n} E(R_i) x_i \geqslant \rho, \\ & \sum_{i=1}^{n} x_i = 1, \\ & l_i \leqslant x_i \leqslant u_i, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{1}
$$

where x_i represents the percentage of money invested in asset i, R_i is the random variable representing the return of asset i, σ_{ij} is the covariance between returns of asset i and of asset j, and ρ is a parameter representing the minimal rate of return required by an investor. Also, u_j (resp. l_j) is the maximum (minimum) amount of money which can be invested in asset j.

The average vector of returns, and the elements of the covariance matrix over T periods, can be approximated by

$$\begin{aligned}
\widehat{E(R_i)} &= \frac{1}{T}\sum_{k=1}^{T} r_{ik}, \quad i = 1,\ldots,n, \\
\widehat{\sigma_{ij}} &= \frac{1}{T}\sum_{k=1}^{T}(r_{ik} - \widehat{E(R_i)})(r_{jk} - \widehat{E(R_j)}), \quad i,j = 1,\ldots,n,
\end{aligned} \tag{2}$$

where r_{ik} is the realization of the random variable R_i during the period k and is obtainable through historical data.

Sharpe [15] showed that the problem MV can be formulated as a simplified quadratic programming model by using market indices to express the asset returns. It is well known that the portfolio models initiated by Markowitz gave rise to a variety of regression models, including the extensively used CAPM, which was subsequently developed by Sharpe and Lintner (see [16] for instance). Different models coexist to select the best portfolio according to their respective objective functions and many of them are linearly constrained (see for instance [2,4,7,17,22]).

Konno and Yamazaki [6] proposed a linear optimization model for portfolio selection: the L_1 risk model. This measure of risk minimizes the sum of absolute deviations from the averages associated with the x_j choices (LMAD model). By using the same notation as in the (MV) problem we have

$$\begin{aligned}
\text{(LMAD)} \qquad \min \quad & \frac{1}{T}\sum_{k=1}^{T} y_k \\
\text{s.t.} \quad & y_k + \sum_{j=1}^{n}\left(r_{jk} - \widehat{E(R_j)}\right)x_j \geqslant 0, \quad 1 \leqslant k \leqslant T, \\
& y_k - \sum_{j=1}^{n}(r_{jk} - \widehat{E(R_j)})x_j \geqslant 0, \quad 1 \leqslant k \leqslant T, \\
& \sum_{i=1}^{n}\widehat{E(R_j)}x_j \geqslant \rho M_0, \\
& \sum_{i=1}^{n} x_j = M_0, \\
& l_j \leqslant x_j \leqslant u_j, \quad 1 \leqslant j \leqslant n,
\end{aligned} \tag{3}$$

where M_0 is the total fund. It has been shown in [6] that the MV and LMAD models usually generate similar portfolios. If the expected return, ρ, belongs to

$$\left[\min_{1\leqslant i\leqslant n}\{\widehat{E(R_i)}\},\ \max_{1\leqslant i\leqslant n}\{\widehat{E(R_i)}\}\right]$$

and there are no diversification conditions (bound-type constraints on the assets) it is well known that the (MV) and (LMAD) problems are always feasible. But in many real life problems, when attempting to reflect the diversification proposed by the investor, infeasibility appears.

We then propose to use a specialization of a fuzzy method that we have developed to repair infeasibility in linearly constrained problems [10]. Our version takes into account the special structure of the constraints in the linear and quadratic programming models for the portfolio selection problem, in such a way that the diversification and the expected return conditions are considered as soft constraints, while the remaining are hard constraints.

3. Analysis of infeasibility of the portfolio selection problem

Let us denote the linearly constrained portfolio selection problem by

$$(\mathscr{P}) \qquad \min \quad \{f(x) : \mathscr{A}_1 x \geqslant \mathscr{B}^1,\ \mathscr{A}_2 x \leqslant \mathscr{B}^2,\ \mathscr{A}_3 x = \mathscr{B}^3,\ x \geqslant 0\}, \tag{4}$$

where $x \in \mathbb{R}^n$, $\mathscr{A}_i \in M_{p_i \times n}(\mathbb{R})$, $\mathscr{B}^i \in \mathbb{R}^{p_i}$ $(i = 1, 2, 3)$ and f is a real-valued function in $\mathbb{R}^n$, and assume that we have an infeasible instance of $\mathscr{P}$. A conceptual assumption underlying our approach is that the problem $\mathscr{P}$ is correctly formulated. In particular, we assume that the investor diversification conditions are "logical" in the sense that the set of bounds $\{l_i, u_i,\ i = 1, \ldots, n\}$ verifies that

$$\left\{ x \in \mathbb{R}^n : l_i \leqslant x_i \leqslant u_i,\ 1 \leqslant i \leqslant n,\ \sum_{i=1}^{n} x_i = 1 \right\} \neq \emptyset. \tag{5}$$

It is clear that this assumption does not imply any loss of generality: every reasonable investor would provide a set of bounds like this in a natural way.

The infeasibility arises when the expected benefit is not compatible with the diversification constraints. Then, to remove or change one single constraint (that associated to ρ, for instance) is not specially appealing.

Our approach to repairing these infeasible instances is a fuzzy method in the conceptual framework by Zimmermann [21]. As we apply our method when the current instance is not viable, we do not have any solution of the problem for performing a parametric or post-optimality analysis.

Let us denote by X the set of hard constraints, i.e.

$$X := \left\{ x \in \mathbb{R}^n : H_1 x \geqslant \beta^1,\ H_2 x \leqslant \beta^2,\ \mathscr{A}_3 x = \mathscr{B}^3,\ x \geqslant 0 \right\}, \tag{6}$$

where $H_i \in M_{(p_i - m_i) \times n}(\mathbb{R})$, $\beta^i \in \mathbb{R}^{p_i - m_i}$ $(i = 1, 2)$ and $\mathscr{A}_3 \in M_{p_3 \times n}(\mathbb{R})$, $\mathscr{B}^3 \in \mathbb{R}^{p_3}$. We assume that X is a non-empty set. Then, $\mathscr{P}$ can be rewritten as

$$\begin{aligned} (\mathscr{P}) \qquad \min \quad & f(x) \\ \text{s.t.} \quad & A_1 x \geqslant b^1, \\ & A_2 x \leqslant b^2, \\ & x \in X, \end{aligned} \tag{7}$$

where $A_1 \in M_{m_1 \times n}(\mathbb{R})$, $A_2 \in M_{m_2 \times n}(\mathbb{R})$, $b^1 \in \mathbb{R}^{m_1}$, $b^2 \in \mathbb{R}^{m_2}$.

In order to attain feasibility in $\mathscr{P}$ these constraints will be verified at a "certain degree". Therefore, assume that we are given m_1 greater or equal type fuzzy constraints $\underset{\sim}{B}_1, \underset{\sim}{B}_2, \ldots, \underset{\sim}{B}_{m_1}$, and m_2 less or equal type fuzzy constraints $\underset{\sim}{C}_1, \underset{\sim}{C}_2, \ldots, \underset{\sim}{C}_{m_2}$, $m_i \leqslant p_i$ $(i = 1, 2)$. We denote by $\mu_{\underset{\sim}{B}_i}(x)$ and $\mu_{\underset{\sim}{C}_j}(x)$ the membership functions for $\underset{\sim}{B}_i$ and $\underset{\sim}{C}_j$, respectively. Now, we need to fuzzify the concept of feasibility (see [10]):

Definition. We define the fuzzy set of feasible solutions of $\mathscr{P}$ as $\underset{\sim}{E} := \{(x, \mu_{\underset{\sim}{E}}(x)),\ x \in X\}$, where

$$\mu_{\underset{\sim}{E}}(x) = \min\{\mu_{\underset{\sim}{B}_1}(x), \ldots, \mu_{\underset{\sim}{B}_{m_1}}(x), \mu_{\underset{\sim}{C}_1}(x), \ldots, \mu_{\underset{\sim}{C}_{m_2}}(x)\}. \tag{8}$$

Notice that the fuzzy set $\underset{\sim}{E}$ is non-empty, so the following fuzzy program is consistent:

$$\text{(FP)} \qquad \text{Find}\{x \in X : A_1 x \gtrsim b^1,\ A_2 x \lesssim b^2\}. \tag{9}$$

The solution with the highest degree of membership $\underset{\sim}{E}$ is given by

$$x_{\max} = \arg(\max_{x \in X} \min_{i,j} \{\mu_{\underset{\sim}{B}_i}(x), \mu_{\underset{\sim}{C}_j}(x)\}). \tag{10}$$

If the investor accepts the solution $x_{\max}$ we have a viable portfolio selection.

In this approach the objective function does not intervene in the proposed selection, which is associated with a degree of investor satisfaction with respect to the constraints. If a decision maker also has aspiration levels for the risk, a symmetric fuzzy multi-objective formulation that no longer distinguishes between objectives and constraints can be used [20].

4. Fuzzy portfolio selection

Under the assumption that the decision maker has goal values both for the expected return rate (ρ) and the diversification conditions (l_i, u_i) before solving the portfolio selection, and that their proposal leads to an infeasible instance of the problem, we propose using an interactive system to solve the problem of getting a viable portfolio selection. Firstly, we apply the scheme introduced in Section 3 which associates certain related membership functions to the soft inequality constraints while leaving the hard constraints unchanged. At the second stage investor opinion is used to select a portfolio in the framework of trade-off analysis.

If we use $(Ax)_i \gtrsim b_i$ to represent a fuzzy inequality relation, for $x \in \mathbb{R}^n$, the degree of satisfaction of this ith constraint is

$$\mu_{\underset{\sim}{B}_i}(x) = \begin{cases} 0 & \text{if } (Ax)_i < b_i - r_i, \\ g_i((Ax)_i) & \text{if } b_i - r_i \leqslant (Ax)_i < b_i, \\ 1 & \text{if } (Ax)_i \geqslant b_i, \end{cases}$$

where r_i is the maximum "violation" allowed for the ith constraint and it is usually assumed that $g_i(x) \in [0, 1]$ is such that the higher the violation of the constraint, the lower the value of $g_i(x)$. Analogously for the fuzzy relation $(Ax)_j \lesssim b_j$ the membership function is

$$\mu_{\underset{\sim}{C}_j}(x) = \begin{cases} 0 & \text{if } (Ax)_j > b_j + s_j, \\ g_j((Ax)_j) & \text{if } b_j \leqslant (Ax)_j < b_j + s_j, \\ 1 & \text{if } (Ax)_j \leqslant b_j. \end{cases}$$

Several types of the function $g_i(x)(g_j(x))$ can be used to construct the membership functions for the fuzzy constraints. Then, the tolerances (r_i, s_j) should be provided by the decision maker or alternatively determined by the analyst. We compute the tolerances by means of the shadow prices of the solution of the Phase I problem (PI) associated to the infeasible instance $\mathscr{P}$:

$$
\begin{array}{llllll}
\text{(PI)} & \min & \sum_{i=1}^{m_1} a_i & & & \\
& \text{s.t.} & A_1x - I^{m_1}h + I^{m_1}a & = & b^1 & (w), \\
& & A_2x + I^{m_2}h' & = & b^2 & (\pi), \\
& & H_1x & \geqslant & \beta^1 & (q), \\
& & H_2x & \leqslant & \beta^2 & (p), \\
& & \mathscr{A}_3x & = & \mathscr{B}^3 & (y), \\
& & h, h', a \geqslant 0, \quad x \in X, & & &
\end{array}
\tag{11}
$$

where the h's are the slack variables, and the a's are the artificial ones. The dual variables associated to the soft constraints are w and π and the dual variables corresponding to the hard constraints are denoted by q, p and y.

Let z^* be the optimal value of (PI) that provides the sum of infeasibilities and let $(w^*, \pi^*, q^*, p^*, y^*)$ be an optimal solution of its dual. The tolerances are defined by

$$
r_i = \begin{cases} 0 & \text{if } w_i^* = 0, \\ z^*/w_i^* & \text{if } w_i^* > 0, \end{cases} \qquad s_j = \begin{cases} 0 & \text{if } \pi_j^* = 0, \\ -z^*/\pi_j^* & \text{if } \pi_j^* < 0. \end{cases} \tag{12}
$$

Now, for each soft constraint, we have both the goal of the investor and the tolerance. We can therefore construct their membership functions, except for those with null tolerance for which it is known that they do not need to be perturbed (see [13] for instance).

Concerning membership functions, for the soft diversification constraints, $x_i \geqslant l_i$ or $x_j \leqslant u_j$ for some i, j we propose to use linear functions, i.e.,

$$
\mu_{\underset{\sim}{B}_i}(x) = g_i((Ax)_i) = 1 - \frac{l_i - x_i}{r_i} \quad \text{for } l_i - r_i < x_i < l_i
$$

and

$$
\mu_{\underset{\sim}{C}_j}(x) = g_j((Ax)_j) = 1 - \frac{x_j - u_j}{s_j} \quad \text{for } u_j < x_j < u_j + s_j.
$$

Let us denote the expected return constraint, $\sum_{i=1}^{n} \widehat{E(R_i)} x_i \gtrsim \rho$, by $E(x) \gtrsim \rho$. We will also use a linear membership function for this constraint. In case the decision maker should want to assign more preference levels to the expected return than the diversification goals, another type of membership function should be used, but this must be done at the second stage of our algorithm.

4.1. The conceptual algorithm

Let us propose a procedure to make an infeasible instance of $\mathscr{P}$ viable. At the first stage we determine a feasible portfolio selection, then we ask for investor opinion in order to modify our proposal or not.

Stage I: *Viability*

- **Step 1:** Classifying the constraints.

Firstly we determine which the soft and the hard constraints are in the linearly constrained problem $\mathscr{P}$. They could be different depending on the model that we are considering (MV, LMAD, etc.).

- **Step 2:** Computing the tolerances and defining the membership functions.

Let z^* be the optimal value of (PI), and let $(w_1^*, \dots, w_{m_1}^*, \pi_1^*, \dots, \pi_{m_2}^*)$ be an optimal solution of its dual. Calculate the tolerances and the vectors:

$$R = (r_1, r_2, \dots, r_{m_1}) \quad \text{and} \quad S = (s_1, s_2, \dots, s_{m_2}), \tag{13}$$

where $r_i = 0$ (resp. $s_j = 0$) if the corresponding dual price is null. As we will show in Theorem 1, if the user considers it acceptable to modify the RHS terms of the soft constraints by at least $1/k$ in the direction (R, S), where k is the number of non-null components of vector (R, S), it makes sense to construct the membership functions in order to obtain a viable instance. We consider a linear membership function for every soft diversification constraint with non-null shadow price.

- **Step 3:** Auxiliary linear problem.

In order to determine the "best solution", i.e. the solution with the highest degree of satisfaction in $\underset{\sim}{E}$ (8), we solve the following auxiliary crisp problem:

$$\begin{aligned} \text{(AP)} \qquad & \min \quad \phi \\ & \text{s.t.} \quad A_1 x + \phi R \geqslant b^1, \\ & \qquad\ \ A_2 x - \phi S \leqslant b^2, \\ & \qquad\ \ x \in X. \end{aligned} \tag{14}$$

Let $(x_{\max}, \phi_{\min})$ be the optimal solution of (AP), the degree of satisfaction of $x_{\max}$ is $\lambda^* = 1 - \phi^*$, where $\phi^* = \min\{\phi_{\min}, 1\}$.

- **Step 4:** Solving the fuzzy reformulation.

We apply a parametric programming formulation for the resource set to our models [8], and then we obtain a set of solutions depending on the values of a parameter ϕ, i.e.

$$x(\phi) \in \operatorname{Argmin}\{f(x) : A_1 x \geqslant b_1 - \phi R,\ A_2 x \leqslant b_2 + \phi S,\ x \in X\} \quad \text{for } \phi^* \leqslant \phi \leqslant 1.$$

Stage II: *Investor opinion*

- **Option 1:** Satisfying solution.

We can ask the decision maker for the values of the parameter ϕ which they feel reasonable and solve the corresponding crisp problem (MV or LMAD). Notice that for any solution x_0^*, obtained for a given ϕ_o, its degree of satisfaction is $\lambda_o = 1 - \phi_o$.

- **Option 2:** Non-satisfying solution.

(a) Because of the constraints

Let us suppose that at the original instance we had the condition $x_{i_0} \leqslant u_{i_0}$, and that the new values proposed at the end of Stage I, $[u_{i_0}, \tilde{u}_{i_0}]$, do not seem appropriate to the decision maker, then we suggest that they to choose a quantity u'_{i_0}. We then fix $x_{i_0} = u'_{i_0}$, and substitute it into the model. These arguments are clearly extensible to the case of more than one constraint being involved. We should tell the investor that these new modifications could provoke other disagreements.

(b) Because of the risk

In this case, we think that the most appropriate is a fuzzy multi-objective decision approach [11].

The following results justify that the algorithm is well defined.

Theorem 1. *If the problem* (AP) *is feasible, then* $\phi_{\min} \geqslant 1/k$, *where* k *denotes the number of non-null components of the vector* (R, S).

Proof. It suffices to construct a feasible solution for the dual of (AP) with objective value $1/k$, because this provides us with a lower bound for the optimal value of (AP).

Let us consider the dual problem of (PI):

$$\begin{array}{lll}\text{(DPI)} & \text{Max} & wb^1 + \pi b^2 + q\beta^1 + p\beta^2 + y\mathscr{B}^3 \\ & \text{s.t.} & wA_1 + \pi A_2 + qH_1 + pH_2 + y\mathscr{A}_3 \leqslant 0, \\ & & 0 \leqslant w_i \leqslant 1, \quad \pi_j \leqslant 0 \quad \forall i,j, \\ & & q \geqslant 0, \quad p \leqslant 0,\end{array}$$

whose optimal solution was denoted by $(w^*, \pi^*, q^*, p^*, y^*)$ and its optimal value as z^*.

And now let us consider the dual problem of (AP):

$$\begin{array}{lll}\text{(DAP)} & \text{Max} & \gamma b^1 + \tau b^2 + g\beta^1 + t\beta^2 + v\mathscr{B}^3 \\ & \text{s.t.} & \gamma A_1 + \tau A_2 + gH_1 + tH_2 + v\mathscr{A}_3 \leqslant 0, \\ & & \gamma R - \tau S \leqslant 1, \\ & & \gamma \geqslant 0, \quad \tau \leqslant 0, \\ & & g \geqslant 0, \quad t \leqslant 0.\end{array}$$

Fixing

$$\gamma = \frac{w^*}{kz^*}, \quad \tau = \frac{\pi^*}{kz^*}, \quad g = \frac{q^*}{kz^*}, \quad t = \frac{p^*}{kz^*}, \quad v = \frac{y^*}{kz^*},$$

we have a feasible solution for (DAP) with objective value $1/k$.

Corollary 1. *Given an infeasible instance of the portfolio selection problem* ($\mathscr{P}$), *our algorithm obtains a solution with satisfaction degree* $\lambda^* \in [0, 1 - (1/k)]$.

Proof. Let us see, first, that the problem (AP) is feasible. We have assumed that the diversification conditions provided by the investor are consistent (5). Thus infeasibility is provoked by the conflict between the expected return and the diversification constraints (both in MV and in LMAD model).

The shadow price associated to the expected return constraint must be non-null because otherwise it would mean that it does not belong to any irreducible inconsistent set. Therefore, by taking ϕ sufficiently large we can reduce the expected value enough to achieve feasibility. By applying Theorem 1 we have $\phi_{\min} \geqslant 1/k$, and $\phi^* = \min\{\phi_{\min}, 1\}$.

Finally, by definition of $\lambda^* = 1 - \phi^*$, we have the result.

5. Numerical example

In order to show the performance of our method, let us use the set of historical data shown in Table 1 introduced by Markowitz in 1959 (see [18]). The columns 2–10 represent American Tobacco, A.T.&T.,

Table 1
Returns of nine securities

Year	Am.T.	A.T.&T.	U.S.S.	G.M.	A.T.&S.	C.C.	Bdn	Frstn.	S.S.
1937	−0.305	−0.173	−0.318	−0.477	−0.457	−0.065	−0.319	−0.4	−0.435
1938	0.513	0.098	0.285	0.714	0.107	0.238	0.076	0.336	0.238
1939	0.055	0.200	−0.047	0.165	−0.424	−0.078	0.381	−0.093	−0.295
1940	−0.126	0.030	0.104	−0.043	−0.189	−0.077	−0.051	−0.090	−0.036
1941	−0.280	−0.183	−0.171	−0.277	0.637	−0.187	0.087	−0.194	−0.240
1942	−0.003	0.067	−0.039	0.476	0.865	0.156	0.262	0.113	0.126
1943	0.428	0.300	0.149	0.225	0.313	0.351	0.341	0.580	0.639
1944	0.192	0.103	0.260	0.290	0.637	0.233	0.227	0.473	0.282
1945	0.446	0.216	0.419	0.216	0.373	0.349	0.352	0.229	0.578
1946	−0.088	−0.046	−0.078	−0.272	−0.037	−0.209	0.153	−0.126	0.289
1947	−0.127	−0.071	0.169	0.144	0.026	0.355	−0.099	0.009	0.184
1948	−0.015	0.056	−0.035	0.107	0.153	−0.231	0.038	0	0.114
1949	0.305	0.038	0.133	0.321	0.067	0.246	0.273	0.223	−0.222
1950	−0.096	0.089	0.732	0.305	0.579	−0.248	0.091	0.650	0.327
1951	0.016	0.090	0.021	0.195	0.040	−0.064	0.054	−0.131	0.333
1952	0.128	0.083	0.131	0.390	0.434	0.079	0.109	0.175	0.062
1953	−0.010	0.035	0.006	−0.072	−0.027	0.067	0.21	−0.084	−0.048
1954	0.154	0.176	0.908	0.715	0.469	0.077	0.112	0.756	0.185

United States Steel, General Motors, Atcheson & Topeka & Santa Fe, Coca-Cola, Borden, Firestone and Sharon Steel securities data, respectively.

Let us suppose that an investor wants to allocate one unit of wealth among the nine assets. Their expected return ρ must be greater than or equal to 16.5%. The portfolio must be selected in such a way that the minimum investments in assets 1, 3 and 6 must be of 5%, 7.5% and 7.5%, respectively, and the maximum investments in assets 4 and 5 must be of 33% and 25%, respectively, i.e.

$$l_1 = 0.05, \quad l_3 = l_6 = 0.075, \quad u_4 = 0.33, \quad u_5 = 0.25. \tag{15}$$

So we have an infeasible instance of $\mathscr{P}$. The Phase I linear program associated to $\mathscr{P}$ is

$$\begin{aligned} \text{(PI)} \qquad \min \quad & a_1 + a_3 + a_6 + a'_1 \\ \text{s.t.} \quad & E(x) + a'_1 - h'_1 = \rho, \\ & x_i - h_i + a_i = l_i, \quad i = 1, 3, 6, \\ & x_j + h_j = u_j, \quad j = 4, 5, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x_i, h_i, h'_i, a_i, a'_1 \geqslant 0, \quad i = 1, \dots, n. \end{aligned} \tag{16}$$

The optimal value of (PI) is $z^* = 0.771861 \times 10^{-2}$, and the “dual prices” are

$$\begin{aligned} & w_1 = 1, && w_2 = 0.080111, && w_3 = 0, \\ & w_4 = 0.090944, && \pi_1 = -0.027389, && \pi_2 = -0.052056. \end{aligned}$$

Then, the tolerances that define the vectors R, S in (13) are

$$R = (0.007719, 0.096349, 0, 0.084872),$$
$$S = (0.281815, 0.148276).$$

Table 2
Results of the procedure of viability

Original values			Transformed values
$\rho = 0.165$	$\Rightarrow$	$\tilde{\rho} = \rho - r_1\phi^*$	$= 0.163456$
$l_1 = 0.05$	$\Rightarrow$	$\tilde{l}_1 = 0.05 - r_2\phi^*$	$= 0.030730$
$l_3 = 0.075$	$\Rightarrow$	$\tilde{l}_3 = 0.075 - r_3\phi^*$	$= 0.075$
$l_6 = 0.075$	$\Rightarrow$	$\tilde{l}_6 = 0.075 - r_4\phi^*$	$= 0.058026$
$u_4 = 0.33$	$\Rightarrow$	$\tilde{u}_4 = 0.33 + s_1\phi^*$	$= 0.386363$
$u_5 = 0.25$	$\Rightarrow$	$\tilde{u}_5 = 0.25 + s_2\phi^*$	$= 0.279655$

Table 3
Parametric portfolio selection

ϕ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Return	0.16346	0.16268	0.16191	0.16114	0.16037	0.15960	0.1588	0.15805	0.15728
ObjMV	0.06450	0.05423	0.05003	0.04667	0.04475	0.04320	0.04176	0.04050	0.03970
ObjLMAD	0.21051	0.19371	0.18461	0.17551	0.16979	0.16593	0.16206	0.15849	0.15642
Investment									
x_1	0.03073	0.02110	0.01146	0.00183	0	0	0	0	0
x_2	0	0	0	0	0	0	0	0	0
x_3	0.24523	0.075	0.075	0.075	0.075	0.075	0.075	0.075	0.075
x_4	0.38636	0.38883	0.32279	0.25676	0.21592	0.20902	0.20212	0.19555	0.19064
x_5	0.27966	0.29448	0.30931	0.32414	0.32941	0.31422	0.29903	0.28505	0.27729
x_6	0.05803	0.04954	0.04105	0.03256	0.02408	0.01559	0.00710	0	0
x_7	0	0.17105	0.24039	0.30972	0.35559	0.3862	0.41675	0.44440	0.45706
x_8	0	0	0	0	0	0	0	0	0
x_9	0	0	0	0	0	0	0	0	0

All the linear and non-linear programs in the numerical example have been solved by using the Microsoft Excel spreadsheet. We have developed a computer application with the Visual Basic language to manage all these programs efficiently.

The optimal value of the associated auxiliary problem (14) is $\phi_{\min} = \frac{1}{5}$, the lower bound in Theorem 1. If the investor accepts this proposal, the new RHS values appear in Table 2. These results are valid both for MV and LMAD objectives, because the objective function has not intervened until now.

Table 3 shows the portfolio selection obtained by applying the parametric programming formulation, both for MV and LMAD models.

5.1. Stage II of the algorithm

Let us suppose that the investor wants to reduce the risk value associated to the portfolio with satisfaction $\lambda^* = 0.8$ by approximately 10%, i.e. "desired risk" (DS) $\simeq 0.058$. However, it is more important to them not to decrease (too much) the expected benefits. As the risk value associated with $\lambda = 0.7$ is lower than DS, a portfolio selection with a satisfaction level greater than 0.7 and lower than 0.8 could be determined by considering a grid for $\lambda \in [0.7, 0.8]$. As the investor considers that it is "more important" to attain the expected benefits constraint than the remaining constraints, we can consider the risk as a soft constraint with a non-linear membership function as (17).

We calculate the tolerances analogously in Step 2 (it is not exactly the same because in this case we have a non-linear constraint). We add to (PI) the following constraint: $V(x) + h^* = \text{DS}$, where $V(x) =$

Table 4
Tolerances of the soft constraints

RHS prices	Shadow prices		Tolerances
DS = 0.058049	−0.226865	⇒	0.036476
$\rho = 0.165$	1.0	⇒	0.008275
$l_1 = 0.05$	0.068907	⇒	0.120092
$l_3 = 0.075$	0.0	⇒	0.0
$l_6 = 0.075$	0.074182	⇒	0.111553
$u_4 = 0.33$	−0.023036	⇒	0.359230
$u_5 = 0.25$	−0.049085	⇒	0.168588

Table 5
Portfolio selection

Investment								
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0.026548	0	0.161655	0.400153	0.282923	0.053215	0.075506	0	0

$\sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{\sigma_{ij}}x_i x_j$ and h^* is a non-negative slack variable. The new shadow prices and tolerances are included in Table 4.

In order to introduce the decision maker's level of preference for the risk constraint we use an exponential membership function for this goal, i.e.

$$\mu_{\underset{\sim}{V}}(x) = g(V(x)) = \frac{1-\exp(\frac{-k(V^- - V(x))}{\text{tol}})}{1-\mathrm{e}^{-k}} \tag{17}$$

for $\text{DS} < V(x) < V^- := \text{DS} + \text{tol}$, where $\text{tol} = 0.036476$. We take $k = -5$, for instance, and solve the crisp problem:

$$\begin{aligned} (\text{mAP}) \qquad \max \quad & \alpha \\ \text{s.t.} \quad & V(x) - \frac{\text{tol}}{k}\ln(1-\alpha(1-exp^{-k})) \leqslant V^-, \\ & A_1 x + (1-\alpha)R \geqslant b^1, \\ & A_2 x - (1-\alpha)S \leqslant b^2, \\ & x \in X. \end{aligned} \tag{18}$$

Let (x^*, α^*) be the optimal solution of this non-linear programming problem. The degree of satisfaction of x^* is α^*. Table 5 shows the optimal portfolio selection associated to $\alpha^* = 0.804713$, where $V(x^*) = 0.059623$ and the expected return is 0.163384. We have reduced the risk by 8% and the expected benefit by less than 0.1%.

6. Conclusions

On some occasions, the conditions imposed by an investor to select the optimal portfolio give rise to infeasible instances of the corresponding mathematical model. It does not necessarily mean that the decision maker has no experience and is not able to think of a coherent set of conditions. It may happen that a certain instance is feasible while a slightly different one is not. It is very likely that the conditions in this "new" feasible instance would also be valid and reasonable for the investor.

Under this assumption we have developed our procedure. We construct a new set of constraints which is quite similar to the original one (according to investor preference) and we ask the decision maker if they feel it is reasonable. If the answer is YES the optimal portfolio selection can be determined.

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