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Abstract: We provide a lower bound construction showing that the union of unit balls in $\mathbb{R}^3$ has quadratic complexity, even if they all contain the origin. This settles a conjecture of Sharir.

Key-words: Computational geometry, Union of balls, Geometric example

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Une union de boules unités d'intersection non vide a une complexité quadratique

Résumé : Nous proposons une construction d'un ensemble de boules de $\mathbb{R}^3$ dont l'union a un nombre quadratique de faces et d'arêtes. Ces boules ont la particularité d'avoir le même rayon et d'avoir une intersection non vide. Cette note démontre une conjecture énoncé par M. Sharir.

Mots-clés : Géométrie algorithmique, Union de boules, Exemple géométrique

1 Introduction

The union of a set of n balls in $\mathbb{R}^3$ has quadratic complexity $\Theta(n^2)$, even if they all have the same radius. All the already known constructions have balls scattered around, however, and Sharir posed the problem whether a quadratic complexity could be achieved if all the balls (of same radius) contained the origin.

In this note, we show a construction of n unit balls, all containing the origin, whose union has complexity $\Theta(n^2)$. As a trivial observation, we observe that the centers are arbitrarily close to the origin in our construction. In fact, if the centers are forced to be at least pairwise ε apart, for some constant $\varepsilon > 0$, then no more than $O(\frac{1}{\varepsilon^3})$ can meet in a single point, and hence the union has complexity at most $O(\frac{1}{\varepsilon^3}n) = O_\varepsilon(n)$. It is an interesting open question what a condition should be so that the union have subquadratic complexity and yet the balls have arbitrarily close centers.

By contrast, the *intersection* of n balls can have quadratic complexity if their radii are not constrained, but the complexity is linear if all the radii are the same [2]. Similarly, the convex hull of n balls can have also quadratic complexity [1], but that complexity is linear if they all have the same radius.

2 Construction

Let m and k be any integers. We define two families of unit balls: the first consists of k unit balls whose centers lie on a small vertical segment; the second consists of m unit balls whose centers lie on a small circle under the segment. (See Figure 3.) We show below that their union has quadratic $O(km)$ complexity.

The balls $B_1 \ldots B_k$. We denote by $B(p,r)$ the ball centered at p and of radius r. Let $n = k + m$ and P_i denote the point of coordinates $(0, 0, (i - 1)/n^4)$, and $B_i = B(P_i, 1)$, for $i = 1, \ldots, k$. It is clear that the boundary of $\cup_{1 \le i \le k} B_i$ consists of two hemispheres belonging to B_1 and B_k linked by a narrow cylinder of height less than $k/n^4 \le 1/n^3$. This cylinder contains all the circles $\partial B_i \cap \partial B_{i+1}$ for $i = 1, \ldots, k-1$. (See Figure 1.)

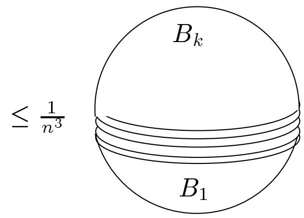

Figure 1: The union $\cup_{1\leq i\leq k} B_i$.

The balls $B_{k+1} \ldots B_{k+m}$. Let R be the point of coordinates $(x, 0, z)$ with

$$x = \frac{2n^2 - 4}{n^4}, \qquad z = -\frac{2n^2 - 4}{n^3}.$$

(Any values satisfying the constraints $P_k H < 1$ in (1) and $\ell < \frac{2}{n}$ in (2) below would do.) We define θ as the rotation around the z-axis of angle $2\pi/m$, and for each $j = 1, \ldots, m$, $R_{k+j} = \theta^{j-1}(R)$ and $B_{k+j} = B(R_{k+j}, 1)$.

3 Analysis

By our choice of x and z, we prove below that the boundaries of B_{k+1} and of the union $\cup_{i=1}^{k} B_i$ depicted in Figure 1 meet along a curve γ which satisfies the two claims below. The situation is depicted on Figure 2.

Claim 1 *The curve γ intersects all the balls B_i for $i = 0, \ldots, k$, .*

Claim 2 *The portion of γ which does not belong to B_1 (equivalently, which belongs to the union $\cup_{i=2}^{k} B_i$) is contained in an angular sector of angle at most $2\pi/m$.*

From claim 2, we conclude that the portion of γ which does not belong to B_1 is contained in the boundary of the union of the $n = k + m$ balls, From claim 1, we conclude that the portion of γ which does not belong to B_1 has complexity $\Omega(k)$. From claim 2, that it is contained in a small angular sector, hence appears completely on the boundary of the union of the $n = k+m$ balls, and it is replicated m times, once for each of the balls B_j, $j = 1, \ldots, m$. It

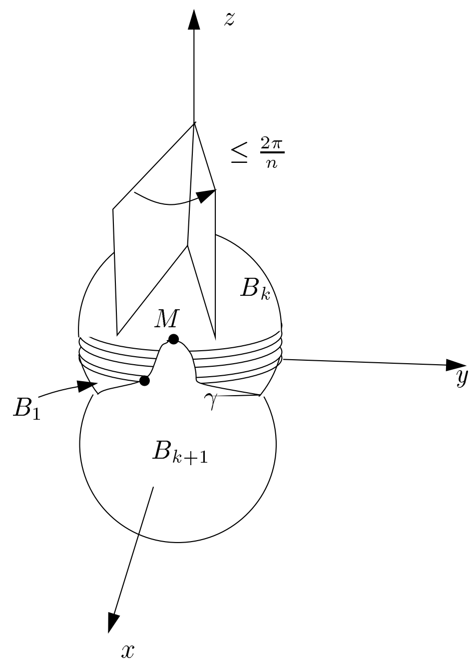

Figure 2: The union $\bigcup_{1\leq i\leq k} B_i \cup B_{k+1}$. The curve γ consists of a portion that belongs to B_1 and of another portion which is contained in a dihedral sector of angle less than π/m.

follows that the union of all the balls B_i for $i = 1, \ldots, k+m$ has quadratic complexity $\Omega(km)$. Moreover, all the balls contain the origin. The union of the n balls is depicted on Figure 3.

The proofs involve only elementary geometry and trigonometry. The situation is depicted in Figure 4 and 5. Figure 4 depicts a section in the xz-plane of the spheres ∂B_i and ∂B_{k+1} and the point M, the highest point of intersection of the bounding spheres. The point M is also depicted on Figure 2.

Proof of Claim 1. It suffices to prove that M is higher than P_k, since then γ extends higher than P_k as well and passes through M by symmetry. The lowest point of γ belongs to B_1 and is clearly below the origin. The two facts together prove that γ must intersect all the balls between B_1 and B_k.

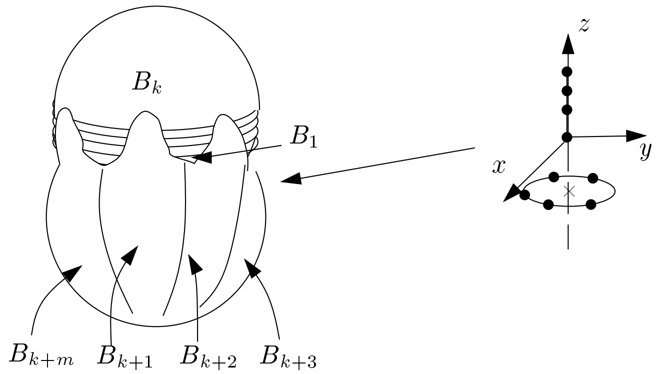

Figure 3: The union $\cup_{1\le i\le k+m} B_i$. On the right, a blow-up of the centers.

Let H be the point in the xz-plane on the median bisector of R and P_k, with same z-ordinate as P_k. (See Figure 4.) In order to prove that M is higher than P_k, it suffices to prove that H belongs to B_k, since then M is farther along the bisector. The two triangles QP_kH and KRP_k have equal angles, hence they are similar. It follows that

$$P_kH = P_kR\frac{P_kQ}{RK} = \frac{P_kR^2}{2RK} = \frac{x^2 + (z - z_k)^2}{2x}, \tag{1}$$

where $z_k = \frac{k-1}{n^4}$. For x and z as given in the construction, we have

$$P_kH = 1/16\,\frac{-40\,n^4 - 15\,n^2 + 68 + 16\,n^6 - 16\,n^3 + 28\,n}{n^4\,(n^2 - 2)}$$

which is smaller than 1 for $n \ge 2$.

Proof of Claim 2. It is easy to see that the intersection of γ and a ball B_i $(2 \le i \le k)$ consists of at most two arcs of circle, any of which is monotone in angular coordinates around the z-axis, and that any such arc is entirely above the plane $z = 0$. Hence the intersections of γ with the xy-plane belong to B_1 and B_{k+1}. It suffices to show that these intersections are at a distance ℓ at most $\frac{2}{n} \le \sin\frac{\pi}{m}$ from the x-axis. (See Figure 5.)

In the xy-plane section, B_1 is a unit circle, and B_{k+1} is a circle of radius $r = \sqrt{1 - z^2}$ and center R' of coordinates $(x, 0)$. (Recall that the center of

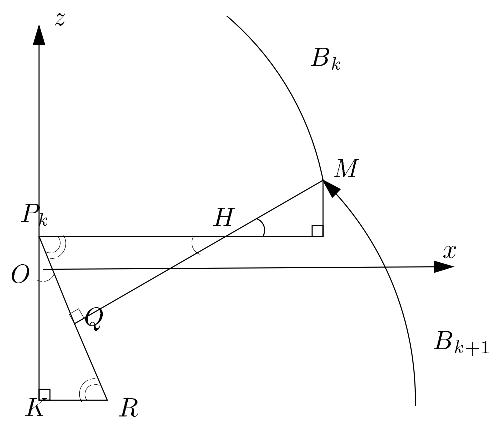

Figure 4: Figure for Claim 1.

B_{k+1} has coordinates $(x, 0, z)$.) Hence ℓ is the height of a triangle with base x and sides 1 and $r < 1$. It is elementary to compute that

$$\ell = \sqrt{1 - \left(\frac{z^2 + x^2}{2x}\right)^2}. \tag{2}$$

For our choice or x and z, this yields

$$\ell = \sqrt{\frac{2\,n^6 + 3\,n^4 - 4\,n^2 - 4}{n^8}}$$

which is smaller than $2/n$ for $n \geq 2$.

Acknowledgments. Thanks to Micha Sharir for pointing out the problem the us. It was also pointed out that Alon Efrat might have a construction which leads to a quadratic lower bound as well. We have derived our construction independently.

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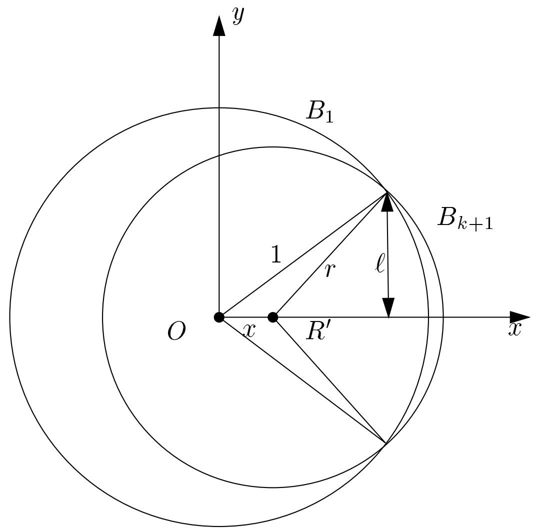

Figure 5: Figure for Claim 2.

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