Monotonicity and Relative Scope Entailments[1]

first version, comments welcome!

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Abstract

This paper explores the hypothesis that simple monotonicity properties of quantifiers in natural language determine to a large extent the entailment relations between their wide/narrow scope readings. We prove that the *disjunctive normal form* of upward monotone quantifiers using principal ultrafilters correlates with whether an object narrow scope reading entails an object wide scope reading. This result naturally extends the familiar entailment relations between ∃∀ and ∀∃ quantification in first order logic into arbitrary "finitely based" upward monotone determiners (over possibly infinite models), which are precisely defined.

Given a simple sentence of the form *Subject-Verb-Object*, we are interested in the logical relations between the *object narrow scope* (ONS) and the *object wide scope* (OWS) readings of the sentence. In [4], Zimmermann (1993) fully characterizes the class of "scopeless" object quantifiers – those for which the ONS and OWS readings are equivalent for any subject. Zimmermann shows that this class is closely related to the class of (principal) ultrafilters (names). In [3], Westerståhl (1996) fully characterizes the class of "self-commuting" quantifiers, i.e. the quantifiers Q for which ONS and OWS readings are equivalent when Q is substituted for both subject and object. However, as far as we know, the more general problem of characterizing (possibly one-way) entailment relations between ONS and OWS readings has not been given serious attention.

Global determiners (see [2]) are functors that map any non-empty domain E to a binary relation over $\wp(E)$. Any set $Q_E \subseteq \wp(E)$ is called a *generalized quantifier* (GQ) *over* E. Thus, a determiner D_E over E maps any $A \subseteq E$ to the generalized quantifier $D_E(A)$ over E.

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Let Q_1 and Q_2 be the GQs over E that the subject and object respectively denote in a given model. The ONS and OWS readings of the sentence in this model with respect to a binary relation $R \subseteq E \times E$, are defined using the following polyadic GQs over $E \times E$:

$$(1) \quad \begin{array}{lcll} Q_1\text{-}Q_2(R) & \stackrel{def}{=} & Q_1(\{x \in E : Q_2(\{y \in E : R(y)(x)\})\}) & \text{(ONS reading)} \\ Q_1{\sim}Q_2(R) & \stackrel{def}{=} & Q_2(\{y \in E : Q_1(\{x \in E : R(y)(x)\})\}) & \text{(OWS reading)} \end{array}$$

Let D_1 and D_2 be global determiners that correspond to the subject and object determiners respectively. The polyadic determiners D_1-D_2 and $D_1{\sim}D_2$, which give rise to ONS and OWS readings respectively, are defined as ternary relations between $A, B \subseteq E$ and $R \subseteq E \times E$.

$$(2) \quad \begin{array}{lcl} D_1\text{-}D_2(A)(B)(R) & \stackrel{def}{=} & ((D_1(A))\text{-}(D_2(B))(R) \\ D_1{\sim}D_2(A)(B)(R) & \stackrel{def}{=} & ((D_1(A)){\sim}(D_2(B))(R) \end{array}$$

A quantifier Q_E is called *upward* (*downward*) *monotone* if it is closed under supersets (subsets). A global determiner D is called *upward* (*downward*) *right monotone* if for every $A \subseteq E$, the quantifier $D_E(A)$ is upward (downward) monotone. Symmetrically, D is called *upward* (*downward*) *left monotone* if for every $A \subseteq E$, the quantifier $\{B \subseteq E : D(B)(A)\}$ is upward (downward) monotone. We use the following abbreviations:

$Q_E \in \text{MON}\uparrow/\text{MON}\downarrow$: Q_E is upward/downward monotone

$D \in \text{MON}\uparrow/\text{MON}\downarrow$: D is upward/downward right monotone

$D \in \uparrow\text{MON}/\downarrow\text{MON}$: D is upward/downward left monotone

We would like to characterize whether the relation D_1-D_2 is contained in the relation $D_1{\sim}D_2$. When D_1 is *every* (*some*) and D_2 is *some* (*every*), it is well-known that the answer is negative (positive) respectively. For instance, the ONS reading of the sentence *some student saw every teacher* entails, but is not entailed by, its OWS reading. We show that in fact, in sentences with upward monotone subjects and objects, the *existential* determiner is the basis for the class of *subject* determiners that guarantee entailment from the ONS reading to the OWS reading. Symmetrically, for upward monotone subjects and objects, the *universal* determiner is the basis for the class of *object* determiners that guarantee entailment from the ONS reading to the OWS reading.

We use the fact (cf. [1]) that any upward monotone quantifier Q_E can be represented as a union of intersections of principal ultrafilters.

Fact 1 *Let Q_E be an upward monotone GQ over E. Then $Q_E = \cup_{N \in M} \cap_{x \in N}$*

I_x, *for some subset* M *of* $\wp(E)$, *where* I_y *is the principal ultrafilter* $\{A \subseteq E : y \in A\}$ *generated by* $y \in E$.

Proof Let M be Q_E. If $A \in Q_E$, then clearly $A \in \cap_{x \in A} I_x$, thus $Q_E \subseteq \cup_{N \in Q_E} \cap_{x \in N} I_x$. In the other direction, if $A \in \cup_{N \in Q_E} \cap_{x \in N} I_x$, then there is $N \in Q_E$ s.t. $A \in \cap_{x \in N} I_x$. Thus, there is $N \in Q_E$ s.t. $N \subseteq A$. Due to upward monotonicity of Q_E, we have: $A \in Q_E$. $\square$

We call M the *signature* of a *disjunctive normal form* (DNF) of an upward monotone quantifier. We define a hierarchy of the upward monotone quantifiers by requiring a DNF for a quantifier $Q_E \in$ MON$\uparrow$, with a signature M that satisfies certain conditions. The classes in the hierarchy, with the respective conditions on M that define them, are listed below.

TRIV$_0$:	$M = \emptyset$: Q_E is empty
TRIV$_1$:	$\emptyset \in M$: Q_E is equal to $\wp(E)$
PUF:	$M = \{\{a\}\}$ for some $a \in E$: Q_E is the *principal ultrafilter* I_a *generated by* a;
PUF$_\cap$:	$M = \{A\}$ for some (possibly empty) $A \subseteq E$: Q_E is an intersection of PUFs: the *principal filter* F_A *generated by* A
PUF$_\cup$:	M is a (possibly empty) collection of singletons in $\wp(E)$: Q_E is a *union of PUFs.*

Obviously, the following relations hold between these classes of GQs: PUF $\subset$ PUF$_\cap$ $\subset$ MON$\uparrow$; PUF $\subset$ PUF$_\cup$ $\subset$ MON$\uparrow$; TRIV$_0$ $\subset$ PUF$_\cup$; TRIV$_1$ $\subset$ PUF$_\cap$.

Further, observe the following simple facts.

Fact 2 *A quantifier* Q *is in* PUF$_\cup$ *iff* $Q = \cup_{\{x\} \in Q} I_x$.

Fact 3 *A quantifier* Q *is in* PUF$_\cap$ *iff* $Q = \cap_{x \in \cap Q} I_x$ $(= F_{\cap Q})$.

Consider now the following simple relation between the above hierarchy and scope entailments.

Lemma 4 *Let* $Q_1, Q_2 \subseteq \wp(E)$ *be upward monotone GQs over* E. *If* $Q_1 \in$ PUF$_\cup$ *or* $Q_2 \in$ PUF$_\cap$ *then* Q_1-$Q_2 \subseteq Q_1{\sim}Q_2$.

Proof Assume first that $Q_1 \in$ PUF$_\cup$.
Assume that $R \in Q_1$-Q_2. That is: $Q_1(\{x \in E : Q_2(\{y \in E : R(y)(x)\})\})$. We conclude that $Q_1 \neq \emptyset$. Because Q_1 is in PUF$_\cup$, we have by fact 2:

$Q_1 = \cup_{\{t\} \in Q_1} I_t$. Thus, there exists $\{t\} \in Q_1$ s.t. $\{x \in E : Q_2(\{y \in E : R(y)(x)\})\} \in I_t$. Let $t_0 \in E$ satisfy $\{t_0\} \in Q_1$ and $Q_2(\{y \in E : R(y)(t_0)\})$. But $\{y \in E : R(y)(t_0)\} \subseteq \{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\}$. Hence, from $Q_2 \in$ MON↑ we conclude: $Q_2(\{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\})$. (i)
From $Q_1 \in$ MON↑ we conclude $\{y \in E : \exists\{t\} \in Q_1[R(y)(t)]\} \subseteq \{y \in E : Q_1(\{x \in E : R(y)(x)\})\}$.
Hence, from $Q_2 \in$ MON↑ and (i) we conclude: $Q_2(\{y \in E : Q_1(\{x \in E : R(y)(x)\})\})$.
Therefore, $R \in Q_1{\sim}Q_2$.
We have shown that if $Q_1 \in \mathrm{PUF}_\cup$ then $Q_1\text{-}Q_2 \subseteq Q_1{\sim}Q_2$. The proof for $Q_2 \in \mathrm{PUF}_\cap$ is analogous. □

We use our classification of MON↑ local quantifiers in order to classify MON↑ global determiners as follows. For any global determiner D:

D is $\mathrm{PUF}^1_\cup$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_\cup \cup \mathrm{TRIV}_1$.

D is $\mathrm{PUF}^0_\cap$ iff for all $A \subseteq E$: $D_E(A)$ is in $\mathrm{PUF}_\cap \cup \mathrm{TRIV}_0$.

D is TRIV_0 (TRIV_1) iff for all $A \subseteq E$: $D_E(A)$ is in TRIV_0 (TRIV_1).
When D is TRIV_0 or TRIV_1 we say that D is *trivial*.

D is $\mathrm{TRIV}^\exists_0$ ($\mathrm{TRIV}^\exists_1$) iff there exist $A \subseteq E$ s.t. $D_E(A)$ is in TRIV_0 (TRIV_1).

D is PUF_0 (PUF_1) iff for all $x \in E$: $D_E(\{x\}) = I_x$, while for all $A \subseteq E$ s.t. $|A| \neq 1$: $D_E(A)$ is in TRIV_0 (TRIV_1).
When D is PUF_0 or PUF_1 we say that D is PUF.

Thus, a determiner is called $\mathrm{PUF}^1_\cup$ ($\mathrm{PUF}^0_\cap$) when it generates only $\mathrm{PUF}_\cup$ ($\mathrm{PUF}_\cap$) and trivial quantifiers. Note that a determiner is classified as $\mathrm{PUF}^1_\cup$ ($\mathrm{PUF}^0_\cap$) or TRIV_0 (TRIV_1) according to its behavior on *all* domains and arguments. By contrast, for classifying a determiner as $\mathrm{TRIV}^\exists_0$ ($\mathrm{TRIV}^\exists_1$), it is sufficient to find *one* domain and one argument for which it is TRIV_0 (TRIV_1). The usefulness of both "universal" and "existential" classifications of determiners will be clarified as we go along.

Meanwhile, to illustrate these definitions, consider the following simple facts.

- The determiner *some* is $\mathrm{TRIV}^\exists_0$ and $\mathrm{PUF}^1_\cup$, but neither $\mathrm{TRIV}^\exists_1$ nor $\mathrm{PUF}^0_\cap$;
- The determiner *every* is $\mathrm{TRIV}^\exists_1$ and $\mathrm{PUF}^0_\cap$, but neither $\mathrm{TRIV}^\exists_0$ nor $\mathrm{PUF}^1_\cup$;
- The determiner *some and (in fact) every* is $\mathrm{TRIV}^\exists_0$ and $\mathrm{PUF}^0_\cap$, but neither $\mathrm{TRIV}^\exists_1$ nor $\mathrm{PUF}^1_\cup$;

- The determiner *some or (perhaps even) every* is $\text{TRIV}^{\exists}_{1}$ and $\text{PUF}^{1}_{\cup}$, but neither $\text{TRIV}^{\exists}_{0}$ nor $\text{PUF}^{0}_{\cap}$.

Of course, the determiners *some and/or every* are respectively the intersection/union of the standard relations for the existential and universal determiners.

Our main claim is that this typology of determiners allows us to determine in which cases of MON↑ determiners D_1 and D_2, the ONS reading D_1-D_2 entails (or is entailed by) the OWS reading $D_1{\sim}D_2$. Before proving that, there is one qualification concerning this result that we should explain. We will assume that both D_1 and D_2 are *finitely based*, in a sense that is defined below. This restriction is needed because MON↑ determiners such as *infinitely many* behave with respect to re lative scope entailments differently than MON↑ determiners such as *at least three*. Consider the following examples.

(3) a. Infinitely many students saw John or Mary.

b. At least three students saw John or Mary.

(4) a. Infinitely many students saw at least one of the two students.

b. At least three students saw at least one of the two students.

In (3a), the ONS reading entails the OWS reading: if there are infinitely many students that have the property *saw John or saw Mary*, then either John or Mary has the property *was seen by infinitely many students*. But this is obviously not the case in (3b). A similar contrast is observed between (4a) and (4b), under a Russellian treatment of the definite article. For instance:

(5) $\textbf{at\_least\_one\_of\_the\_n}'(A)(B) = 1 \;\Leftrightarrow\; |A| = n \wedge A \cap B \neq \emptyset$

(6) $\textbf{each\_of\_the\_n}'(A)(B) = 1 \;\Leftrightarrow\; |A| = n \wedge A \subseteq B$

We have seen that MON↑ determiners such as *infinitely many* show scope entailments that are different than those of similar "finite" determiners. Such "infinite" determiners, which are common in the mathematical jargon, are much less common – and have a much less defined meaning – in everyday speech. This is in contrast to more ordinary determiners such *at least three* or *every*, which English speakers use by and large with the same meaning as logicians do. The formal distinction between determiners that is held responsible for this difference is defined as follows.

Definition 1 (FB **quantifiers**) *Let E be a denumerable non-empty domain. A sequence $A_i|_{i=1}^{\infty}$ of subsets of E is called* properly monotone *if $A_i \subset A_{i+1}$ for every $i \geq 1$, or $A_i \supset A_{i+1}$ for every $i \geq 1$.*
Two properly monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ are called mutually monotone *if $A_i \subset B_j$ for all $i, j \geq 1$, or $A_i \supset B_j$ for all $i, j \geq 1$.*
A quantifier Q_E over E is called finitely based (FB) *iff for any two mutually monotone sequences $A_i|_{i=1}^{\infty}$ and $B_j|_{j=1}^{\infty}$ s.t. Q_E is constant on both sequences, Q_E sends both sequences to the same value.*

By "constancy" of a quantifier Q_E on a set $\mathcal{X} \subseteq \wp(E)$, we of course mean: $\mathcal{X} \subseteq Q_E$ or $Q_E \cap \mathcal{X} = \emptyset$. In the first case say we say that Q_E sends $\mathcal{X}$ to 1. In the second case say we say that Q_E sends $\mathcal{X}$ to 0.

The definition of FB determiners is derived from the definition of FB quantifiers.

Definition 2 (FB **determiners**) *A global determiner D is* FB *iff for any denumerable non-empty domain E, $D_E(E)$ is an* FB *quantifier.*

Note that this definition pays attention only to the behavior of D_E on the whole E domain, and does not take into account proper subsets of E. Thus, a determiner such as *all* is provably FB, even though on the domain of natural numbers, the quantifier *all odd natural numbers* is *not* FB. This is in accordance with the intuition that the determiner *all* does not inherently pertain to infinite sets. By contrast, the determiner *all but finitely many* provably maps any infinite domain to a non-FB quantifier, hence it is not FB itself.

Let us consider an example for a pair of FB/non-FB determiners that belong in the same class of the above hierarchy. Consider first the determiner *infinitely many*. Let $\mathbf{N}$ by the set of natural numbers, with $\mathbf{N}_O \subset \mathbf{N}$ the set of odd natural numbers. Consider two sequences $(\mathbf{N}_O \cap [1..2i])|_{i=1}^{\infty}$ – the increasing sequence of sets of odd numbers; and $(\mathbf{N}_O \cup [1..2i])|_{i=1}^{\infty}$ – the unions of the odd numbers with elements in the increasing sequence of sets of even numbers. These two sequences are mutually monotone, but the denotation of *infinitely many natural numbers* on the domain $E = \mathbf{N}$ is constantly false on the first sequence but constantly true on the second sequence. Consequently, the determiner *infinitely many* is *not* FB. It is impossible to find two such sequences for the determiner *at least three*: trivially, for any domain E, the quantifier $\mathbf{at\_least\_3'}_E(E)$ cannot be false over an infinite properly monotone sequence. Consequently, the determiner *at least three* is FB. Note however that, for each of the determiners *infinitely many* and *at least three*, there are quantifiers that the determiner forms that belong in the class MON$\uparrow$

$\setminus(\mathrm{PUF}_\cup \cup \mathrm{PUF}_\cap)$. Hence, both determiners are in the class MON$\uparrow \setminus(\mathrm{PUF}^1_\cup \cup \mathrm{PUF}^0_\cap)$. Some more examples for FB and non-FB determiners are given in table 1. We note without proof that the class of FB determiners is closed under complements and finite intersections and unions.

FB	non-FB
at least three	finitely many
at most three	infinitely many
exactly three	
all	
all but (at least) three	all but finitely many
all but at most three	all but infinitely many

Table 1: FB and non-FB determiners

We observe the following fact about upward monotone FB quantifiers.

Lemma 5 *Let Q be an FB upward monotone quantifier over a denumerable domain E. If $C_1 \supset C_2 \supset \ldots$ is a properly decreasing infinite sequence of sets in Q, then there is a finite set $A \subseteq C_1$ in Q.*

Proof Assume for contradiction that every $A \subseteq C_1$ in Q is infinite. We will show that Q is not FB.
Assume first that there is $A \subseteq C_1$ in Q s.t. $E \setminus A$ is infinite. Let us denote $A = \{a_1, a_2, ...\}$, $E \setminus A = \{e_1, e_2, ...\}$. Consider the following two sequences:

$$X_1 = A;\ \ X_{i+1} = X_i \cup \{e_i\} \text{ for every } i \geq 1.$$

$$Y_1 = \{a_1\};\ \ Y_{j+1} = Y_j \cup \{a_{j+1}\} \text{ for every } j \geq 1.$$

These two infinite sequences are mutually monotone.
For every $i \geq 1$ we have: $X_i \in Q$ (by monotonicity of Q and $X_1 = A \in Q$).
For every $j \geq 1$ we have: $Y_j \notin Q$ (by finiteness of Y_j and $Y_j \subseteq A \subseteq C_1$).
Hence Q is not FB, in contradiction to the assumption that it is.
Assume now that for every $A \subseteq C_1$ in Q: $E \setminus A$ is finite. Consider the following two sequences:

$$X_1 = C_1;\ \ X_{i+1} = X_i \setminus (C_{2i} \setminus C_{2i+1}).$$

$$Y_1 = C_1 \setminus C_2;\ \ Y_{j+1} = Y_j \cup (C_{2j+1} \setminus C_{2j+2}).$$

These two infinite sequences are mutually monotone.
For every $i \geq 1$ we have: $X_i \in Q$, because $C_{2i-1} \in Q$, $C_{2i-1} \subseteq X_i$, and $Q \in \mathrm{MON}\!\uparrow$.
For every $j \geq 1$ we have: $Y_j \subseteq \cup_{k=1}^{\infty}(C_{2k-1} \setminus C_{2k}) \stackrel{def}{=} D$. But $E \setminus D \supseteq \cup_{n=1}^{\infty}(C_{2k} \setminus C_{2k+1})$ is infinite and $D \subseteq C_1$, hence by our assumption: $D \notin Q$. We conclude $Y_j \notin Q$ by monotonicity of Q.
Therefore we proved again that Q is not FB, in contradiction to the assumption that it is. $\square$

For the statement of our main claim, recall the following definitions, which are standard in GQ theory ([2]). For any global determiner D:

D satisfies *extension* (EXT) iff for all $A, B \subseteq E \subseteq E'$: $D_E(A)(B) = D_{E'}(A)(B)$.

D is *isomorphism invariant* (ISOM) iff for all bijections $\pi : E \rightarrow E'$, for all $A, B \subseteq E$: $D_{E'}(\{\pi(x) : x \in A\})(\{\pi(y) : y \in B\}) = D_E(A)(B)$.

D is *conservative* (CONS) iff for all $A, B \subseteq E$: $D_E(A)(B) = D_E(A)(A \cap B)$.

As in other works on GQ theory, we restrict our attention to determiners in natural language that are EXT, ISOM and CONS.

It is now possible to move on to our main claim.

Theorem 6 *Let D_1 and D_2 be two global* MON↑ *determiners that satisfy* FB, EXT *and* CONS. *Then D_1-$D_2 \subseteq D_1{\sim}D_2$ for any domain denumerable non-empty E iff both following conditions hold: (1) D_1 is $\mathrm{PUF}^1_{\cup}$ or D_2 is $\mathrm{PUF}^0_{\cap}$; and (2) D_1 is not $\mathrm{TRIV}^{\exists}_1$ or D_2 is not $\mathrm{TRIV}^{\exists}_0$.*

Proof (if)
We prove that if D_1 is $\mathrm{PUF}^1_{\cup}$ and condition 2 holds then D_1-$D_2 \subseteq D_1{\sim}D_2$. The proof in case that D_2 is $\mathrm{PUF}^0_{\cap}$ is analogous.
If D_1 is not $\mathrm{TRIV}^{\exists}_1$ then for all $A \subseteq E$: $D_{1E}(A) \in \mathrm{PUF}_{\cup}$. Hence, by lemma 4, for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)){\sim}(D_2(B))$. In other words: D_1-$D_2 \subseteq D_1{\sim}D_2$ for any domain E.
Otherwise, D_1 is $\mathrm{TRIV}^{\exists}_1$ and by condition 2: D_2 is not $\mathrm{TRIV}^{\exists}_0$. (i)
For all $A \subseteq E$: $D_{1E}(A) \in \mathrm{PUF}_{\cup} \cup \mathrm{TRIV}_1$. If $D_{1E}(A) \in \mathrm{PUF}_{\cup}$, then again by lemma 4, for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)){\sim}(D_2(B))$.
If $D_{1E}(A) \in \mathrm{TRIV}_1$, then $\{y \in E : D_1(A)(\{x \in E : R(y)(x)\})\} = E$ for any $R \in E \times E$. But for all $B \subseteq E$: $E \in D_{2E}(B)$, because $D_2(B)$ is not TRIV_0 (by (i)) and upward monotone. We conclude that for all $B \subseteq E$: $(D_1(A))$-$(D_2(B)) \subseteq (D_1(A)){\sim}(D_2(B))$.

We have shown that if $D_1 \in \mathrm{PUF}^1_\cup$ and condition 2 holds, then D_1-$D_2 \subseteq D_1{\sim}D_2$. The proof for $D_2 \in \mathrm{PUF}^0_\cap$ is analogous. □

To prove the "only if" direction of theorem 6, we will first prove the following two lemmas, which rely on the FB property.

Lemma 7 *Let D be an* FB *determiner in* $\mathrm{MON}\!\uparrow \setminus \mathrm{PUF}^1_\cup$ *that satisfies* EXT *and* CONS*. Then there are $A \subseteq E$, for which there is $B \in D_E(A)$ s.t. $|B| \geq 2$ and for every $X \subset B$: $X \notin D_E(A)$.*

Proof We first show that there is a domain A s.t. $D_A(A) \in \mathrm{MON}\!\uparrow \setminus (\mathrm{PUF}_\cup \cup \mathrm{TRIV}_1)$. The proof is routine in usages of CONS and EXT. By the assumption that D is not $\mathrm{PUF}^1_\cup$, there are $A \subseteq E$ s.t. $D_E(A) \notin \mathrm{PUF}_\cup \cup \mathrm{TRIV}_1$. By fact 2 and upward monotonicity of $D_E(A)$, there is $B \in D_E(A)$ s.t. for every $x \in B$: $\{x\} \notin D_E(A)$. By conservativity of D: $A \cap B \in D_E(A)$. Because $D_E(A) \in \mathrm{MON}\!\uparrow \setminus \mathrm{TRIV}_1$ we conclude $A \cap B \neq \emptyset$. Because D satisfies EXT: $A \cap B \in D_A(A)$. Hence by assumption on B and extension, for all $x \in A \cap B$: $\{x\} \notin D_A(A)$. By fact 2 and upward monotonicity of $D_A(A)$: $D_A(A)$ is not $\mathrm{PUF}_\cup$. Because $A \cap B \neq \emptyset$, $D_A(A)$ is not TRIV_1 either.
Let us denote $Q = D_A(A)$.
Let us further denote $Q' \overset{def}{=} \{X \in Q : \text{for every } x \in X\text{: } \{x\} \notin Q\}$.
Because $Q \in \mathrm{MON}\!\uparrow \setminus \mathrm{PUF}_\cup$ and fact 2: $Q' \neq \emptyset$.
Because $Q \in \mathrm{MON}\!\uparrow \setminus \mathrm{TRIV}_1$: $\emptyset \notin Q'$.
Assume that there is no properly decreasing infinite sequence in Q'. Then for every $C \in Q'$, there is $B \subseteq C$ s.t. $B \in Q'$ and for every $X \subset B$: $X \notin Q'$. Any such $B \in Q'$ is non-empty and by definition of Q': $|B| \geq 2$. Because $Q' \neq \emptyset$, we proved existence of B as required.
Assume now that there is a properly decreasing infinite sequence $C_1 \supset C_2 \supset \ldots$ in $Q' \subseteq Q$. By lemma 5, there is a finite set $C \subseteq C_1$ in Q. From definition of Q' we conclude $C \in Q'$. Hence, by finiteness of C, there is $B \subseteq C$ s.t. $B \in Q'$ and for every $X \subset B$: $X \notin Q'$. Thus, by $\emptyset \notin Q'$ and definition of Q' we again conclude $|B| \geq 2$. □

Lemma 8 *Let D be an* FB *determiner in* $\mathrm{MON}\!\uparrow \setminus \mathrm{PUF}^0_\cap$ *that satisfies* EXT *and* CONS*. Then there is a domain E and $A \subseteq E$, for which there are $B_1, B_2 \in D_E(A)$ s.t. $B_1 \cap B_2 \notin D_E(A)$.*

Proof For similar considerations as in the proof of lemma 7, we have $D_A(A) \in \mathrm{MON}\!\uparrow \setminus (\mathrm{PUF}_\cap \cup \mathrm{TRIV}_0)$ for some A. Let us denote $Q = D_A(A)$. Assume that there is no properly decreasing infinite sequence in Q. Let us

denote $Q_{min} \stackrel{def}{=} \{A \in Q$: for every $B \subset A$: $B \notin Q\}$. By assumption on Q, we get that for every $A \in Q$ there is $B \subseteq A$ s.t. $B \in Q_{min}$. Therefore, by monotonicity of Q: $Q = \cup_{A \in Q_{min}} F_A$. Because $Q \notin \mathrm{TRIV}_0$: $Q_{min} \neq \emptyset$. Because $Q \notin \mathrm{PUF}_\cap$: $|Q_{min}| \neq 1$. Hence, any two sets $B_1, B_2 \in Q_{min}$ are as required.
Assume now that there is a properly decreasing infinite sequence in Q. Because Q is MON↑ and FB, it follows by lemma 5 that there is a finite set $A_0 \in Q$. Assume for contradiction that for all $B_1, B_2 \in Q$: $B_1 \cap B_2 \in Q$ (assumption (i)). The set $Q_0 \stackrel{def}{=} \{A \cap A_0 : A \in Q\} \subseteq Q$ is finite and non-empty, thus by assumption (i): $\cap Q_0 \in Q$. But $\cap Q_0 = \cap Q$, and therefore $\cap Q \in Q$. By monotonicity of Q and fact 3: Q is in $\mathrm{PUF}_\cap$, in contradiction to our assumption that it is not. We conclude that there are $B_1, B_2 \in Q$ s.t. $B_1 \cap B_2 \notin Q$. □

We can now finally prove the "only if" direction of theorem 6.

Proof of theorem 6 (only if)
Let D_1 and D_2 be two global MON↑ determiners that satisfy FB, EXT and CONS.
We assume that at least one of the two conditions 1 and 2 does not hold, and will show that $D_1\text{-}D_2 \nsubseteq D_1{\sim}D_2$.
Assume first that condition 1 does not hold: D_1 is not $\mathrm{PUF}^1_\cup$ and D_2 is not $\mathrm{PUF}^0_\cap$. According to lemma 7, there are $A_1, B_0 \subseteq E$ s.t. $|B_0| \geq 2$ and $Q_1 \stackrel{def}{=} D_1(A_1)$ on E satisfies: $B_0 \in Q_1$, and for all $X \subset B_0$: $X \notin Q_1$. Let us denote $a_1, a_2 \in B_0$, for arbitrary $a_1 \neq a_2$.
According to lemma 8, there are $A_2, B_1, B_2 \subseteq E'$ s.t. $B_1, B_2 \in Q_2 \stackrel{def}{=} D_2(A_2)$ on E', but $B_1 \cap B_2 \notin Q_2$.
Because D_1 and D_2 satisfy EXT, we can assume without loss of generality that $E = E'$ (otherwise, choose $E'' = E \cup E'$).
Let us define $R \subseteq E \times E$ as follows:

$$R(y)(x) \Leftrightarrow (x = a_1 \wedge y \in B_1) \vee (x \in B_0 \setminus \{a_1\} \wedge y \in B_2).$$

We shall now show that $(A_1, A_2, R) \in D_1\text{-}D_2$: From definition of R, $\forall x \in B_0 : \{y \in E : R(y)(x)\} \in \{B_1, B_2\} \subseteq Q_2$. Hence: $B_0 \subseteq \{x \in E : \{y \in E : R(y)(x)\} \in Q_2\}$. From monotonicity of Q_1 and $B_0 \in Q_1$ we get that $(A_1, A_2, R) \in D_1\text{-}D_2$.
To prove that $(A_1, A_2, R) \notin D_1{\sim}D_2$, we will show that $\{y \in E : \{x \in E : R(y)(x)\} \in Q_1\} = B_1 \cap B_2$, which is sufficient because $B_1 \cap B_2 \notin Q_2$.
Direction "$\supseteq$": We assume $b \in B_1 \cap B_2$. We have for every $x \in E$:

$R(b)(x) \Leftrightarrow x = a_1 \vee x \in B_0 \setminus \{a_1\} \Leftrightarrow x \in B_0$. Hence $\{x : R(b)(x)\} = B_0$. But $B_0 \in Q_1$, hence we conclude $b \in \{y \in E : \{x \in E : R(y)(x)\} \in Q_1\}$.
Direction "$\subseteq$": We assume $b \in \{y \in E : \{x \in E : R(y)(x)\} \in Q_1\}$. Hence, $\{x \in E : R(b)(x)\} \in Q_1$. From definition of R, we conclude $\{x \in E : R(b)(x)\} \subseteq B_0$. From our assumption about minimality of B_0 in Q_1, it follows that $\{x : R(b)(x)\} = B_0$. Especially: $R(b)(a_1)$ and $R(b)(a_2)$. Hence, by definition of R: $b \in B_1 \cap B_2$.
From the assumption that condition 1 does not hold, we have shown $\langle A_1, A_2, R\rangle \in D_1\text{-}D_2 \setminus D_1{\sim}D_2$, which means that $D_1\text{-}D_2 \not\subseteq D_1{\sim}D_2$.

Assume now that condition 2 does not hold: D_1 is $\mathrm{TRIV}_1^{\exists}$ and D_2 is $\mathrm{TRIV}_0^{\exists}$. Let $A \subseteq E'$ be s.t. $D_1(A)$ on E' is $\wp(E')$ and let $B \subseteq E''$ be s.t. $D_2(B)$ on E'' is empty. On $E \overset{def}{=} E' \cup E''$ we have:
$D_1(A) = \wp(E)$, by MON$\uparrow$ and EXT of D_1.
$D_2(B) = \emptyset$, by CONS and EXT of D_2.
Thus, for every $R \in E \times E$:
$((D_1(A))\text{-}(D_2(B)))(R)$ trivially holds; but
$((D_1(A)){\sim}(D_2(B)))(R)$ trivially does not hold.
From the assumption that condition 2 does not hold, we have again shown that $D_1\text{-}D_2 \not\subseteq D_1{\sim}D_2$. $\square$

Theorem 6 characterizes all the FB logical cases of upward right-monotone subject and object determiners that make the ONS reading entail (or be entailed by) the OWS reading. Simple cases like that are when the subject determiner is $\mathrm{PUF}_{\cup}^{1} \setminus \mathrm{TRIV}_1^{\exists}$ or when the object determiner is $\mathrm{PUF}_{\cap}^{0} \setminus \mathrm{TRIV}_0^{\exists}$. That is: when the subject always denotes a $\mathrm{PUF}_{\cup}$ quantifier or the object always denotes a $\mathrm{PUF}_{\cap}$ quantifier. This is the case in the following sentences.

(7) a. Some student saw every/most/at least two teachers.
 b. Every/most/at least two student(s) saw every teacher.

However, to characterize completely the cases of MON$\uparrow$ logical determiners for which the ONS reading entails the OWS reading, we have also considered some more complex cases of global determiners. An example for a member in $\mathrm{PUF}_{\cup}^{1} \cap \mathrm{TRIV}_1^{\exists}$ is the determiner *some or every*. Examples for members in $\mathrm{PUF}_{\cap}^{0} \cap \mathrm{TRIV}_0^{\exists}$ are the determiner *some and every* and the determiner *each of the five* (cf. definition (6)). These determiners show entailments from the ONS reading to the OWS reading in sentences such as the following.

(8) Some or (perhaps even) every student saw some or (perhaps even) every teacher.

(9) a. At least two teachers saw some and (in fact) every student.

b. At least two teachers saw each of the five students.

The complete characterization of scope entailments with MON$\uparrow$ determiners explains why there is no entailment from the ONS reading to the OWS reading in simple cases such as the following.

(10) Every/most/at least two student(s) saw some/most/at least two teacher(s).

Also in more complex cases such as the following, there is no entailment from the ONS reading to the OWS reading, as theorem 6 expects.

(11) Some or (perhaps even) every student saw some teacher.

(12) Every student saw some and (in fact) every teacher.

In both cases, when there are no students and no teachers, the ONS reading is true but the OWS reading is false.

Another result concerns the following fact that is mentioned in [3] about *local* quantifiers. Westerståhl calls two quantifiers Q_1 and Q_2 over E *independent* when Q_1-$Q_2 = Q_1{\sim}Q_2$. Then he makes the following claim.

Proposition 9 (Westerståhl) *Let Q_1 and Q_2 be two quantifiers over E that are* MON$\uparrow$, *non-trivial and* ISOM. *Then Q_1 and Q_2 are independent iff $Q_1 = Q_2 = \mathbf{every}'_E(E)$ $(= \{E\})$ or $Q_1 = Q_2 = \mathbf{some}'_E(E)$ $(= \wp(E) \setminus \emptyset)$.*

When we consider global determiners, we call D_1 and D_2 *independent* if D_1-$D_2 = D_1 \sim D_2$ for any domain E. Theorem 6 entails the following fact about independent determiners. Note the ISOM requirement (as in Westerståhl's proposition), in addition to the requirements in theorem 6.

Corollary 10 *Let D_1 and D_2 be two global* MON$\uparrow$ *determiners that satisfy* FB, EXT, ISOM *and* CONS.
Then D_1 and D_2 are independent iff both of the following conditions hold:

1. *At least one of the following holds: (a) D_1 and D_2 are both* $\mathrm{PUF}^1_{\cup}$; *or (b) D_1 and D_2 are both* $\mathrm{PUF}^0_{\cap}$; *or (c) D_1 is trivial; or (d) D_2 is trivial; or (e) D_1 is* PUF; *or (f) D_2 is* PUF.
2. *At least one of the following holds: (a) Neither D_1 nor D_2 are* $\mathrm{TRIV}^{\exists}_0$; *or (b) Neither D_1 nor D_2 are* $\mathrm{TRIV}^{\exists}_1$.

Proof In the "if" direction, if neither D_1 nor D_2 are trivial or PUF, the claim follows directly from theorem 6. We first assume without loss of generality that D_1 is trivial. If D_1 is TRIV_0, we conclude that D_1-$D_2 = \emptyset$. From condition 2 above we conclude that D_2 is not $\text{TRIV}_1^{\exists}$. From monotonicity of D_2, it follows that for all $A \subseteq E$: $\emptyset \notin D_{2E}(A)$. From triviality of D_1 we know that for all $A \subseteq E$, $R \subseteq E \times E$: $\{y \in E : D_{1E}(A)(\{x \in E : R(y)(x)\})\} = \emptyset$. Thus, $D_1 \sim D_2 = \emptyset$. The proof for the case that D_1 is TRIV_1 is analogous.

We now assume without loss of generality that D_1 is PUF. Let E an arbitrary domain. We shall look at an arbitrary $\langle A, B, R\rangle \in D_{1E}$-$D_{2E}$ and prove that $\langle A, B, R\rangle \in D_{1E} \sim D_{2E}$. If $|A| \neq 1$, then from definition of PUF, $D_1(A)$ is trivial. From PUF we also know that if $D_1(A)$ is TRIV_1 then D_1 is not $\text{TRIV}_1^{\exists}$, and if $D_1(A)$ is TRIV_0 then D_1 is not $\text{TRIV}_0^{\exists}$. As above, we can now see that $\langle A, B, R\rangle \in D_1 \sim D_2$. For $A = \{a\}$, we assumed $D_1(\{a\})(\{x : D_2(B)(\{y : R(y)(x)\})\})$. From PUF, $a \in \{x : D_2(B)(\{y : R(y)(x)\})\}$, therefore $D_2(B)(\{y : R(y)(a)\})$, now from PUF, $D_2(B)(\{y : D_1(\{a\})(\{x : R(y)(x)\})\})$, therefore $\langle \{a\}, B, R\rangle \in D_1 \sim D_2$. The other direction is analogous, thus proving D_1-$D_2 = D_1 \sim D_2$.

In the "only-if" direction, we assume that D_1-$D_2 = D_1 \sim D_2$ and prove that both conditions (1) and (2) above hold. From theorem 6, we conclude that the following four propositions hold:

(i) D_1 is $\text{PUF}_{\cup}^1$ or D_2 is $\text{PUF}_{\cap}^0$ (from D_1-$D_2 \subseteq D_1 \sim D_2$).

(ii) D_2 is $\text{PUF}_{\cup}^1$ or D_1 is $\text{PUF}_{\cap}^0$ (from D_1-$D_2 \supseteq D_1 \sim D_2$).

(iii) D_1 is not $\text{TRIV}_1^{\exists}$ or D_2 is not $\text{TRIV}_0^{\exists}$ (from D_1-$D_2 \subseteq D_1 \sim D_2$).

(iv) D_2 is not $\text{TRIV}_1^{\exists}$ or D_1 is not $\text{TRIV}_0^{\exists}$ (from D_1-$D_2 \supseteq D_1 \sim D_2$).

Assume first for contradiction that condition 2 above does not hold. Thus, D_1 or D_2 are $\text{TRIV}_0^{\exists}$, and D_1 or D_2 are $\text{TRIV}_1^{\exists}$. From propositions (iii) and (iv) above, we conclude that either D_1 or D_2 are neither $\text{TRIV}_0^{\exists}$ nor $\text{TRIV}_1^{\exists}$. But, both D_1 and D_2 are CONS, hence for any $E \neq \emptyset$: $D_{1E}(\emptyset), D_{2E}(\emptyset) \in \{\emptyset, \wp(E)\}$, which contradicts our assumption, hence condition 2 holds.

Assume now for contradiction that condition 1 above does not hold. Thus, either D_1 or D_2 are not $\text{PUF}_{\cup}^1$, either D_1 or D_2 are not $\text{PUF}_{\cap}^0$, and D_1 and D_2 are both neither trivial nor PUF.

From propositions (i) and (ii) above, we conclude that either D_1 or D_2 must be both $\text{PUF}_{\cup}^1$ and $\text{PUF}_{\cap}^0$. Without loss of generality, assume that D_1 is both $\text{PUF}_{\cup}^1$ and $\text{PUF}_{\cap}^0$. By definition of $\text{PUF}_{\cup}^1$ and $\text{PUF}_{\cap}^0$, for every $A \subseteq E : D_{1E}(A) \in \{\emptyset, I_x, \wp(E)\}$ for some $x \in E$.

First, we shall prove that for all $A \subseteq E \subseteq E'$ and for all $y \in E$, if $D_{1E}(A) = I_y^E$, then also $D_{1E'}(A) = I_y^{E'}$. For any $B \in \wp(E')$, if $y \in B$ then also $y \in B \cap E$, therefore $D_{1E}(A)(B \cap E) = 1$, and thus from MON↑ and EXT, $D_{1E'}(A)(B)$. We have shown $I_y^E \subseteq D_{1E'}(A)$. But, from EXT, $D_{1E'}(A)(\emptyset) = D_{1E}(A)(\emptyset) = 0$, thus $D_{1E'}(A)$ is not trivial, therefore $D_{1E'}(A)$ must be $I_y^{E'}$.

If $A = \emptyset$, then from CONS, $D_{1E}(A) \in \{\emptyset, \wp(E)\}$.

If $A = \{x\} \subseteq E$, from CONS, $D_{1E}(\{x\}) \in \{\emptyset, \wp(E), I_x, \overline{I_x}\}$. From MON↑, $D_{1E}(\{x\}) \in \{\emptyset, \wp(E), I_x\}$.

If $|A| \geq 2$, assume $D_{1E}(A) = I_x$. From CONS, $x \in A$. From $|A| \geq 2$, there is $y \in A$ s.t. $y \neq x$. Let π be a permutation of E that swaps x and y and maps any other element of E to itself. From ISOM, $D_{1E}(\pi(A)) = I_y$. But, $D_{1E}(\pi(A)) = D_{1E}(A) = I_x$, hence a contradiction.

From condition 2 proven above, we know that either D_1 is not $\mathrm{TRIV}_0^{\exists}$ or D_1 is not $\mathrm{TRIV}_1^{\exists}$. Therefore, D_1 must be trivial with the same value for all $A \subseteq E$ for which it is trivial. From nontriviality, we know that there is $A \subseteq E$ s.t. $D_{1E}(A) = I_x$ for some x. We have shown that this can occur only for $A = \{x\}$. From EXT, $D_{\{x\}}(\{x\}) = I_x^{\{x\}}$. From the claim above, we know that $D_{E'}(\{x\}) = I_x^{E'}$ for all E' s.t. $x \in E'$. We will now show that $D_E(\{y\}) = I_y^E$ for all E and for all $y \in E$. Let E be some domain and let $y \in E$ be some element. Let E' be a domain s.t. $|E'| = |E|$ and $x \in E'$. Let π be a bijection $E' \to E$ that maps x to y. From ISOM, $D_{1E}(\{y\})(B) \leftrightarrow D_{1E}(\pi(\{x\}))(\pi^{-1}(B)) \leftrightarrow D_{1E'}(\{x\})(\pi^{-1}(B)) \leftrightarrow \pi^{-1}(B) \in I_x \leftrightarrow x \in \pi^{-1}(B) \leftrightarrow y \in B$. Thus, $D_{1E}(\{y\}) = I_y$ for all $y \in E$. We have already shown that for all $A \subseteq E$ s.t. $|A| \neq 1$, $D_{1E}(\{x\})$ is trivial with the same value, thus D_1 must be PUF, contradicting our assumption.

We have shown that both conditions hold, thus proving the corollary. □

Examples for identical D_1 and D_2 that are independent are the following cases: $D_1 = D_2 =$ *some, every, some-or-every, some-and-every.* However, independent determiners do not have to be identical. For instance: *each of the two* and *each of the five* are independent determiners, since according to the Russellian definition in (6), they are both in $\mathrm{PUF}_{\cap}^{0} \setminus \mathrm{TRIV}_1^{\exists}$.

References

[1] E. Keenan and J. Stavi. A semantic characterization of natural language determiners. *Linguistics and Philosophy*, 9:253–326, 1986.

[2] E. Keenan and D. Westerståhl. Generalized quantifiers in linguistics and logic. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*. Elsevier, Amsterdam, 1996.

[3] D. Westerståhl. Self-commuting quantifiers. *The Journal of Symbolic Logic*, 61:212–224, 1996.

[4] T. E. Zimmermann. Scopeless quantifiers and operators. *Journal of Philosophical Logic*, 22:545–561, 1993.