An axiomatic approach to noncompensatory sorting methods in MCDM, II: The general case

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Abstract

This paper is devoted to the study of methods allowing to sort multi-attributed alternatives between several ordered categories. Within a general framework that encompasses most sorting models proposed in the literature, we provide an axiomatic analysis of what we call noncompensatory sorting models, with or without veto effects. These models contain the pessimistic version of ELECTRE TRI as a particular case. Our analysis can be seen as attempt to give a firm axiomatic basis to ELECTRE TRI emphasizing its specific features, i.e., the rather poor information that is used on each attribute.

Keywords: Decision with multiple attributes, Sorting models, Noncompensation, Conjoint measurement, ELECTRE TRI.

Contents

1 Introduction and motivation

In most MCDM models, a recommendation is built using a preference relation comparing alternatives in terms of desirability. Hence in these models, the recommendation is derived on the basis of a *relative evaluation model* as given by the preference relation. This is not always appropriate since, e.g., the best alternatives may not be desirable at all. This calls for evaluation models having a more absolute character. In response to this need, the MCDM community has recently developed a number of techniques designed to sort alternatives between ordered categories defined by norms. Recent reviews of this trend of research can be found in Greco et al. (2002) or Zopounidis and Doumpos (2000a, 2002). Contrary to more traditional approaches based on binary relations, these techniques have often been proposed on a more or less ad hoc basis.

This paper takes a more theoretical view on these sorting techniques. Our approach is inspired by the analysis of Goldstein (1991) who showed how standard conjoint measurement techniques could be extended to deal with ordered partitions of multi-attributed alternatives.

In a companion paper (Bouyssou and Marchant, 2004), we proposed an axiomatic analysis of several sorting models between two categories, concentrating on what we called noncompensatory sorting models. These models contain the pessimistic version of ELECTRE TRI as a particular case. The main aim of this paper is to extend this analysis to the case of an arbitrary number of ordered categories. We refer the reader to Bouyssou and Marchant (2004) for a detailed motivation of such an analysis and its possible implications for the practice of MCDM. This paper mostly concentrates on technical results. It is organized as follows. We introduce our setting and some background material in section 2. Section 3 deals with the case of noncompensatory sorting models in the absence of veto effects. Section 4 extends this analysis to include the possibility of such effects. A final section concludes.

2 Background

2.1 The setting

Let $n \geq 2$ be an integer and $X = X_1 \times X_2 \times \cdots \times X_n$ be a set of objects. Elements $x, y, z, \ldots$ of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. For any nonempty subset J of the set of attributes N, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$).

With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$ we shall simply write X_{-i} and (x_i, y_{-i}).

2.2 Primitives

Let $r \geq 2$ be an integer and define $R = \{1, 2, \ldots, r\}$ and $R^+ = \{2, 3, \ldots, r\}$. Our primitives consist in an r-fold partition $\langle C^1, C^2, \ldots, C^r \rangle$ of the set X (the sets C^k are therefore nonempty and pairwise disjoint; their union is the entire set X). We often abbreviate $\langle C^1, C^2, \ldots, C^r \rangle$ as $\langle C^k \rangle_{k \in R}$. Note that throughout the paper superscripts are used to distinguish between categories and not, unless otherwise specified, to denote exponentiation.

We interpret the partition $\langle C^k \rangle_{k \in R}$ as the result of a sorting model between ordered categories applied to the alternatives in X. We suppose that the ordering of these categories is known beforehand and that they have been labelled in such a way that the desirability of a category increases with its label: the worst category is C^1 and the best one is C^r. Our central aim is to study various models allowing to represent the information contained in $\langle C^k \rangle_{k \in R}$.

Remark 1
The fact that we suppose the ordering of categories is known beforehand is in line with the type of data that is likely to be collected in order to test the conditions that will be introduced below. Furthermore, this does not involve any serious loss of generality.

Indeed, suppose that the ordering of categories is unknown and, consequently, that the categories have been labelled arbitrarily. In such a case, it will be extremely unlikely that the conditions introduced below are satisfied since they implicitly use the fact that categories have been labelled according to their desirability. In this case, we should reformulate our conditions saying that it is possible to label the categories in such a way that these conditions hold. This would clearly imply a much more cumbersome notation and formulation of the conditions with almost no additional insight. Hence, we stick to the framework in which the ordering of the categories is known and categories are labelled accordingly. •

For all $k \in R^+$, we define $C^k_{\geq} = \bigcup_{j=k}^{r} C^j$ and $C^k_{<} = \bigcup_{j=1}^{k-1} C^j$. The set $C^k_{\geq}$ (resp. $C^k_{<}$) is therefore a category grouping all categories that are at least as good as (resp. worse than) than C^k. The $r-1$ twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ for $k \in R^+$ will play an important role in what follows.

We say that an attribute $i \in N$ is influent for $\langle C^k \rangle_{k \in R}$ if there are $x_i, y_i \in X_i$ and $a_{-i} \in X_{-i}$ such that (x_i, a_{-i}) and (y_i, a_{-i}) do not belong to the same category. We say that an attribute is degenerate if it is not influent. Clearly, a degenerate attribute has no influence whatsoever on the sorting of the alternatives and may be suppressed from N. Note however that an attribute that is influent for $\langle C^k \rangle_{k \in R}$ may well be degenerate for some, but not all, of the twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$.

Remark 2
The fact that not all attributes are influent for all twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ will complicate the analysis. The reader willing to have a feeling of the results without entering details is invited to skip the parts linked with the treatment of degenerate attributes. He/she is also invited to devise the much simpler proofs that are available if it is supposed that all attributes are influent for all twofold partitions induced by $\langle C^k \rangle_{k \in R}$. This is a strong hypothesis however. •

2.3 Binary relations

We use a standard vocabulary for binary relations. An equivalence (resp. a weak order, a semiorder) is a reflexive, symmetric and transitive (resp. complete and transitive, complete, Ferrers and semi-transitive) relation. If R is an equivalence on a set A, A/R will denote the set of equivalence classes of R on A.

Following, e.g., Krantz et al. (1971, Chapter 2), we say that B is dense in A for R if, for all $a, b \in A$, $[a \; R \; b$ and $Not[b \; R \; a]] \Rightarrow [a \; R \; c$ and $c \; R \; b$, for some $c \in B]$.

Let R be a binary relation on A. It is well-known (Fishburn, 1970; Krantz et al., 1971) that there is a real-valued function f on A such that, for all $a, b \in A$,

$$a \; R \; b \Leftrightarrow f(a) \geq f(b),$$

if and only if R is a weak order and there is a finite or countably infinite set $B \subseteq A$ that is dense in A for R.

Let R and R' be two weak orders on A. We say that R' *refines* R if, for all $a, b \in A$, $a \; R' \; b \Rightarrow a \; R \; b$. If R' refines R and there is a set B that is dense in A for R', B is also dense in A for R.

2.4 The decomposable sorting model

Goldstein (1991) was the first to suggest the use of conjoint measurement techniques for the analysis of twofold and threefold partitions of a set of multi-attributed alternatives through decomposable models. His analysis was later generalized in Greco et al. (2001) to the general case of r-fold partitions. For the convenience of the reader and because our proofs are somewhat simpler than the one proposed in the above-mentioned papers, we briefly recall here the central points of this analysis.

Consider a measurement model in which, for all $x \in X$ and all $k \in R$,

$$x \in C^k \Leftrightarrow s_k < F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) < s_{k+1}, \tag{1}$$

where $s_1, s_2, \ldots, s_{r+1}$ are real numbers such that $s_1 < s_2 < \ldots < s_{r+1}$, u_i is a real-valued function on X_i and F is a real-valued function on $\prod_{i=1}^n u_i(X_i)$ that is increasing in all its arguments.

Define on each X_i the binary relation $\succsim_i^R$ letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i^R y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i} \text{ and all } k \in R, (y_i, a_{-i}) \in C^k \Rightarrow (x_i, a_{-i}) \in C^k_{\geq}].$$

We use $\succ_i^R$ and $\sim_i^R$ as is usual. By construction, $\succsim_i^R$ is reflexive and transitive. When $x_i \succsim_i^R y_i$ any alternative (x_i, a_{-i}) must belong to a category that is at least as good as the category containing the alternative (y_i, a_{-i}). Hence, $\succsim_i^R$ may be interpreted as an "at least as good as" relation induced on X_i by the partition $\langle C^k \rangle_{k \in R}$. We have:

Lemma 1
For all $k \in R$ and all $x, y \in X$,

1. $[y \in C^k$ *and* $x_i \succsim_i^R y_i] \Rightarrow (x_i, y_{-i}) \in C^k_{\geq}$,
2. $[x_i \sim_i^R y_i$, *for all* $i \in N] \Rightarrow [x \in C^k \Leftrightarrow y \in C^k]$.

Proof
Part 1 is clear form the definition of $\succsim_i^R$. Part 2 follows. □

We say that the partition $\langle C^k \rangle_{k \in R}$ is R-linear on attribute $i \in N$ (condition *linear*$_i^R$) if, for all $x_i, y_i \in X_i$, all $k, \ell \in R$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$\left.\begin{array}{c}(x_i, a_{-i}) \in C^k \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \in C^k_{\geq} \\ \text{or} \\ (x_i, b_{-i}) \in C^\ell_{\geq}\end{array}\right. \tag{$linear_i^R$}$$

We say that $\langle C^k \rangle_{k \in R}$ is R-linear if it is R-linear for all $i \in N$. R-linearity is a condition that was first proposed by Goldstein (1991) (for the case of twofold and threefold partitions) and was generalized in Greco et al. (2001) for the analysis of r-fold partitions.

Remark 3
Observe that, in the expression of *linear*$_i^R$, it is possible to replace the premises $(x_i, a_{-i}) \in C^k$ and $(y_i, b_{-i}) \in C^\ell$ by $(x_i, a_{-i}) \in C^k_{\geq}$ and $(y_i, b_{-i}) \in$

$C^{\ell}_{\geq}$. Indeed, suppose that $(x_i, a_{-i}) \in C^k_{\geq}$ and $(y_i, b_{-i}) \in C^{\ell}_{\geq}$ while $(y_i, a_{-i}) \in C^k_{<}$ and $(x_i, b_{-i}) \in C^{\ell}_{<}$. Since $(x_i, a_{-i}) \in C^k_{\geq}$, we have $(x_i, a_{-i}) \in C^{\alpha}$, for some $\alpha \geq k$. Similarly, we have $(y_i, b_{-i}) \in C^{\beta}$, for me $\beta \geq \ell$. Applying *linear*$_i^R$ to $(x_i, a_{-i}) \in C^{\alpha}$ and $(y_i, b_{-i}) \in C^{\beta}$ leads to either $(y_i, a_{-i}) \in C^{\alpha}_{\geq}$ or $(x_i, b_{-i}) \in C^{\beta}_{\geq}$. Because $\alpha \geq k$ and $\beta \geq \ell$, we know this implies either $(y_i, a_{-i}) \in C^k_{\geq}$ or $(x_i, b_{-i}) \in C^{\ell}_{\geq}$, a contradiction. •

The consequences of R-linearity on attribute $i \in N$ are noted below.

Lemma 2
A partition $\langle C^k \rangle_{k \in R}$ satisfies linear$_i^R$ iff $\succsim_i^R$ is complete.

Proof
The partition $\langle C^k \rangle_{k \in R}$ violates *linear*$_i^R$ on $i \in N$ if and only if, for some $x_i, y_i \in X_i$ and some $a_{-i}, b_{-i} \in X_{-i}$, $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^{\ell}$, $(y_i, a_{-i}) \notin C^k_{\geq}$ and $(x_i, b_{-i}) \notin C^{\ell}_{\geq}$. This is equivalent to saying that $\succsim_i^R$ is not complete. □

This leads to:

Proposition 1 (Goldstein (1991), Greco et al. (2001))
A partition $\langle C^k \rangle_{k \in R}$ has a representation in model (1) *iff it is R-linear and, for all $i \in N$, there is a finite or countably infinite set $X'_i \subseteq X_i$ that is dense in X_i for $\succsim_i^R$.*

Proof
The necessity of R-linearity is easily established. The weak order induced on X_i by u_i refines $\succsim_i^R$, so that model (1) implies that $\succsim_i^R$ has a numerical representation. Hence, there is a finite or countably infinite set $X'_i \subseteq X_i$ that is dense in X_i for $\succsim_i^R$.

Sufficiency. Using lemma 2, we know that $\succsim_i^R$ is a weak order. Since there is a finite or countably infinite set $X'_i \subseteq X_i$ that is dense in X_i for $\succsim_i^R$, there is a real-valued function u_i on X_i such that, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i^R y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i). \tag{2}$$

Consider, on each $i \in N$ any function u_i satisfying (2). Take any $s_1, s_2, \ldots, s_{r+1}$ such that $s_1 < s_2 < \cdots < s_{r+1}$. For all $k \in R$, consider any increasing function ϕ_k mapping $\mathbb{R}$ into (s_k, s_{k+1}). Define F on $\prod_{i=1}^n u_i(X_i)$ letting, for all $x \in X$ and all $k \in R$,

$$F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) = \phi_k \left(\textstyle\sum_{i=1}^n u_i(x_i)\right) \text{ if } x \in C^k.$$

The well-definedness of F follows from part 2 of lemma 1. Its increasingness is easily shown using the definition of u_i and part 1 of lemma 1. □

This result prompts a series of remarks.

1. The uniqueness of the representation of $\langle C^k \rangle_{k \in R}$ in model (1) is quite weak. It can easily be analyzed along the lines sketched in Bouyssou and Marchant (2004).

2. Model (1) contains as particular cases many sorting models that have been proposed in the literature. Notice, in particular, that when F is taken to be a sum, model (1) is nothing but the additive sorting model used in the UTADIS technique (Jacquet-Lagrèze, 1995; Zopounidis and Doumpos, 2000b). As shown below, it also contains the pessimistic version of ELECTRE TRI as a particular case.

3. Using the above result, it is easy to see that the variant of model (1) in which F is only supposed to be nondecreasing in each variable is, in fact, equivalent to model (1).

2.5 ELECTRE TRI

For the ease of future reference, we briefly recall here the main points of the ELECTRE TRI sorting technique. We suppose below that preference and indifference thresholds are equal and that discordance effects occur in an "all or nothing" way. This will allow to keep things simple while preserving what we believe to be the general spirit of the method. Furthermore, for reasons detailed in Bouyssou and Marchant (2004), we restrict our attention to the pessimistic version of the method. We refer the reader to Mousseau et al. (2000), Roy and Bouyssou (1993, ch. 6) or Wei (1992) for more detailed descriptions.

The aim of ELECTRE TRI is to sort alternatives evaluated on several attributes between r ordered categories $C^1, C^2, \dots, C^r$. This is done as follows. For all $k \in R^+$, there is a profile p^k being the lower limit of category C^k and the upper limit of C^{k-1}. Each of these profiles p^k is defined by its evaluations $(p^k_1, p^k_2, \dots, p^k_n)$ on the attributes in N. Define $\widehat{X}_i = X_i \cup \{p^2_i, p^3_i, \dots, p^r_i\}$ and $\widehat{X} = \prod_{i=1}^{N} \widehat{X}_i$.

Let g_i be a real-valued function on $\widehat{X}_i$ such that, for all $x_i, y_i \in \widehat{X}_i$, $g_i(x_i) \geq g_i(y_i)$ implies that x_i is judged at least as good as y_i, i.e. the functions g_i are what is usually called "criteria". Because categories are supposed to be ordered, it seems obvious to require that the definition of the profiles p^k is such that, for all $i \in N$,

$$g_i(p^r_i) \geq g_i(p^{r-1}_i) \geq \cdots \geq g_i(p^2_i).$$

Using these functions g_i, we derive two relations two relations S_i and V_i on $\widehat{X}_i$ letting, for all $x_i, y_i \in \widehat{X}_i$,

$$x_i \; S_i \; y_i \Leftrightarrow g_i(x_i) \geq g_i(y_i) - q_i,$$
$$x_i \; V_i \; y_i \Leftrightarrow g_i(x_i) > g_i(y_i) + v_i,$$

with $q_i \leq v_i$. The number q_i (resp. v_i) is called the preference (resp. veto) threshold in ELECTRE TRI (although we take them here to be constant, they could well be allowed to vary, see Roy and Bouyssou (1993)). The relation S_i is interpreted as an "at least as good" relation on $\widehat{X}_i$, while the relation V_i is a "far better than relation". Clearly, V_i is included in the asymmetric part of S_i.

A nonnegative weight w_i is assigned to each attribute $i \in N$. We suppose wlog that weights are normalized so that $\sum_{i=1}^{n} w_i = 1$. For all $k \in R^+$, let λ^k be a real number between 1/2 and 1. It is supposed that:

$$\lambda^r \geq \lambda^{r-1} \geq \cdots \geq \lambda^2.$$

In ELECTRE TRI, for all $k \in R^+$, we build a binary relation S^k on $\hat{X}$ letting, for all $x, y \in \hat{X}$,

$$x \; S^k \; y \Leftrightarrow \sum_{i \in S(x,y)} w_i \geq \lambda^k \text{ and } [Not[y_i \; V_i \; x_i], \text{ for all } i \in N], \qquad (3)$$

where $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$. Hence, we have $x \; S^k \; y$ when x is judged "at least as good as" y on a qualified weighted majority of attributes (concordance condition) and there is no attribute on which y is judged "far better" than x (non-discordance condition). Observe that, by construction, we have:

$$S^r \subseteq S^{r-1} \subseteq \cdots \subseteq S^2 .$$

The sorting of an alternative $x \in X$ is based the comparison of x with the profiles p^k using the relations S^k. In the pessimistic version of ELECTRE TRI, we have, for all $x \in X$,

$$x \in C^k_{\geq} \Leftrightarrow x \; S^k \; p^k.$$

Remark 4

In the original presentation of the method (see Roy and Bouyssou, 1993, p. 390), the assignment of an alternative to one of the categories is presented slightly differently: for $k = r, r-1, \ldots 2$, it is tested whether $x \; S^k \; p^k$ and x is assigned to the highest category C^k such that this test is positive or to C^1 if the test is never positive. Our presentation is clearly equivalent to the original one. •

3 The noncompensatory sorting model

This section studies a particular case of the decomposable sorting model (1) that will turn to have close links with (the pessimistic version of) ELECTRE TRI when there are no veto effects.

3.1 Definitions

We say that $\langle C^k \rangle_{k \in R}$ has a representation in the *noncompensatory sorting model* if:

- for all $i \in N$ there are sets $\mathscr{A}_i^r \subseteq \mathscr{A}_i^{r-1} \subseteq \cdots \subseteq \mathscr{A}_i^2 \subseteq X_i$,
- there are subsets $\mathscr{F}^r$, $\mathscr{F}^{r-1}$, …, $\mathscr{F}^2$ of 2^N that are monotonic wrt inclusion (i.e. such that, for all $k \in R^+$, $[I \in \mathscr{F}^k$ and $I \subset J] \Rightarrow J \in \mathscr{F}^k$) and such that $\mathscr{F}^r \subseteq \mathscr{F}^{r-1} \subseteq \cdots \subseteq \mathscr{F}^2$,

such that:

$$x \in C^k_{\geq} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i^k\} \in \mathscr{F}^k. \tag{4}$$

In this case, we say that $\langle \mathscr{F}^k, \langle \mathscr{A}_i^k \rangle_{i \in N} \rangle$ is a representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model. We note $A^k(x)$ instead of $\{i \in N : x_i \in \mathscr{A}_i^k\}$ when there is no ambiguity on the underlying sets $\mathscr{A}_i^k$. We define, in this section, $\mathscr{U}_i^k = X_i \setminus \mathscr{A}_i^k$.

The interpretation of the noncompensatory sorting model is clear. For all $k \in R^+$, we isolate within the set X_i a subset $\mathscr{A}_i^k$ that we interpret as containing the elements $x_i \in X_i$ that are judged "satisfactory at the level k". In order for $x \in X$ to belong at least to C^k, it is necessary that x is judged satisfactory at the level k on a subset of attributes that is "sufficiently important at the level k", as indicated by the set $\mathscr{F}^k$. The fact that $\mathscr{F}^k$ is monotonic means that replacing an unsatisfactory evaluation at the level k by a satisfactory one cannot turn an alternative in C^k into an alternative in $C^k_{<}$. Because the categories are ordered, the hypothesis that $\mathscr{A}_i^k \subseteq \mathscr{A}_i^{k-1}$ simply means that an evaluation that is satisfactory at the level k must be judged satisfactory at any lower level. Similarly imposing that $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$ means that a subset of attributes that is judged "sufficiently important at level k" must be so at any lower level.

When no discordance is involved, i.e. when $V_i = \varnothing$, for all $i \in N$, the pessimistic version of ELECTRE TRI is a particular case of the noncompensatory sorting model. Indeed, remember from section 2.5 that in the pessimistic version of ELECTRE TRI we have, for all $x \in X$,

$$x \in C^k_{\geq} \Leftrightarrow \sum_{i \in S(x, p^k)} w_i \geq \lambda_k.$$

Define $\mathscr{A}_i^k = \{i \in N : x_i \; S_i \; p^k\}$ and let $I \in \mathscr{F}^k$ whenever $\sum_{i \in I} w_i \geq \lambda_k$. By construction of the profiles p^k and of the relations S_i, we have $\mathscr{A}_i^k \subseteq \mathscr{A}_i^{k-1}$. Because $\lambda_{k+1} \geq \lambda_k$, we have $\mathscr{F}^{k+1} \subseteq \mathscr{F}^k$. Hence $\langle \mathscr{F}^k, \langle \mathscr{A}_i^k \rangle_{i \in N} \rangle$ is a representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model.

Our aim in this section is to characterize the partitions $\langle C^k \rangle_{k \in R}$ that can be represented in the noncompensatory sorting model.

3.2 Axioms

Let us first observe that, if $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model then it is R-linear.

Lemma 3
If a partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model then it is R-linear.

Proof
Suppose that $linear_i^R$ is violated so that $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^\ell$, $(y_i, a_{-i}) \in C_<^k$ and $(x_i, b_{-i}) \in C_<^\ell$. This implies $x_i \in \mathscr{A}_i^k$, $y_i \notin \mathscr{A}_i^k$, $y_i \in \mathscr{A}_i^\ell$ and $x_i \notin \mathscr{A}_i^\ell$. This violates the fact that we have either $\mathscr{A}_i^k \subseteq \mathscr{A}_i^\ell$ or $\mathscr{A}_i^\ell \subseteq \mathscr{A}_i^k$. □

In the noncompensatory sorting model all elements in $\mathscr{A}_i^k$ are treated in a similar way. Therefore, if $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model, then, for all $i \in N$, the relation $\succsim_i^R$ can have at most r distinct equivalence classes. Hence, for all $i \in N$, the set $X_i/\sim_i^R$ is finite. In view of section 2.4, this shows that the noncompensatory sorting model is a particular case of the decomposable sorting model (1). We provide below conditions that will allow to isolate the noncompensatory sorting model within model (1).

Using, e.g., an additive sorting model, it is easy to build partitions $\langle C^k \rangle_{k \in R}$ in which all relations $\succsim_i^R$ have at most r equivalence classes that cannot be represented in the noncompensatory sorting model. In order to capture the specific features of the noncompensatory sorting model, consider, for all $k \in R^+$ and all $i \in N$, the binary relation $\succsim_i^k$ on X_i such that, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i^k y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in C^k \Rightarrow (x_i, a_{-i}) \in C_{\geq}^k].$$

By construction, $\succsim_i^k$ is reflexive and transitive. The relation $\succsim_i^R$ always refines $\succsim_i^k$. R-linearity is equivalent to saying that each $\succsim_i^k$ is complete and that these relations are compatible, i.e. that $\succsim_i^R = \bigcap_{k \in R} \succsim_i^k$ is complete.

On top of the fact that all relations $\succsim_i^R$ can have at most r distinct equivalence classes, the noncompensatory sorting model also implies that all relations $\succsim_i^k$ can have at most 2 distinct equivalence classes. This is the key to the following condition. We say that $\langle C^k \rangle_{k \in R}$ is R-2-graded on attribute $i \in N$ (condition 2-$graded_i^R$) if:

$$\left.\begin{array}{c}(x_i, a_{-i}) \in C^k_{\geq} \\ \text{and} \\ (y_i, a_{-i}) \in C^k_{\geq} \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell_{\geq}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, b_{-i}) \in C^\ell_{\geq} \\ \text{or} \\ (z_i, a_{-i}) \in C^k_{\geq}\end{array}\right. \qquad (2\text{-}graded_i^R)$$

for all $x_i, y_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $k, \ell \in R^+$ with $\ell \leq k$. We say that $\langle C^k \rangle_{k \in R}$ is R-2-graded if it is R-2-graded on all $i \in N$. As shown below, R-2-gradedeness is necessary for the noncompensatory sorting model.

Lemma 4
If $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model then it is R-2-graded.

Proof
Suppose that condition 2-$graded_i^R$ is violated so that, for some $\ell \leq k$, $(x_i, a_{-i}) \in C^k_{\geq}$, $(y_i, a_{-i}) \in C^k_{\geq}$, $(y_i, b_{-i}) \in C^\ell_{\geq}$, $(x_i, b_{-i}) \in C^\ell_{<}$ and $(z_i, a_{-i}) \in C^k_{<}$. In the noncompensatory sorting model, $(y_i, b_{-i}) \in C^\ell_{\geq}$ and $(x_i, b_{-i}) \in C^\ell_{<}$ imply $x_i \notin \mathscr{A}_i^\ell$. Similarly, $(x_i, a_{-i}) \in C^k_{\geq}$ and $(z_i, a_{-i}) \in C^k_{<}$ imply $x_i \in \mathscr{A}_i^k$. This is contradictory since $\mathscr{A}_i^k \subseteq \mathscr{A}_i^\ell$. Hence, condition 2-$graded_i^R$ holds. □

The following lemma makes clear the consequences of conditions $linear_i^R$ and 2-$graded_i^R$ using the relations $\succsim_i^k$.

Lemma 5
Conditions $linear_i^R$ and 2-$graded_i^R$ hold iff

1. *$\succsim_i^k$ is a weak order having at most two distinct equivalence classes,*
2. *$[x_i \succ_i^k y_i] \Rightarrow [x_i \succsim_i^\ell y_i$, for all $\ell \in R^+]$,*
3. *$[x_i \sim_i^k z_i$ and $x_i \succ_i^k y_i] \Rightarrow [x_i \sim_i^\ell z_i$, for all $\ell < k]$.*

for all $k \in R^+$ and all $x_i, y_i, z_i \in X_i$.

Proof
Part [$\Leftarrow$]. Suppose that $linear_i^R$ is violated so that, for some $x_i, y_i \in X_i$ and some $a_{-i}, b_{-i} \in X_{-i}$, $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^\ell$, $(y_i, a_{-i}) \in C^k_{<}$ and

$(x_i, b_{-i}) \in C^\ell_<$. Because, for all $k \in R$, $\succsim^k_i$ is a weak order, we have $x_i \succ^k_i y_i$ and $y_i \succ^\ell_i x_i$, a contradiction.

Suppose that 2-*graded*^R_i is violated so that, for some $\ell \leq k$, $(x_i, a_{-i}) \in C^k_\geq$, $(y_i, a_{-i}) \in C^k_\geq$, $(y_i, b_{-i}) \in C^\ell_\geq$, $(x_i, b_{-i}) \in C^\ell_<$ and $(z_i, a_{-i}) \in C^k_<$.

Suppose that $k = \ell$. This would imply $y_i \succ^k_i x_i$ and $x_i \succ^k_i z_i$, violating the fact that $\succsim^k_i$ has only two distinct equivalence classes. Suppose henceforth that $k > \ell$. We obtain, $y_i \succ^\ell_i x_i$, $x_i \succ^k_i z_i$ and $y_i \succ^k_i z_i$. Since $y_i \succ^\ell_i x_i$, we cannot have $x_i \succ^k_i y_i$. If $x_i \sim^k_i y_i$, $x_i \succ^k_i z_i$ would imply $x_i \sim^\ell_i y_i$, a contradiction. But supposing that $y_i \succ^k_i x_i$, would imply that $\succsim^k_i$ has three distinct equivalence classes.

Part [⇒]. Using *linear*^R_i, we know that $\succsim^R_i$ is complete. Since $\succsim^R_i$ refines $\succsim^k_i$, it follows that $\succsim^k_i$ is complete and, hence, a weak order. Clearly, $x_i \succ^k_i y_i$ and $y_i \succ^\ell_i x_i$ would violate *linear*^R_i.

Suppose that, for some $k \in R^+$, $\succsim^k_i$ has at least three equivalence classes so that, for some $x_i, y_i, z_i \in X_i$, we have $x_i \succ^k_i y_i$ and $y_i \succ^k_i z_i$. Using the definition of $\succsim^k_i$, we have, for some $a_{-i}, b_{-i} \in X_{-i}$, $(x_i, a_{-i}) \in C^k$, $(y_i, a_{-i}) \in C^k_<$, $(y_i, b_{-i}) \in C^k$, $(z_i, b_{-i}) \in C^k_<$. Using *linear*^R_i, $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^k$ and $(y_i, a_{-i}) \in C^k_<$ imply $(x_i, b_{-i}) \in C^k_\geq$. Using 2-*graded*$^R_i$ with $\ell = k$, $(y_i, b_{-i}) \in C^k$, $(x_i, b_{-i}) \in C^k_\geq$ and $(x_i, a_{-i}) \in C^k$ imply, either $(y_i, a_{-i}) \in C^k_\geq$ or $(z_i, b_{-i}) \in C^k_\geq$, a contradiction.

Suppose now that, for some $\ell < k$ and some $x_i, y_i, z_i \in X_i$, $x_i \sim^k_i z_i$, $x_i \succ^k_i y_i$ and $x_i \succ^\ell_i z_i$. By definition, $x_i \sim^k_i z_i$ and $x_i \succ^k_i y_i$ imply that $(x_i, a_{-i}) \in C^k$, $(z_i, a_{-i}) \in C^k$ and $(y_i, a_{-i}) \in C^k_<$, for some $a_{-i} \in X_{-i}$. Similarly $x_i \succ^\ell_i z_i$ implies $(x_i, b_{-i}) \in C^\ell$ and $(z_i, b_{-i}) \in C^\ell_<$, for some $b_{-i} \in X_{-i}$. Using 2-*graded*^R_i, $(z_i, a_{-i}) \in C^k$, $(x_i, a_{-i}) \in C^k$ and $(x_i, b_{-i}) \in C^\ell$ imply $(z_i, b_{-i}) \in C^\ell_\geq$ or $(y_i, a_{-i}) \in C^k_\geq$, a contradiction. □

3.3 Background on twofold partitions

If a partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model, all the twofold partitions $\langle C^k_\geq, C^k_< \rangle$ will have a representation in the noncompensatory sorting model. Hence, the analysis of the twofold partitions $\langle C^k_\geq, C^k_< \rangle$ will be crucial in what follows. With this objective in mind, we briefly recall below the main points of the analysis of twofold partitions as given in Bouyssou and Marchant (2004).

Consider a twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of X. We define on each X_i the binary relation $\succsim_i$ letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i y_i \Leftrightarrow [\text{for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathscr{A} \Rightarrow (x_i, a_{-i}) \in \mathscr{A}].$$

In Bouyssou and Marchant (2004), we prove the following:

Proposition 2
Let $\langle \mathscr{A}, \mathscr{U} \rangle$ be a twofold partition X. Then there are subsets $\mathscr{B}_i \subseteq X_i$ and a subset $\mathscr{G} \subseteq 2^N$ that is monotonic wrt inclusion such that, for all $x \in X$,

$$x \in \mathscr{A} \Leftrightarrow \{i \in N : x_i \in \mathscr{B}_i\} \in \mathscr{G}, \tag{5}$$

iff

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \in \mathscr{A} \\ \text{or} \\ (x_i, b_{-i}) \in \mathscr{A} \end{array}\right. \tag{6}$$

and

$$\left.\begin{array}{c} (x_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathscr{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in \mathscr{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathscr{A} \end{array}\right. \tag{7}$$

for all $i \in N$, all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$. Furthermore:

1. *Conditions* (6) *and* (7) *are independent.*

2. *The representation $\langle \mathscr{G}, \langle \mathscr{B}_i \rangle_{i \in N} \rangle$ of $\langle \mathscr{A}, \mathscr{U} \rangle$ is unique iff all attributes are influent for $\langle \mathscr{A}, \mathscr{U} \rangle$.*

3. *Suppose that $i \in N$ is influent for $\langle \mathscr{A}, \mathscr{U} \rangle$. In all representations $\langle \mathscr{G}, \langle \mathscr{B}_i \rangle \rangle$ of $\langle \mathscr{A}, \mathscr{U} \rangle$, $\mathscr{B}_i$ must coincide with the first equivalence class of $\succsim_i$.*

4. *If $i \in N$ is degenerate for $\langle \mathscr{A}, \mathscr{U} \rangle$, it is always possible to take $\mathscr{B}_i = \varnothing$. With such a choice, we may always choose $\mathscr{G}$ in such a way that $I \in \mathscr{G}$ whenever there is some $x \in \mathscr{A}$ such that $\{i \in N : x_i \in \mathscr{B}_i\} \subseteq I$.*

5. *Furthermore, keeping the set $\mathscr{G}$ as above, on each degenerate attribute we may modify $\mathscr{B}_i$ taking it to be an arbitrary subset of X_i. If this subset is taken to be strict, after this modification, we still have that $I \in \mathscr{G}$ whenever there is some $x \in \mathscr{A}$ such that $\{i \in N : x_i \in \mathscr{B}_i\} \subseteq I$.*

Taking $\mathscr{A} = C^k_{\geq}$ shows that, if a partition $\langle C^k \rangle_{k \in R}$ is R-linear, then all twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ will satisfy (6). Similarly, still taking $\mathscr{A} = C^k_{\geq}$, if $\langle C^k \rangle_{k \in R}$ is R-2-graded, then all twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ will satisfy (7). Hence, if $\langle C^k \rangle_{k \in R}$ is R-linear and R-2-graded, all twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ will have a representation in the noncompensatory sorting model. These representations of the twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$ will form the basis of our analysis.

3.4 Result

Our main result in this section says that R-linearity and R-2-gradedness characterize the noncompensatory sorting model.

Theorem 1
An r-fold partition $\langle C^k \rangle_{k \in R}$ of X has a representation in the noncompensatory sorting model iff it is R-linear and R-2-graded.

PROOF
Necessity results from lemmas 3 and 4. We show sufficiency. Because $\langle C^k \rangle_{k \in R}$ is a partition, it is clear that, for all $k \in R^+$, $\langle C^k_{\geq}, C^k_{<} \rangle$ is a partition, so that there is at least one attribute that is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$. Since $\langle C^k \rangle_{k \in R}$ is R-linear and R-2-graded, for all $k \in R^+$, the twofold partition $\langle C^k_{\geq}, C^k_{<} \rangle$ satisfies (6) and (7). Using proposition 2, there are subsets $\mathscr{B}^k_i \subseteq X_i$ and a monotonic subset $\mathscr{G}^k \subseteq 2^N$ such that, for all $x \in X$,

$$x \in C^k_{\geq} \Leftrightarrow \{i \in N : x_i \in \mathscr{B}^k_i\} \in \mathscr{G}^k.$$

We define $\mathscr{F}^k$ and $\mathscr{A}^k_i$ and the basis of $\mathscr{G}^k$ and $\mathscr{B}^k_i$.

Case $k = r$
Let $\langle \mathscr{G}^r, \langle \mathscr{B}^r_i \rangle_{i \in N} \rangle$ be the representation of $\langle C^r_{\geq}, C^r_{<} \rangle$ derived from proposition 2. It is such that if $i \in N$ is degenerate for $\langle C^r_{\geq}, C^r_{<} \rangle$, then $\mathscr{B}^r_i = \varnothing$. Take $\mathscr{F}^r = \mathscr{G}^r$ and, for all $i \in N$, $\mathscr{A}^r_i = \mathscr{B}^r_i$. By construction, $\langle \mathscr{F}^r, \langle \mathscr{A}^r_i \rangle_{i \in N} \rangle$ is a representation of $\langle C^r_{\geq}, C^r_{<} \rangle$ in the noncompensatory sorting model.

Case $k < r$
For $k = r-1, r-2, \ldots, 2$, let $\langle \mathscr{G}^k, \langle \mathscr{B}^k_i \rangle_{i \in N} \rangle$ be the representation of $\langle C^k_{\geq}, C^k_{<} \rangle$ derived from proposition 2. We build $\mathscr{F}^k$ and $\mathscr{A}^k_i$ in sequence, starting with $k = r - 1$.

If $i \in N$ is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$, take $\mathscr{A}^k_i = \mathscr{B}^k_i$. If $i \in N$ is degenerate for $\langle C^k_{\geq}, C^k_{<} \rangle$, we have $\mathscr{B}^k_i = \varnothing$. In such a case, we take $\mathscr{A}^k_i = \mathscr{A}^{k+1}_i$. We take $\mathscr{F}^k = \mathscr{G}^k$. Using parts 4 and 5 of proposition 2, we know that $\langle \mathscr{F}^k, \langle \mathscr{A}^k_i \rangle_{i \in N} \rangle$ is a representation of $\langle C^k_{\geq}, C^k_{<} \rangle$ in the noncompensatory sorting model. We have $I \in \mathscr{F}^k$ whenever there is some $x \in C^k_{\geq}$ such that $\{i \in N : x_i \in \mathscr{A}^k_i\} \subseteq I$.

Proof that $\mathscr{A}^k_i \subseteq \mathscr{A}^{k-1}_i$
Let us prove that, for all $i \in N$ and all $k \in R^+$ we have $\mathscr{A}^k_i \subseteq \mathscr{A}^{k-1}_i$. If attribute $i \in N$ is not influent for $\langle C^\ell_{\geq}, C^\ell_{<} \rangle$ for $\ell = r, r-1, \ldots, k$, we have $\mathscr{A}^k_i = \varnothing$ and there is nothing to prove. Similarly if $i \in N$ is not influent

for $\langle C^{k-1}_{\geq}, C^{k-1}_{<}\rangle$, we have $\mathscr{A}_i^{k-1} = \mathscr{A}_i^k$ and there is nothing to prove either. Suppose henceforth that $i \in N$ is influent for $\langle C^{k-1}_{\geq}, C^{k-1}_{<}\rangle$ and let ℓ be the smallest $k \in \{r, r-1, \ldots, k\}$ such that i is influent for $\langle C^{\ell}_{\geq}, C^{\ell}_{<}\rangle$. By construction, we have $\mathscr{A}_i^k = \mathscr{A}_i^\ell$. Suppose, in contradiction with the thesis, that $x_i \in \mathscr{A}_i^\ell$ and $x_i \notin \mathscr{A}_i^{k-1}$ with $\ell \geq k$.

Since $i \in N$ is influent for $\langle C^{\ell}_{\geq}, C^{\ell}_{<}\rangle$ and $x_i \in \mathscr{A}_i^\ell$, we know that $(x_i, a_{-i}) \in C^\ell_{\geq}$ and $(y_i, a_{-i}) \notin C^\ell_{\geq}$, for some $y_i \in X_i$ and some $a_{-i} \in X_{-i}$. Similarly, since $i \in N$ is influent for $\langle C^{k-1}_{\geq}, C^{k-1}_{<}\rangle$ and $x_i \notin \mathscr{A}_i^{k-1}$, we know that $(z_i, b_{-i}) \in C^{k-1}_{\geq}$ and $(x_i, b_{-i}) \notin C^{k-1}_{\geq}$, for some $z_i \in X_i$ and some $b_{-i} \in X_{-i}$.

Using linear_i^R, $(x_i, a_{-i}) \in C^\ell_{\geq}$, $(z_i, b_{-i}) \in C^{k-1}_{\geq}$ and $(x_i, b_{-i}) \notin C^{k-1}_{\geq}$ imply $(z_i, a_{-i}) \in C^\ell_{\geq}$. Using 2-$\mathit{graded}_i^R$, $(x_i, a_{-i}) \in C^\ell_{\geq}$, $(z_i, a_{-i}) \in C^\ell_{\geq}$ and $(z_i, b_{-i}) \in C^{k-1}_{\geq}$ imply $(x_i, b_{-i}) \in C^{k-1}_{\geq}$ or $(y_i, a_{-i}) \in C^\ell_{\geq}$, a contradiction.

Proof that $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$

Let us now prove that $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$. By construction, we know that, for all $k \in R^+$, $I \in \mathscr{F}^k$ whenever there is some $x \in C^k_{\geq}$ such that $A^k(x) \subseteq I$. Let $I \in \mathscr{F}^k$ and let $x \in X$ be such that $x \in C^k_{\geq}$ and $A^k(x) \subseteq I$. Starting with such an alternative $x \in X$, let us build an alternative $x' \in X$ as follows. For all $i \in N$ such that $x_i \in \mathscr{A}_i^k$, let $x'_i = x_i$. Because $\mathscr{A}_i^k \subseteq \mathscr{A}_i^{k-1}$, we know that on these attributes $x'_i \in \mathscr{A}_i^{k-1}$. For all $i \in N$ such that $x_i \notin \mathscr{A}_i^k$, we consider two cases:

1. if i is influent for $\langle C^{k-1}_{\geq}, C^{k-1}_{<}\rangle$, by construction there is a $z_i \in X_i$ such that $z_i \notin \mathscr{A}_i^{k-1}$. In this case, let $x'_i = z_i$, so that $x'_i \notin \mathscr{A}_i^{k-1}$.
2. If i is not influent for $\langle C^{k-1}_{\geq}, C^{k-1}_{<}\rangle$, we have $\mathscr{A}_i^k = \mathscr{A}_i^{k-1}$. In this case, take x'_i equal to x_i, so that $x'_i \notin \mathscr{A}_i^{k-1}$.

By construction, we have $A^k(x) = A^k(x') = A^{k-1}(x') \subseteq I$. Because $A^k(x) = A^k(x')$ and $x \in C^k_{\geq}$, we know that $x' \in C^k_{\geq}$ so that $x' \in C^{k-1}_{\geq}$. Since $x \in C^{k-1}_{\geq}$ and $A^{k-1}(x') \subseteq I$, we have $I \in \mathscr{F}^{k-1}$. This completes the proof. □

The construction of the representation in the noncompensatory sorting model is illustrated below.

Example 1

Suppose that $n = 3$, $X_1 = X_2 = X_3 = \{9, 10, 11\}$. We consider a threefold partition $\langle C^1, C^2, C^3\rangle$ such that $C^3 = \{(9,10,10)$, $(9,10,11)$, $(9,11,10)$, $(9,11,11)$, $(10,9,10)$, $(10,9,11)$, $(10,10,9)$, $(10,10,10)$, $(10,10,11)$, $(10,11,9)$, $(10,11,10)$, $(10,11,11)$, $(11,9,10)$, $(11,9,11)$, $(11,10,9)$, $(11,10,10)$, $(11,10,11)$, $(11,11,9)$, $(11,11,10)$, $(11,11,11)\}$, $C^2 = \{(9,10,9)$, $(9,11,9)$, $(10,9,9)$, $(11,9,9)\}$ and $C^1 = \{(9,9,9)$, $(9,9,10)$, $(9,9,11)\}$.

This partition can be obtained with the pessimistic version of ELECTRE TRI with $(10, 10, 10)$ as the limiting profile between C^3 and C^2 and $(10, 10, 9)$ as the limiting profile between C^2 and C^1, $S_i = \geq$ and $V_i = \varnothing$ for all $i \in N$, $w_1 = w_2 = w_3 = 1/3$, $\lambda^3 = \lambda^2 = 2/3$. This shows that it is R-linear and R-2-graded.

All attributes are influent for the twofold partition $\langle C^3_{\geq}, C^3_{<} \rangle$. Let $\langle \mathscr{G}^3, \langle \mathscr{B}^3_1, \mathscr{B}^3_2, \mathscr{B}^3_3 \rangle\rangle$ be the unique representation of $\langle C^3_{\geq}, C^3_{<} \rangle$ derived from proposition 2. We have: $\mathscr{B}^3_1 = \mathscr{B}^3_2 = \mathscr{B}^3_3 = \{10, 11\}$ and $\mathscr{G}^3 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. We take $\mathscr{A}^3_i = \mathscr{B}^3_i$, for all $i \in N$, and $\mathscr{F}^3 = \mathscr{G}^3$.

Only attributes 1 and 2 are influent for the twofold partition $\langle C^2_{\geq}, C^2_{<} \rangle$. The representation $\langle \mathscr{G}^2, \langle \mathscr{B}^2_1, \mathscr{B}^2_2, \mathscr{B}^2_3 \rangle\rangle$ of $\langle C^2_{\geq}, C^2_{<} \rangle$ derived from proposition 2 is such that $\mathscr{B}^2_1 = \mathscr{B}^2_2 = \{10, 11\}$, $\mathscr{B}^2_3 = \varnothing$ and $\mathscr{G}^2 = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. As explained in the proof of theorem 1, we take $\mathscr{A}^2_1 = \mathscr{B}^2_1$, $\mathscr{A}^2_2 = \mathscr{B}^2_2$, $\mathscr{A}^2_3 = \mathscr{A}^3_3$ and $\mathscr{F}^2 = \mathscr{G}^2$.

It can easily be checked that $\langle \mathscr{F}^k, \langle \mathscr{A}^k_i \rangle_{i \in N} \rangle$ is a representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model. This is detailed in table 1. $\diamond$

3.5 Independence and uniqueness

This section discusses the independence of the conditions used in theorem 1 and the uniqueness of the representation in the noncompensatory sorting model.

3.5.1 Independence of conditions

Let us show that none of the two conditions used in theorem 1 can be dispensed with. Consider first the following condition:

$$\left.\begin{array}{c} (x_i, a_{-i}) \in C^k_{\geq} \\ \text{and} \\ (y_i, a_{-i}) \in C^k_{\geq} \\ \text{and} \\ (y_i, b_{-i}) \in C^k_{\geq} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in C^k_{\geq} \\ \text{or} \\ (z_i, a_{-i}) \in C^k_{\geq} \end{array}\right. \tag{8}$$

for all $x_i, y_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $k \in R^+$. Condition (8) is nothing but condition 2-$graded^R_i$ restricted to the case $\ell = k$. The following example shows that the conjunction of R-linearity and condition (8), for all $i \in N$, is not sufficient to precipitate the noncompensatory sorting model.

Example 2
Let $n = 3$, $X = \{x_1, y_1, z_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$ and $r = 3$. Let $C^3 = \{(x_1, x_2, x_3), (y_1, x_2, x_3)\}$, $C^2 = \{(x_1, x_2, y_3), (x_1, y_2, x_3), (y_1, x_2, y_3), (y_1, y_2, x_3), (y_1, y_2, y_3), (z_1, x_2, x_3), (z_1, x_2, y_3), (z_1, y_2, x_3)\}$ and $C^1 = \{(z_1, y_2, y_3), (x_1, y_2, y_3)\}$.

x	Category	$B^3(x)$	$B^2(x)$	$A^2(x)$
$(9,10,10)$	C^1	$\{2,3\}$	$\{2\}$	$\{2,3\}$
$(9,10,11)$	C^1	$\{2,3\}$	$\{2\}$	$\{2,3\}$
$(9,11,10)$	C^1	$\{2,3\}$	$\{2\}$	$\{2,3\}$
$(9,11,11)$	C^1	$\{2,3\}$	$\{2\}$	$\{2,3\}$
$(10,9,10)$	C^1	$\{1,3\}$	$\{1\}$	$\{2,3\}$
$(10,9,11)$	C^1	$\{1,3\}$	$\{1\}$	$\{2,3\}$
$(10,10,9)$	C^1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$(10,10,10)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(10,10,11)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(10,11,9)$	C^1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$(10,11,10)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(10,11,11)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(11,9,10)$	C^1	$\{1,3\}$	$\{1\}$	$\{1,3\}$
$(11,9,11)$	C^1	$\{1,3\}$	$\{1\}$	$\{1,3\}$
$(11,10,9)$	C^1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$(11,10,10)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(11,10,11)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(11,11,9)$	C^1	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$(11,11,10)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
$(11,11,11)$	C^1	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$
....	....	....	.....	....
$(9,10,9)$	C^2	$\{2\}$	$\{2\}$	$\{2\}$
$(9,11,9)$	C^2	$\{2\}$	$\{2\}$	$\{2\}$
$(10,9,9)$	C^2	$\{1\}$	$\{1\}$	$\{1\}$
$(11,9,9)$	C^2	$\{1\}$	$\{1\}$	$\{1\}$
....	....	....	.....	....
$(9,9,9)$	C^3	$\varnothing$	$\varnothing$	$\varnothing$
$(9,9,10)$	C^3	$\{3\}$	$\varnothing$	$\{3\}$
$(9,9,11)$	C^3	$\{3\}$	$\varnothing$	$\{3\}$

Table 1: Details of example 1
$B^3(x) = \{i \in N : x_i \in \mathscr{B}_i^3\}$, $B^2(x) = \{i \in N : x_i \in \mathscr{B}_i^2\}$,
$\mathscr{B}_1^3 = \mathscr{B}_2^3 = \mathscr{B}_2^3 = \{10,11\}$, $\mathscr{A}_1^3 = \mathscr{A}_2^3 = \mathscr{A}_3^3 = \{10,11\}$,
$\mathscr{F}^3 = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$,
$\mathscr{B}_1^2 = \mathscr{B}_2^2 = \{10,11\}$, $\mathscr{B}_3^2 = \varnothing$, $\mathscr{A}_1^2 = \mathscr{A}_2^2 = \{10,11\}$, $\mathscr{A}_3^2 = \{10,11\}$,
$\mathscr{F}^2 = \{\{1\}, \{2\}, \{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

We have $y_1 \succ_1^R x_1 \succ_1^R z_1$, $x_2 \succ_2^R y_2$ and $x_3 \succ_3^R y_3$, which shows that $\langle C^k \rangle_{k \in R}$ is R-linear.

The twofold partition $\langle C^3_{\geq}, C^3_{<} \rangle$ has a representation in the noncompensatory sorting model with $\mathscr{B}_1^3 = \{x_1, y_1\}$, $\mathscr{B}_2^3 = \{x_2\}$, $\mathscr{B}_3^3 = \{x_3\}$ and $\mathscr{G}^3 = \{\{1, 2, 3\}\}$. This representation is unique since all attributes are influent for $\langle C^3_{\geq}, C^3_{<} \rangle$. Indeed, we have $(x_1, x_2, x_3) \in C^3_{\geq}$ and $(z_1, x_2, x_3) \in C^3_{<}$, $(x_1, x_2, x_3) \in C^3_{\geq}$ and $(x_1, y_2, x_3) \in C^3_{<}$, $(x_1, x_2, x_3) \in C^3_{\geq}$ and $(x_1, x_2, y_3) \in C^3_{<}$.

Similarly, the twofold partition $\langle C^2_{\geq}, C^2_{<} \rangle$ has a representation in the noncompensatory sorting model with $\mathscr{B}_1^1 = \{y_1\}$, $\mathscr{B}_2^2 = \{x_2\}$, $\mathscr{B}_3^2 = \{x_3\}$ and $\mathscr{G}^2 = 2^N \setminus \{\varnothing\}$. This representation is unique since all attributes are influent for $\langle C^2_{\geq}, C^2_{<} \rangle$. Indeed, we have $(y_1, y_2, y_3) \in C^2_{\geq}$ and $(x_1, y_2, y_3) \in C^2_{<}$, $(x_1, x_2, y_3) \in C^2_{\geq}$ and $(x_1, y_2, y_3) \in C^2_{<}$, $(x_1, y_2, x_3) \in C^2_{\geq}$ and $(x_1, y_2, y_3) \in C^2_{<}$.

Since each of the twofold partitions induced by $\langle C^1, C^2, C^3 \rangle$ has a representation in the noncompensatory sorting model, condition (8) holds, for all $i \in N$. However the partition $\langle C^1, C^2, C^3 \rangle$ cannot be represented in the noncompensatory sorting model. Indeed, $(x_1, x_2, x_3) \in C^3_{\geq}$ and $(z_1, x_2, x_3) \in C^3_{<}$ would imply $x_1 \in \mathscr{A}_i^3$. Similarly, $(y_1, y_2, y_3) \in C^2_{\geq}$ and $(x_1, y_2, y_3) \in C^2_{<}$ would imply $x_1 \notin \mathscr{A}_i^2$, violating $\mathscr{A}_i^3 \subseteq \mathscr{A}_i^2$.

On can check, e.g. using lemma 5, that the partition $\langle C^1, C^2, C^3 \rangle$ satisfies 2-*graded*$_2^R$ and 2-*graded*$_3^R$. Condition 2-*graded*$_1^R$ is violated since $(x_1, x_2, x_3) \in C^3_{\geq}$, $(y_1, x_2, x_3) \in C^3_{\geq}$ and $(y_1, y_2, y_3) \in C^2_{\geq}$ but $(x_1, y_2, y_3) \notin C^2_{\geq}$ and $(z_1, x_2, x_3) \notin C^3_{\geq}$. $\diamond$

Consider now the following condition

$$\left. \begin{array}{c} (x_i, a_{-i}) \in C^k \\ \text{and} \\ (y_i, b_{-i}) \in C^k \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (y_i, a_{-i}) \in C^k_{\geq} \\ \text{or} \\ (x_i, b_{-i}) \in C^k_{\geq} \end{array} \right. \tag{9}$$

for all $x_i, y_i \in X_i$, all $k \in R$ and all $a_{-i}, b_{-i} \in X_{-i}$. Condition (9) is nothing but condition *linear*$_i^R$ restricted to the case $\ell = k$. It is clearly equivalent to requiring that all relations $\succsim_i^k$ are complete. The following example shows that the conjunction of R-2-gradedness and condition (9), for all $i \in N$, is not sufficient to precipitate the noncompensatory sorting model.

Example 3

Let $n = 3$, $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$ and $r = 3$. Let $C^3 = \{(x_1, x_2, x_3),\ (x_1, y_2, x_3),\ (y_1, y_2, x_3)\}$, $C^2 = \{(y_1, y_2, y_3),\ (y_1, x_2, x_3)\}$ and $C^1 = \{(x_1, x_2, y_3),\ (x_1, y_2, y_3),\ (y_1, x_2, y_3)\}$.

Because each X_i has only two elements, this partition is trivially R-2-graded. We have $y_2 \succ_i^R x_2$ and $x_3 \succ_i^R y_3$, so that *linear*$_2^R$ and *linear*$_3^R$ hold.

Since $(x_1, x_2, x_3) \in C^3$ and $(y_1, y_2, y_3) \in C^2$ but neither $(y_1, x_2, x_3) \in C^3_{\geq}$ nor $(x_1, y_2, y_3) \in C^2_{\geq}$, condition $linear_1^R$ is violated. Observe that $x_1 \succ_1^3 y_1$ and $y_1 \succ_1^2 x_1$, so that condition (9) is satisfied for attribute 1. $\diamond$

3.5.2 Uniqueness of representation

Let $\langle \mathscr{A}, \mathscr{U} \rangle$ be a twofold partition of X having a representation in the noncompensatory sorting model. In Bouyssou and Marchant (2004), we prove that $\langle \mathscr{A}, \mathscr{U} \rangle$ has a unique representation in the noncompensatory sorting model if and only if all attributes are influent for $\langle \mathscr{A}, \mathscr{U} \rangle$.

Using the above observation, it is clear that if, for all $i \in N$ and all $k \in R^+$, attribute i is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$, the r-fold partition $\langle C^k \rangle_{k \in R}$ will have a unique representation in the noncompensatory sorting model. To examine the converse, suppose that, for some $\ell \in R^+$, there is an attribute $j \in N$ that is degenerate for $\langle C^\ell_{\geq}, C^\ell_{<} \rangle$. Because, by hypothesis, $j \in N$ is influent for $\langle C^k \rangle_{k \in R}$, it is influent for some of the twofold partitions $\langle C^k_{\geq}, C^k_{<} \rangle$. Let τ_j be the largest $k \in R^+$ such that $j \in N$ is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$. Similarly, let β_j be the smallest $k \in R^+$ such that $j \in N$ is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$. The sets $\mathscr{A}_j^{\tau_j}$ and $\mathscr{A}_j^{\beta_j}$ are nonempty strict subsets of X_j and are uniquely defined.

Suppose first that $\ell > \tau_j$. We have defined in the proof of theorem 1 $\mathscr{A}_j^k = \varnothing$, for all $k > \tau_j$. Clearly, we can freely choose the sets $\mathscr{A}_i^k$ for all $k > \tau_j$ to be arbitrary subsets of $\mathscr{A}_j^{\tau_j}$ provided that this arbitrary choice is consistent with the constraints $\mathscr{A}_j^k \subseteq \mathscr{A}_j^{k-1}$. In this case, the representation will not be unique.

Suppose now that $\ell < \beta_j$. We have defined in the proof of theorem 1 $\mathscr{A}_j^k = \mathscr{A}_j^{\beta_j}$, for all $k < \beta_j$. Clearly, we can freely choose the sets $\mathscr{A}_j^k$ for all $k < \beta_j$ to be arbitrary supersets of $\mathscr{A}_j^{\beta_j}$ provided that this arbitrary choice is consistent with the constraints $\mathscr{A}_j^k \subseteq \mathscr{A}_j^{k-1}$. Because $\mathscr{A}_j^{\nu_j}$ is a strict subset X_j, this shows that the representation will not be unique.

Otherwise, let $\ell_+(j) \in R^+$ be the smallest $k > \ell$ such that $j \in N$ is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$ and $\ell_-(j) \in R^+$ be the largest $k < \ell$ such that $j \in N$ is influent for $\langle C^k_{\geq}, C^k_{<} \rangle$. We know that $\mathscr{A}_j^{\ell+(j)}$ and $\mathscr{A}_j^{\ell-(j)}$ are uniquely defined and that $\mathscr{A}_j^{\ell+(j)} \subseteq \mathscr{A}_j^{\ell-(j)}$. We claim that if $\mathscr{A}_j^{\ell+(j)} \subsetneq \mathscr{A}_j^{\ell-(j)}$, the representation will not be unique. Indeed, for all k such that $\ell_-(j) < k < \ell_+(j)$, we have defined in the proof of theorem 1, $\mathscr{A}_j^k = \mathscr{A}_j^{\ell+(j)}$. Clearly, we might as well have taken all sets $\mathscr{A}_j^k$ to be such that $\mathscr{A}_j^{\ell+(j)} \subseteq \mathscr{A}_j^k \subseteq \mathscr{A}_j^{\ell-(j)}$, provided that this arbitrary choice is consistent with the constraints $\mathscr{A}_j^k \subseteq \mathscr{A}_j^{k-1}$. This proves the claim.

The uniqueness of the representation in the noncompensatory sorting model is therefore stronger than what it is in the particular case of twofold partitions. It is nevertheless apparent that, unless in rather particular cases, the representation of a partition in the noncompensatory sorting model will not be unique when an attribute is degenerate for one of the twofold partitions $\langle C^k_\geq, C^k_< \rangle$. As detailed in Bouyssou and Marchant (2004), this shows that methods designed to infer the parameters of an ELECTRE TRI model on the basis of assignment examples (e.g. Dias et al., 2002; Mousseau et al., 2001; Mousseau and Słowiński, 1998; Ngo The and Mousseau, 2002), should be prepared to deal with such situations.

We give below an example showing that there are instances of partitions in which some attributes are degenerate for some of the twofold partitions $\langle C^k_\geq, C^k_< \rangle$, while the representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model is unique. It also shows that an attribute may be influent for $\langle C^{k+1}_\geq, C^{k+1}_< \rangle$ and $\langle C^{k-1}_\geq, C^{k-1}_< \rangle$, while being degenerate for $\langle C^k_\geq, C^k_< \rangle$.

Example 4
Let $n = 3$, $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$ and $r = 4$. Let $C^4 = \{(x_1, x_2, x_3)\}$, $C^3 = \{(y_1, x_2, x_3)\}$, $C^2 = \{(x_1, x_2, y_3), (x_1, y_2, x_3)\}$ and $C^1 = \{(x_1, y_2, y_3), (y_1, x_2, y_3), (y_1, y_2, x_3), (y_1, y_2, y_3)\}$.

All attributes are influent for the twofold partition $\langle C^4_\geq, C^4_< \rangle$. It has the unique representation $\mathscr{A}^4_1 = \{x_1\}$, $\mathscr{A}^4_2 = \{x_2\}$, $\mathscr{A}^4_3 = \{x_3\}$ and $\mathscr{F}^4 = \{N\}$.

All attributes are influent for $\langle C^2_\geq, C^2_< \rangle$. It has the unique representation $\mathscr{A}^2_1 = \{x_1\}$, $\mathscr{A}^2_2 = \{x_2\}$, $\mathscr{A}^2_3 = \{x_3\}$ and $\mathscr{F}^2 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Attributes 2 and 3 are influent $\langle C^3_\geq, C^3_< \rangle$ while attribute 1 is degenerate. In order to satisfy the constraints of the noncompensatory sorting model, the representation of $\langle C^3_\geq, C^3_< \rangle$ must be chosen so that $\mathscr{A}^3_1 = \{x_1\}$, $\mathscr{A}^3_2 = \{x_2\}$, $\mathscr{A}^3_3 = \{x_3\}$ and $\mathscr{F}^2 = \{\{2,3\}, \{1,2,3\}\}$. $\diamond$

3.6 Extensions

The remainder of this section is devoted to the study of two particular cases of the noncompensatory sorting model.

3.6.1 The case $\mathscr{A}^k_i = \mathscr{A}^\ell_i$

We analyze here what must be added to the conditions in theorem 1, in order to ensure that $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model in which $\mathscr{A}^k_i = \mathscr{A}^\ell_i$, for all $k, \ell \in R^+$. With such a model, going from $C^k_\geq$ to $C^{k-1}_\geq$ only involves a change from $\mathscr{F}^k$ to $\mathscr{F}^{k-1}$, i.e. a change in the "strength" of the coalition of attributes needed to ensure that an alternative is judged satisfactory.

Suppose that $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model such that $\mathscr{A}_i^k = \mathscr{A}_i^\ell$, for all $k, \ell \in R^+$. Let $k \in \{3, 4, \ldots, r\}$ and suppose that $(x_i, a_{-i}) \in C^k_{\geq}$ and $(y_i, a_{-i}) \in C^k_{<}$. This implies that $x_i \in \mathscr{A}_i^k$ and $y_i \notin \mathscr{A}_i^k$. Let $\ell \in R^+$ such that $\ell < k$. Since, by hypothesis, $\mathscr{A}_i^k = \mathscr{A}_i^\ell$, this implies that $y_i \notin \mathscr{A}_i^\ell$. Therefore, if $(y_i, b_{-i}) \in C^\ell_{\geq}$, it must be true that $(z_i, b_{-i}) \in C^\ell_{\geq}$, for all $z_i \in X_i$. This shows that $\langle C^k \rangle_{k \in R}$ satisfies the following condition:

$$\left.\begin{array}{c} (x_i, a_{-i}) \in C^k_{\geq} \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell_{\geq} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \in C^k_{\geq} \\ \text{or} \\ (z_i, b_{-i}) \in C^\ell_{\geq} \end{array}\right. \qquad (Eq_i)$$

for all $x_i, y_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $k, \ell \in R^+$ such that $\ell < k$. We say that $\langle C^k \rangle_{k \in R}$ satisfies condition Eq if it satisfies condition Eq_i for all $i \in N$. We have:

Proposition 3
An r-fold partition $\langle C^k \rangle_{k \in R}$ of X has a representation in the noncompensatory sorting model with $\mathscr{A}_i^k = \mathscr{A}_i^\ell$, for all $k, \ell \in R^+$ iff it is R-linear, R-2-graded and satisfies Eq.

Proof
The necessity of R-linearity and R-2-gradedness follows from theorem 1. Suppose that Eq_i is violated so that, for some $k, \ell \in R^+$ with $\ell < k$, $(x_i, a_{-i}) \in C^k_{\geq}$, $(y_i, b_{-i}) \in C^\ell_{\geq}$, $(y_i, a_{-i}) \in C^k_{<}$ and $(z_i, b_{-i}) \in C^\ell_{<}$. The first and the third conditions imply $y_i \notin \mathscr{A}_i^k$. The second and the fourth imply $y_i \in \mathscr{A}_i^\ell$, violating the fact that $\mathscr{A}_i^k = \mathscr{A}_i^\ell$. Hence, Eq_i holds.

Sufficiency. Since $\langle C^k \rangle_{k \in R}$ is R-linear and R-2-graded, it has a representation in the noncompensatory sorting model. Let $\langle \mathscr{F}^k, \langle \mathscr{A}_i^k \rangle_{i \in N} \rangle$ be the representation built in theorem 1. Suppose that, for some $\ell \in R^+$, we have $x_i \in \mathscr{A}_i^\ell$. Since $\mathscr{A}_i^\ell \neq \varnothing$, attribute $i \in N$ is influent for $\langle C^\ell_{\geq}, C^\ell_{<} \rangle$. This implies $(x_i, c_{-i}) \in C^\ell_{\geq}$ and $(z_i, c_{-i}) \in C^\ell_{<}$, for some $z_i \in X_i$ and some $c_{-i} \in X_{-i}$.

Suppose first that attribute $i \in N$ is degenerate for all $k > \ell$. We have $\mathscr{A}_i^k = \varnothing$, for all $k \geq \ell$. However, as shown in section 3.5.2, this representation is not unique and it is always possible to take all sets $\mathscr{A}_i^k$ to be equal to $\mathscr{A}_i^\ell$.

Suppose now that attribute $i \in N$ is influent for for $\langle C^k_{\geq}, C^k_{<} \rangle$, for some $k > \ell$ and that $x_i \notin \mathscr{A}_i^k$. This implies $(y_i, a_{-i}) \in C^k_{\geq}$ and $(x_i, a_{-i}) \in C^k_{<}$, for some $y_i \in X_i$ and some $a_{-i} \in X_{-i}$. Using Eq_i, $k > \ell$, $(y_i, a_{-i}) \in C^k_{\geq}$ and $(x_i, c_{-i}) \in C^\ell_{\geq}$ imply either $(x_i, a_{-i}) \in C^k_{\geq}$ or $(z_i, c_{-i}) \in C^\ell_{\geq}$, a contradiction. Hence, we have shown that for all ℓ and all $k > \ell$, there is a representation such that $\mathscr{A}_i^k = \mathscr{A}_i^\ell$. This completes the proof. □

Let us observe that none of the conditions used in proposition 3 is redundant. Using an ELECTRE TRI model, it is easy to build partitions that are R-linear, R-2-graded and satisfy condition Eq_i holds on all but one attribute. We give below the other two examples.

Example 5
Let $n = 2$, $X = \{x_1, y_1, z_1\} \times \{x_2, y_2\}$ and $r = 3$. Let $C^3 = \{(x_1, x_2), (y_1, x_2), (y_1, y_2)\}$, $C^2 = \{(z_1, x_2)\}$ and $C^1 = \{(x_1, y_2), (z_1, y_2)\}$. We have $y_1 \succ_1^R x_1 \succ_1^R z_1$ and $x_2 \succ_2^R y_2$, so that this partition is R-linear. Condition 2-$graded_2^R$ is trivially satisfied. Condition 2-$graded_1^R$ is violated because $(x_1, x_2) \in C^3$, $(y_1, x_2) \in C^3$ and $(y_1, y_2) \in C^3$ but neither $(x_1, y_2) \in C^3_{\geq}$ not $(z_1, x_2) \in C^3_{\geq}$. A routine check shows that condition Eq holds. ◇

Example 6
Let $n = 3$ and $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$ and $r = 3$. Let $C^3 = \{(x_1, x_2, x_3), (y_1, x_2, x_3)\}$, $C^2 = \{(x_1, x_2, y_3), (y_1, y_2, x_3)\}$ and $C^1 = \{(x_1, y_2, y_3), (y_1, y_2, y_3), (x_1, y_2, x_3), (y_1, x_2, y_3)\}$. Since all sets X_i have two only elements, this partition is trivially R-2-graded. Condition $linear_2$ and $linear_3$ hold with $x_2 \succ_2^R y_2$ and $x_3 \succ_3^R y_3$. Condition $linear_1$ is violated since $(x_1, x_2, y_3) \in C^2$, $(y_1, y_2, x_3) \in C^2$ but neither $(y_1, x_2, y_3) \in C^2_{\geq}$ nor $(x_1, y_2, x_3) \in C^2_{\geq}$. A routine check shows that Eq holds. ◇

3.6.2 The case $\mathscr{F}^k = \mathscr{F}^\ell$

We analyze here what must be added to the conditions in theorem 1 in order to ensure that $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model in which $\mathscr{F}^k = \mathscr{F}^\ell$, for all $k, \ell \in R^+$. In this case, going from $C^k_{\geq}$ to $C^{k-1}_{\geq}$ only involves a change in the definition of the sets $\mathscr{A}_i^k$. In most applications of ELECTRE TRI, this additional hypothesis is used. This case is more difficult to analyze than the preceding one since, in our proofs, the construction of the sets $\mathscr{F}^k$ is left implicit. This is a clear drawback of our technique.

Take any $x \in C^k$ and let $J(x)$ be the subset of attributes $i \in N$ such that:

$$(z_i, x_{-i}) \in C^k_{<}, (y_i, y_{-i}) \in C^{k+1}_{\geq} \text{ and } (x_i, y_{-i}) \in C^{k+1}_{<}, \tag{10}$$

for some $y \in X$ and some $z_i \in X_i$. Hence, $J(x)$ is the subset of attributes such that $x_i \in \mathscr{A}_i^k$, because $(x_i, x_{-i}) \in C^k$ but $(z_i, x_{-i}) \in C^k_{<}$, while $x_i \notin \mathscr{A}_i^{k+1}$, because $(y_i, y_{-i}) \in C^{k+1}_{\geq}$ and $(x_i, y_{-i}) \in C^{k+1}_{<}$.

For all $j \in J(x)$, let $y^j \in X$ be any alternative such that (10) is satisfied. Consider an alternative $w \in X$ such that $w_j = x_j$ if $j \notin J(x)$ and $w_j = y^j_j$ if $j \in J(x)$. The alternative w is identical to x on all attributes such that $x_i \in \mathscr{A}_i^{k+1}$. The same is true on all attributes such that $x_i \notin \mathscr{A}_i^k$. On all attributes such that $x_i \in \mathscr{A}_i^k$ and $x_i \notin \mathscr{A}_i^{k+1}$, we have $w_i \in \mathscr{A}_i^{k+1}$. Therefore, we have $A^{k+1}(w) = A^k(x)$. Because $x \in C^k$, we know that $A^k(x) \subseteq I$ implies $I \in \mathscr{F}^k$. If it is required that $w \in C_{\geq}^{k+1}$, all such subsets I will also belong to $\mathscr{F}^{k+1}$.

We have exhibited a necessary condition for the existence of a representation in which $\mathscr{F}^k = \mathscr{F}^{k+1}$. When added to R-linearity and R-2-gradedness, this condition is also sufficient to ensure the existence of such a representation. Since this new condition is quite cumbersome and the proof is not very instructive, we do not formalize this point further here. We are not presently aware of a more satisfactory characterization of this particular case of the noncompensatory sorting model, in spite of its intuitive appeal.

4 The noncompensatory sorting model with veto

4.1 Definitions

We consider a model generalizing the noncompensatory sorting model in order to allow for possible veto effects. We say that $\langle C^k \rangle_{k \in R}$ has a representation in the *noncompensatory sorting model with veto* if:

- for all $i \in N$ and all $k \in R^+$ there are disjoint sets $\mathscr{A}_i^k, \mathscr{V}_i^k \subseteq X_i$,
- for all $i \in N$ $\mathscr{A}_i^r \subseteq \mathscr{A}_i^{r-1} \subseteq \cdots \subseteq \mathscr{A}_i^2$,
- for all $i \in N$, $\mathscr{V}_i^r \supseteq \mathscr{V}_i^{r-1} \supseteq \cdots \supseteq \mathscr{V}_i^2$,
- for all $k, \ell \in R$ such that $k < \ell$, if $x_i \in \mathscr{A}_i^k$, $y_i \in \mathscr{U}_i^k$ and $x_i \in \mathscr{V}_i^\ell$ then $y_i \in \mathscr{V}_i^\ell$, where, in this section, $\mathscr{U}_i^k = X_i \setminus [\mathscr{A}_i^k \cup \mathscr{V}_i^k]$,
- there are subsets $\mathscr{F}^r$, $\mathscr{F}^{r-1}$, ..., $\mathscr{F}^2$ of 2^N that are monotonic wrt inclusion (i.e. such that $[I \in \mathscr{F}^k$ and $I \subset J] \Rightarrow J \in \mathscr{F}^k)$ such that $\mathscr{F}^r \subseteq \mathscr{F}^{r-1} \subseteq \cdots \subseteq \mathscr{F}^2$,

such that:

$$x \in C_{\geq}^k \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i^k\} \in \mathscr{F}^k \text{ and } \{i \in N : x_i \in \mathscr{V}_i^k\} = \varnothing, \qquad (11)$$

for all $x \in X$ and all $k \in R^+$. As before we note $A^k(x)$ and $V^k(x)$ instead of $\{i \in N : x_i \in \mathscr{A}_i^k\}$ and $\{i \in N : x_i \in \mathscr{V}_i^k\}$ when there is no ambiguity on the underlying sets $\mathscr{A}_i^k$ and $\mathscr{V}_i^k$.

The interpretation of the noncompensatory sorting model with veto is similar to that of the noncompensatory sorting model, the latter being a particular case of the former. The only difference is that, for each $k \in R^+$, there is a set $\mathscr{V}_i^k$ that is repulsive for $C_{\geq}^k$. Since the categories are ordered, the requirement that a level that is repulsive for a given category should be repulsive for all higher categories is not surprising. This explains the introduction of the additional constraints $\mathscr{V}_i^k \supseteq \mathscr{V}_i^{k-1}$. The consistency condition on $\mathscr{A}_i^k$, $\mathscr{U}_i^k$ and $\mathscr{V}_i^\ell$ is interpreted as follows. If $x_i \in \mathscr{A}_i^k$, $y_i \in \mathscr{U}_i^k$, this is an indication that x_i is superior to y_i. Supposing now that, for some $\ell > k$, $x_i \in \mathscr{V}_i^\ell$ and $y_i \notin \mathscr{V}_i^\ell$, would give the inconsistent indication that y_i is superior to x_i.

The pessimistic version of ELECTRE TRI is a particular case of the noncompensatory sorting model with veto. Indeed, remember from section 2.5 that in the pessimistic version of ELECTRE TRI we have, for all $x \in X$,

$$x \in C_{\geq}^k \Leftrightarrow \left[\sum_{i \in S(x,y)} w_i \geq \lambda^k \text{ and } [Not[y_i \; V_i \; x_i], \text{ for all } i \in N] \right].$$

Define $\mathscr{A}_i^k = \{i \in N : x_i \; S_i \; p^k\}$, $\mathscr{V}_i^k = \{i \in N : p^k \; V_i \; x_i\}$ and let $I \in \mathscr{F}^k$ whenever $\sum_{i \in I} w_i \geq \lambda^k$.

By construction, $x_i \in \mathscr{A}_i^k$ implies $g_i(x_i) \geq g_i(p_i^k) - q_i$. Since $g_i(p_i^k > g_i(p_i^{k-1})$, we obtain $g_i(x_i) \geq g_i(p_i^{k-1}) - q_i$, so that $x_i \in \mathscr{A}_i^{k-1}$. The proof that $\mathscr{V}_i^{k-1} \subseteq \mathscr{V}_i^k$ is similar. The sets $\mathscr{A}_i^k$ and $\mathscr{V}_i^k$ are disjoint. Suppose now that $k < \ell$, $x_i \in \mathscr{A}_i^k$, $y_i \in \mathscr{U}_i^k$ and $x_i \in \mathscr{V}_i^\ell$. This implies $g_i(x_i) \geq g_i(p_i^k) - q_i$, $g_i(y_i) < g_i(p_i^k) - q_i$ and $g_i(p_i^\ell) > g_i(x_i) + v_i$. The first two equations imply $g_i(x_i) > g_i(y_i)$. The third equation therefore imply that $g_i(p_i^k) > g_i(y_i) + v_i$, so that $y_i \in \mathscr{V}_i^\ell$. Because $\lambda^k \geq \lambda^{k-1}$, we have $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$. Hence, $\langle \mathscr{F}^k, \langle \mathscr{A}_i^k, \mathscr{V}_i^k \rangle_{i \in N} \rangle$ is a representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model with veto.

4.2 Axioms

The noncompensatory sorting model with veto shares with the noncompensatory sorting model the fact that it implies R-linearity.

Lemma 6

If a partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model with veto then it is R-linear.

PROOF

Suppose that *linear*$_i^R$ is violated so that $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^\ell$, $(y_i, a_{-i}) \in C_{<}^k$ and $(x_i, b_{-i}) \in C_{<}^\ell$. Suppose wlog that $k \leq \ell$. Because $(x_i, a_{-i}) \in C^k$ and $(y_i, a_{-i}) \in C_{<}^k$, we have either $y_i \in \mathscr{U}_i^k$ or $y_i \in \mathscr{V}_i^k$. Because $k \leq \ell$, we know that $\mathscr{V}_i^\ell \supseteq \mathscr{V}_i^k$. It is therefore impossible that $y_i \in \mathscr{V}_i^k$

since this would imply $y_i \in \mathscr{V}_i^\ell$, contradicting $(y_i, b_{-i}) \in C^\ell$. Hence, we must have $y_i \in \mathscr{U}_i^k$, so that $x_i \in \mathscr{A}_i^k$. Because $\mathscr{A}_i^\ell \subseteq \mathscr{A}_i^k$, we know that $y_i \in \mathscr{U}_i^\ell$.

Because $(y_i, b_{-i}) \in C^\ell$, $(x_i, b_{-i}) \in C^\ell_<$ and $y_i \in \mathscr{U}_i^\ell$, we must have $x_i \in \mathscr{V}_i^\ell$. Since we have $x_i \in \mathscr{A}_i^k$, $x_i \in \mathscr{V}_i^\ell$ and $y_i \in \mathscr{U}_i^k$, the definition of the generalized ordinal sorting model with veto implies that $y_i \in \mathscr{V}_i^\ell$, contradicting the fact that $(y_i, b_{-i}) \in C^\ell$. □

Similarly to what was done before, we wish to add to R-linearity conditions that would precipitate the noncompensatory sorting model with veto. We say that $\langle C^k \rangle_{k \in R}$ is R-3v-graded on attribute $i \in N$ (condition *3v-graded*$_i^R$) if:

$$\left.\begin{array}{c} (x_i, a_{-i}) \in C^k_\geq \\ \text{and} \\ (y_i, a_{-i}) \in C^k_\geq \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell_\geq \\ \text{and} \\ (z_i, c_{-i}) \in C^k_\geq \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \in C^\ell_\geq \\ \text{or} \\ (z_i, a_{-i}) \in C^k_\geq \end{array}\right. \qquad (\textit{3v-graded}_i^R)$$

for all $x_i, y_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $k, \ell \in R^+$ such that $\ell \leq k$. We say that $\langle C^k \rangle_{k \in R}$ is R-3v-graded if condition *3v-graded*$_i^R$ holds for all $i \in N$.

Note that condition *3v-graded*$_i^R$ is the weakening of *2-graded*$_i^R$ obtained by adding the premise $(z_i, a_{-i}) \in C^k_\geq$. The intuition behind this weakening is that the noncompensatory sorting model with veto requires condition *2-graded*$_i^R$ to hold for elements that are not repulsive. Adding the premise $(z_i, a_{-i}) \in C^k_\geq$ ensures that $z_i \notin \mathscr{V}_i^k$.

Lemma 7
If a partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model with veto then it is R-3v-graded.

Proof
Suppose that $(x_i, a_{-i}) \in C^k_\geq$, $(z_i, c_{-i}) \in C^k_\geq$ and $(z_i, a_{-i}) \in C^k_<$. By construction, it is impossible that $z_i \in \mathscr{V}_i^k$. Hence, we must have $x_i \in \mathscr{A}_i^k$. Because $\ell \leq k$, we know that $\mathscr{A}_i^k \subseteq \mathscr{A}_i^\ell$. Hence, we have $x_i \in \mathscr{A}_i^\ell$. Since $(y_i, b_{-i}) \in C^\ell_\geq$, the monotonicity of $\mathscr{F}^\ell$ implies $(x_i, b_{-i}) \in C^\ell_\geq$. □

The following lemma summarizes the consequences of conditions *linear*$_i^R$ and *3v-graded*$_i^R$ using the relations $\succsim_i^k$.

Lemma 8
Conditions linear$_i^R$ and 3v-graded$_i^R$ hold iff

1. *$\succsim_i^k$ is a weak order having at most three distinct equivalence classes.*

2. *If* $\succsim_i^k$ *has three distinct equivalence classes and* x_i *is in the last class then, for all* $a_{-i} \in X_{-i}$, $(x_i, a_{-i}) \notin C_{\geq}^k$,

3. $[x_i \succ_i^k y_i] \Rightarrow [x_i \succsim_i^\ell y_i$, *for all* $\ell \in R^+]$,

4. $[x_i \sim_i^k z_i$ *and* $x_i \succ_i^k y_i$ *and* $(y_i, a_{-i}) \in C_{\geq}^k$, *for some* $a_{-i} \in X_{-i}] \Rightarrow [x_i \sim_i^\ell z_i$, *for all* $\ell < k]$.

for all $k \in R^+$ *and all* $x_i, y_i, z_i \in X_i$.

PROOF

Part $[\Leftarrow]$. Suppose that *linear*$_i^R$ is violated so that, for some $x_i, y_i \in X_i$ and some $a_{-i}, b_{-i} \in X_{-i}$, $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^\ell$, $(y_i, a_{-i}) \in C_<^k$ and $(x_i, b_{-i}) \in C_<^\ell$. Because, for all $k \in R$, $\succsim_i^k$ is a weak order, we have $x_i \succ_i^k y_i$ and $y_i \succ_i^\ell x_i$, a contradiction.

Suppose that *3v-graded*$_i^R$ is violated so that, for some $\ell \leq k$, $(x_i, a_{-i}) \in C_{\geq}^k$, $(y_i, a_{-i}) \in C_{\geq}^k$, $(y_i, b_{-i}) \in C_{\geq}^\ell$, $(z_i, c_{-i}) \in C_{\geq}^k$, $(x_i, b_{-i}) \in C_<^\ell$ and $(z_i, a_{-i}) \in C_<^k$.

Suppose that $k = \ell$. This implies $y_i \succ_i^k x_i$ and $x_i \succ_i^k z_i$. This is contradictory since $\succsim_i^k$ has three distinct equivalence classes and z_i belongs to the last class, while $(z_i, c_{-i}) \in C_{\geq}^k$.

Suppose henceforth that $k > \ell$. We have $y_i \succ_i^\ell x_i$, $x_i \succ_i^k z_i$ and $y_i \succ_i^k z_i$. Since $y_i \succ_i^\ell x_i$, we cannot have $y_i \succ_i^k x_i$. Since $(z_i, a_{-i}) \in C_<^k$, we cannot have $x_i \succ_i^k y_i$. If $x_i \sim_i^k y_i$, $x_i \succ_i^k z_i$ and $(z_i, a_{-i}) \in C_<^k$ would imply $x_i \sim_i^\ell y_i$, a contradiction.

Part $[\Rightarrow]$. Using *linear*$_i^R$, we know that $\succsim_i^R$ is complete. Since $\succsim_i^R$ refines $\succsim_i^k$, it follows that $\succsim_i^k$ is complete and, hence, a weak order. Clearly, $x_i \succ_i^k y_i$ and $y_i \succ_i^\ell x_i$ would violate *linear*$_i^R$.

Suppose that, for some $k \in R^+$, $\succsim_i^k$ has at least four distinct equivalence classes so that, for some $x_i, y_i, z_i, w_i \in X_i$, we have $x_i \succ_i^k y_i$, $y_i \succ_i^k z_i$ and $z_i \succ_i^k w_i$. Using the definition of $\succsim_i^k$, we have, for some $a_{-i}, b_{-i}, c_{-i} \in X_{-i}$, $(x_i, a_{-i}) \in C^k$, $(y_i, a_{-i}) \in C_<^k$, $(y_i, b_{-i}) \in C^k$, $(z_i, b_{-i}) \in C_<^k$, $(z_i, c_{-i}) \in C^k$, $(w_i, c_{-i}) \in C_<^k$.

Using *linear*$_i^R$, $(x_i, a_{-i}) \in C^k$, $(y_i, b_{-i}) \in C^k$ and $(y_i, a_{-i}) \in C_<^k$ imply $(x_i, b_{-i}) \in C_{\geq}^k$. Using *3v-graded*$_i^R$ with $\ell = k$, $(y_i, b_{-i}) \in C^k$, $(x_i, b_{-i}) \in C_{\geq}^k$, $(x_i, a_{-i}) \in C^k$ and $(z_i, c_{-i}) \in C^k$ imply either $(y_i, a_{-i}) \in C_{\geq}^k$ or $(z_i, b_{-i}) \in C_{\geq}^k$, a contradiction.

Suppose now that, for some $\ell < k$ and some $x_i, y_i, z_i \in X_i$, $x_i \sim_i^k z_i$, $x_i \succ_i^k y_i$, $(y_i, a_{-i}) \in C_{\geq}^k$ and $x_i \succ_i^\ell z_i$. By definition, $x_i \sim_i^k z_i$ and $x_i \succ_i^k y_i$ imply that $(x_i, b_{-i}) \in C^k$, $(z_i, b_{-i}) \in C^k$ and $(y_i, b_{-i}) \in C_<^k$, for some $b_{-i} \in X_{-i}$. Similarly $x_i \succ_i^\ell z_i$ implies $(x_i, c_{-i}) \in C^\ell$ and $(z_i, c_{-i}) \in C_<^\ell$, for some $c_{-i} \in X_{-i}$. Using *3v-graded*$_i^R$, $(z_i, b_{-i}) \in C^k$, $(x_i, b_{-i}) \in C^k$, $(x_i, c_{-i}) \in C^\ell$ and $(y_i, a_{-i}) \in C_{\geq}^k$ imply $(z_i, c_{-i}) \in C_{\geq}^\ell$ or $(y_i, b_{-i}) \in C_{\geq}^k$, a contradiction. □

4.3 Result

Our main result in this section says that R-linearity and R-3v-gradedness characterize the noncompensatory sorting model with veto.

Theorem 2
An r-fold partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model with veto iff it is R-linear and R-3v-graded.

Proof
Necessity results from lemmas 6 and 7. We show sufficiency. Suppose that $\langle C^k \rangle_{k \in R}$ is R-linear and R-3v-graded. For all $k \in R^+$, define $\mathscr{V}_i^k = \{x_i \in X_i : (x_i, a_{-i}) \in C^k_<, \text{ for all } a_{-i} \in X_{-i}\}$. By construction, the constraints $\mathscr{V}_i^k \subseteq \mathscr{V}_i^\ell$ for $\ell > k$ are always satisfied with such a definition.

Let $Y_i^k = X_i \setminus \mathscr{V}_i^k$ and $Y^k = \prod_{i \in N} Y_i^k$. We have $Y_i^k \subseteq Y_i^{k-1}$, for all $k \in \{r, r-1, \dots, 3\}$. Since $\langle C^k \rangle_{k \in R}$ is a partition, $\langle C^k_\geq, C^k_< \rangle$ is a partition, for all $k \in R^+$. Hence the sets Y^k are nonempty, for all $k \in R^+$. Let $D^k_\geq = C^k_\geq \cap Y^k$ and $D^k_< = C^k_< \cap Y^k$. Our plan is to build a representation of the twofold partitions $\langle D^k_\geq, D^k_< \rangle$ in the noncompensatory sorting model. The set $\mathscr{A}_i^k$ will always be a subset of Y_i^k. Hence, $\mathscr{V}_i^k$ and $\mathscr{A}_i^k$ will be disjoint.

Suppose first that for all $k \in R^+$, all attributes $i \in N$ are degenerate for $\langle D^k_\geq, D^k_< \rangle$. By construction of the sets Y_i^k, we have $D^k_\geq = Y^k$. In this case, define, for all $k \in R^+$ and all $i \in N$, $\mathscr{A}_i^k = Y_i^k$ and $\mathscr{F}^k = \{N\}$. This clearly gives a representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model with veto.

Otherwise, let ℓ^* be the largest $k \in R^+$ such that there is at least one influent attribute for $\langle D^k_\geq, D^k_< \rangle$. In order to build the representation of $\langle C^k \rangle_{k \in R}$, we distinguish several cases.

Case $\ell^* < k$
By hypothesis, all attributes are degenerate for $\langle D^k_\geq, D^k_< \rangle$. By construction of the sets Y^k, we must have $D^k_\geq = Y^k$. In this case, we define:

- $\mathscr{A}_i^k = Y_i^k \cap \mathscr{A}_i^{\ell^*}$, for all $i \in N$,
- $\mathscr{F}^k = \mathscr{F}^{\ell^*}$,

where $\mathscr{A}_i^{\ell^*}$ and $\mathscr{F}^{\ell^*}$ will be defined below. For all $k \geq \ell^*$, the constraints $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$ will be satisfied; since $Y_i^{k+1} \subseteq Y_i^k$, we also have that $\mathscr{A}_i^{k+1} \subseteq \mathscr{A}_i^k$.

Case $k \leq \ell^*$
By hypothesis, some attribute is influent for $\langle D^k_{\geq}, D^k_{<}\rangle$, so that it is a partition. It is clear that this twofold partition satisfies (6). Let us show that it satisfies (7). Suppose that $(x_i, a_{-i}) \in D^k_{\geq}$, $(y_i, a_{-i}) \in D^k_{\geq}$, $(y_i, b_{-i}) \in D^k_{\geq}$, $(x_i, b_{-i}) \in D^k_{<}$ and $(z_i, a_{-i}) \in D^k_{<}$, for some $x_i, y_i, z_i \in Y^k_i$ and some $a_{-i}, b_{-i} \in Y^k_{-i}$. Because $z_i \in Y^k_i$, we know that $(z_i, c_{-i}) \in D^k_{\geq}$, for some $c_{-i} \in Y^k_{-i}$. Using *3v-graded*^R_i, $(x_i, a_{-i}) \in D^k_{\geq}$, $(y_i, a_{-i}) \in D^k_{\geq}$, $(y_i, b_{-i}) \in D^k_{\geq}$ and $(z_i, c_{-i}) \in D^k_{\geq}$ imply $(x_i, b_{-i}) \in D^k_{\geq}$ or $(z_i, a_{-i}) \in D^k_{\geq}$, a contradiction. Hence, using proposition 2, $\langle D^k_{\geq}, D^k_{<}\rangle$ has a representation $\langle \mathscr{G}^k, \langle \mathscr{B}^k_i\rangle_{i\in N}\rangle$ in the noncompensatory sorting model.

Subcase $k = \ell^*$
Consider first the case $k = \ell^*$. Let $\langle \mathscr{G}^{\ell^*}, \langle \mathscr{B}^{\ell^*}_i\rangle_{i\in N}\rangle$ be the representation of $\langle D^{\ell^*}_{\geq}, D^{\ell^*}_{<}\rangle$ built using proposition 2. We define, for all $i \in N$, $\mathscr{A}^{\ell^*}_i = \mathscr{B}^{\ell^*}_i$ and $\mathscr{F}^{\ell^*} = \mathscr{G}^{\ell^*}$. If $i \in N$ is influent for $\langle D^{\ell^*}_{\geq}, D^{\ell^*}_{<}\rangle$, we have $\varnothing \subsetneq \mathscr{A}^{\ell^*}_i \subsetneq Y^{\ell^*}$. If $i \in N$ is degenerate for $\langle D^{\ell^*}_{\geq}, D^{\ell^*}_{<}\rangle$, we have $\mathscr{A}^{\ell^*}_i = \varnothing$. By construction, $\langle \mathscr{F}^{\ell^*}, \langle \mathscr{A}^{\ell^*}_i\rangle_{i\in N}\rangle$ is a representation of $\langle D^{\ell^*}_{\geq}, D^{\ell^*}_{<}\rangle$ in the noncompensatory sorting model. In this representation, we have $I \in \mathscr{F}^{\ell^*}$ whenever there is some $x \in D^{\ell^*}_{\geq}$ such that $A^{\ell^*}(x) \subseteq I$.

Correctness of the choices for $k > \ell^*$
Let $k > \ell^*$. We know that $D^k_{\geq} = Y^k$. We have defined above $\mathscr{A}^k_i = Y^k_i \cap \mathscr{A}^{\ell^*}_i$ and $\mathscr{F}^k = \mathscr{F}^{\ell^*}$. By construction, for all $x \in D^k_{\geq} = Y^k$, we have $A^k(x) = A^{\ell^*}(x)$ and $A^{\ell^*}(x) \in \mathscr{F}^{\ell^*}$, so that $A^k(x) \in \mathscr{F}^k$. Hence, the above choices give, for all $k > \ell^*$, a representation of $\langle D^k_{\geq}, D^k_{<}\rangle$ in the noncompensatory sorting model.

Subcase $k < \ell^*$
Suppose henceforth that $k < \ell^*$ and let $\langle \mathscr{G}^k, \langle \mathscr{B}^k_i\rangle_{i\in N}\rangle$ be the representation of $\langle D^k_{\geq}, D^k_{<}\rangle$ in the noncompensatory sorting model built in proposition 2. We build the sets $\mathscr{F}^k$ and $\mathscr{A}^k_i$ in sequence starting with $k = \ell^* - 1$.

Let $\mathscr{F}^k = \mathscr{G}^k$. If $i \in N$ is influent for $\langle D^k_{\geq}, D^k_{<}\rangle$, we take $\mathscr{A}^k_i = \mathscr{B}^k_i$. We clearly have $\varnothing \subsetneq \mathscr{A}^k_i \subsetneq Y^k_i$. If $i \in N$ is degenerate for $\langle D^k_{\geq}, D^k_{<}\rangle$, we have $\mathscr{B}^k_i = \varnothing$. We take $\mathscr{A}^k_i = \mathscr{A}^{k+1}_i$; since $Y^{k+1}_i \subseteq Y^k_i$, this is always possible. On these attributes, we have $\mathscr{A}^k_i \subsetneq Y^k_i$ since we never have $\mathscr{A}^k_i = Y^k_i$, for all $k \leq \ell^*$.

Because it is always true that $\mathscr{A}^k_i \subsetneq Y^k_i$, we have $I \in \mathscr{F}^k$ whenever there is some $x \in D^k_{\geq}$ such that $A^k(x) \subseteq I$. Clearly, $\langle \mathscr{F}^k, \langle \mathscr{A}^k_i\rangle_{i\in N}\rangle$ is a representation of $\langle D^k_{\geq}, D^k_{<}\rangle$ in the noncompensatory sorting model.

We have now defined the sets $\mathscr{A}_i^k, \mathscr{V}_i^k$ and $\mathscr{F}^k$, for all $i \in N$ and all $k \in R^+$. We know that for all $k \in R^+$, these sets give a representation of $\langle C_{\geq}^k, C_{<}^k \rangle$ in the noncompensatory sorting model with veto. It remains to check that the additional constraints of the noncompensatory sorting model with veto are satisfied.

Proof that $\mathscr{A}_i^{k+1} \subseteq \mathscr{A}_i^k$
This was shown above for $k \geq \ell^*$. Let us prove that, for all $i \in N$ and all $k = 2, 3, \ldots, \ell^* - 1$ we have $\mathscr{A}_i^{k+1} \subseteq \mathscr{A}_i^k$.

If, for all $\ell \in \{\ell^*, \ell^* - 1, \ldots, k+1\}$, attribute $i \in N$ is degenerate for the partition $\langle D_{\geq}^{\ell}, D_{<}^{\ell} \rangle$ induced on $\prod_{i \in N} Y_i^k$, we have $\mathscr{A}_i^{k+1} = \varnothing$ and there is nothing to prove. Similarly if $i \in N$ is degenerate for the partition $\langle D_{\geq}^k, D_{<}^k \rangle$ induced on $\prod_{i \in N} Y_i^k$, we have $\mathscr{A}_i^k = \mathscr{A}_i^{k+1}$ and there is nothing to prove either.

Suppose henceforth that $i \in N$ is influent for the partition $\langle D_{\geq}^k, D_{<}^k \rangle$ induced on $\prod_{i \in N} Y_i^k$ and let ℓ be the smallest $m \in \{\ell^*, \ell^* - 1, \ldots, k+1\}$ such that $i \in N$ is influent for the partition $\langle D_{\geq}^{\ell}, D_{<}^{\ell} \rangle$ induced on $\prod_{i \in N} Y_i^{\ell}$. By construction, we have $\mathscr{A}_i^{k+1} = \mathscr{A}_i^{\ell}$.

Suppose, in contradiction with the thesis, that $x_i \in \mathscr{A}_i^{\ell}$ and $x_i \notin \mathscr{A}_i^k$ with $\ell > k$. Since $i \in N$ is influent for the partition $\langle D_{\geq}^{\ell}, D_{<}^{\ell} \rangle$ and $x_i \in \mathscr{A}_i^{\ell}$, we know that $(x_i, a_{-i}) \in D_{\geq}^{\ell}$ and $(y_i, a_{-i}) \notin D_{\geq}^{\ell}$, for some $y_i \in Y_i^{\ell}$ and some $a_{-i} \in Y_{-i}^{\ell}$. Because, by construction, $y_i \in Y_i^{\ell}$, we know that $(y_i, c_{-i}) \in D_{\geq}^{\ell}$, for some $c_{-i} \in Y_{-i}^{\ell}$. Similarly, since $i \in N$ is influent for $\langle D_{\geq}^k, D_{<}^k \rangle$ and $x_i \notin \mathscr{A}_i^k$, we know that $(z_i, b_{-i}) \in D_{\geq}^k$ and $(x_i, b_{-i}) \notin D_{\geq}^k$, for some $z_i \in Y_i^k$ and some $b_{-i} \in Y_{-i}^k$.

Using *linear*$_i^R$, $(x_i, a_{-i}) \in D_{\geq}^{\ell}$, $(z_i, b_{-i}) \in D_{\geq}^k$ and $(x_i, b_{-i}) \notin D_{\geq}^k$ imply $(z_i, a_{-i}) \in D_{\geq}^{\ell}$. Using *3v-graded*$_i^R$, $(x_i, a_{-i}) \in D_{\geq}^{\ell}$, $(z_i, a_{-i}) \in D_{\geq}^{\ell}$, $(z_i, b_{-i}) \in D_{\geq}^k$ and $(y_i, c_{-i}) \in D_{\geq}^{\ell}$ imply $(x_i, b_{-i}) \in D_{\geq}^k$ or $(y_i, a_{-i}) \in D_{\geq}^{\ell}$, a contradiction. Hence we have $\mathscr{A}_i^{k+1} \subseteq \mathscr{A}_i^k$.

Proof of the consistency condition
Suppose that for some $k < \ell$, $x_i \in \mathscr{A}_i^k$, $y_i \in \mathscr{U}_i^k$ and $x_i \in \mathscr{V}_i^{\ell}$. We have to show that $y_i \in \mathscr{V}_i^{\ell}$. Because $x_i \in \mathscr{A}_i^k$, $y_i \in \mathscr{U}_i^k$, we have $(x_i, a_{-i}) \in D_{\geq}^k$ and $(y_i, a_{-i}) \notin D_{\geq}^k$, for some $a_{-i} \in Y_{-i}^k$. In contradiction with the thesis, suppose that $(y_i, b_{-i}) \in D_{\geq}^{\ell}$, for some $b_{-i} \in Y_{-i}^{\ell}$. Using *linear*$_i^R$, $(x_i, a_{-i}) \in D_{\geq}^k$ and $(y_i, b_{-i}) \in D_{\geq}^{\ell}$ imply $(y_i, a_{-i}) \in D_{\geq}^k$ or $(x_i, b_{-i}) \in D_{\geq}^{\ell}$. This is contradictory since we know that $(y_i, a_{-i}) \notin D_{\geq}^k$ and $x_i \in \mathscr{V}_i^{\ell}$.

Proof that $\mathscr{F}^k \subseteq \mathscr{F}^{k-1}$
This was already shown for $k > \ell^*$. Suppose henceforth that $k \leq \ell^*$. The

proof is identical to that of theorem 1. Indeed, for all $k \leq \ell^*$, we know that $I \in \mathscr{F}^k$ whenever there is some $x \in D^k_{\geq}$ such that $A^k(x) \subseteq I$. Take any such alternative $x \in D^k_{\geq}$. Starting with x, let us build an alternative $x' \in Y^k \subseteq Y^{k-1}$ as follows. For all $i \in N$ such that $x_i \in \mathscr{A}^k_i$, let $x'_i = x_i$. Because $\mathscr{A}^k_i \subseteq \mathscr{A}^{k-1}_i$, we know that on these attributes $x'_i \in \mathscr{A}^{k-1}_i$. For all $i \in N$ such that $x_i \notin \mathscr{A}^k_i$, we consider two cases.

1. Suppose that $i \in N$ is influent for $\langle D^{k-1}_{\geq}, D^{k-1}_{<}\rangle$. By construction, there is a $z_i \in Y^{k-1}_i$ such that $z_i \notin \mathscr{A}^{k-1}_i$. In this case, let $x'_i = z_i$. By construction, we know that $x'_i \notin \mathscr{A}^{k-1}_i$ on these attributes.

2. Suppose that $i \in N$ is not influent for $\langle D^{k-1}_{\geq}, D^{k-1}_{<}\rangle$. By construction, we have $\mathscr{A}^k_i = \mathscr{A}^{k-1}_i$. In this case, take x'_i equal to x_i. Therefore $x'_i \notin \mathscr{A}^{k-1}_i$ on these attributes.

By construction, we have $A^k(x) = A^k(x') = A^{k-1}(x') = I$. Because $A^k(x) = A^k(x')$ and $x \in D^k_{\geq}$, we know that $x' \in D^k_{\geq}$ so that $x' \in D^{k-1}_{\geq}$. We have $x \in D^{k-1}_{\geq}$ and $A^{k-1}(x') \subseteq I$. Hence, it must be true that $I \in \mathscr{F}^{k-1}$. This completes the proof. □

The construction of a representation in the noncompensatory sorting model with veto is illustrated below.

Example 7
Suppose that $n = 3$, $X_1 = X_2 = X_3 = \{8, 9, 10, 11\}$. We consider a threefold partition $\langle C^1, C^2, C^3\rangle$. Let $C^3 = \{(10, 9, 10), (10, 9, 11), (10, 10, 10), (10, 10, 11), (10, 11, 10), (10, 11, 11), (11, 9, 10), (11, 9, 11), (11, 10, 10), (11, 10, 11), (11, 11, 10), (11, 11, 11)\}$, $C^2 = \{(8, 10, 9), (8, 10, 10), (8, 10, 11), (8, 11, 9), (8, 11, 10), (8, 11, 11), (9, 10, 9), (9, 10, 10), (9, 10, 11), (9, 11, 9), (9, 11, 10), (9, 11, 11), (10, 9, 9), (10, 10, 8), (10, 10, 9), (10, 11, 8), (10, 11, 9), (11, 9, 9), (11, 10, 8), (11, 10, 9), (11, 11, 8), (11, 11, 9)\}$ and $C^1 = X \setminus [C^3 \cup C^2]$.

This partition can be obtained with the pessimistic version of ELECTRE TRI with $(10, 10, 10)$ as the limiting profile between C^3 and C^2 and $(10, 10, 9)$ as the limiting profile between C^2 and C^1, $S_i = \geq$ for all $i \in N$, $w_1 = w_3 = 0,4$, $w_2 = 0,2$, $\lambda^3 = 2/3$, $\lambda^2 = 0,55$, $V_1 = \varnothing$ and $V_i = \{(10, 8), (11, 8)\}$, for $i \in \{2, 3\}$. This shows that this partition is R-linear and R-3v-graded.

We have:

- $\mathscr{V}^3_1 = \mathscr{V}^3_3 = \{8, 9\}$, $\mathscr{V}^3_2 = \{8\}$,
- $\mathscr{V}^2_1 = \mathscr{V}^2_3 = \varnothing$, $\mathscr{V}^2_2 = \{8\}$.

For the partition the twofold partition $\langle D^3_{\geq}, D^3_{<}\rangle$ on $Y^3 = \{10, 11\} \times \{9, 10, 11\} \times \{10, 11\}$, all attributes are degenerate. All attributes are influent for the twofold partition $\langle D^2_{\geq}, D^2_{<}\rangle$ on $Y^2 = \{8, 9, 10, 11\} \times \{9, 10, 11\} \times \{8, 9, 10, 11\}$.

Let $\langle \mathscr{G}^2, \langle \mathscr{B}^2_1, \mathscr{B}^2_2, \mathscr{B}^2_3\rangle\rangle$ be the unique representation of $\langle D^2_{\geq}, D^2_{<}\rangle$ induced on Y^2 by $\langle C^2_{\geq}, C^2_{<}\rangle$ derived from proposition 2. We have $\mathscr{B}^2_1 = \{10, 11\}$, $\mathscr{B}^2_2 = \{10, 11\}$, $\mathscr{B}^2_3 = \{9, 10, 11\}$ and $\mathscr{G}^2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. We take $\mathscr{A}^2_i = \mathscr{B}^2_i$, for all $i \in N$ and $\mathscr{F}^2 = \mathscr{G}^2$. As in the above proof, we take $\mathscr{A}^3_i = Y^3_i \cap \mathscr{A}^2_i$, for all $i \in N$ and $\mathscr{F}^3 = \mathscr{F}^2$.

It can easily be checked that $\langle \mathscr{F}^k, \langle \mathscr{A}^k_i, \mathscr{V}^k_i\rangle_{i \in N}\rangle$ is a representation of $\langle \mathscr{A}, \mathscr{U}\rangle$ in the noncompensatory sorting model with veto. ◇

4.4 Remarks

In view of the complexity of the noncompensatory sorting model with veto, we do not pursue here a detailed analysis of particular cases of the noncompensatory sorting model with veto as was done in section 3.6. Such an analysis is likely to be quite cumbersome. We simply analyze below the independence of the conditions used in theorem 2 and the uniqueness of the representation in the noncompensatory sorting model with veto.

4.4.1 Independence of conditions

Example 3 above gives a partition that is R-2-graded and that satisfies *linear*^R_i on all but one attribute. Since R-2-gradedness implies R-3v-gradedness, this gives an example showing that, in theorem 2, no condition *linear*^R_i is redundant.

The following example shows that a partition may be R-linear and may satisfy *3v-graded*^R_i on all but one attribute.

Example 8
Let $n = 3$, $X = \{x_1, y_1, z_1\} \times \{x_2, y_2, z_2\} \times \{x_3, y_3, z_3\}$ and $r = 4$. Let $C^4 = \{(x_1, x_2, x_3), (y_1, x_2, x_3), (z_1, x_2, x_3), (x_1, y_2, x_3), (y_1, y_2, x_3), (z_1, y_2, x_3), (x_1, x_2, y_3), (y_1, x_2, y_3), (z_1, x_2, y_3), (x_1, y_2, y_3), (y_1, y_2, y_3)\}$, $C^3 = \{(z_1, y_2, y_3)\}$, $C^2 = \{(x_1, x_2, z_3), (y_1, x_2, z_3), (x_1, y_2, z_3), (y_1, y_2, z_3), (x_1, z_2, x_3), (y_1, z_2, x_3), (x_1, z_2, y_3), (y_1, z_2, y_3), (y_1, z_2, z_3)\}$ and $C^1 = \{(x_1, z_2, z_3), (z_1, z_2, z_3), (z_1, y_2, z_3), (z_1, z_2, y_3), (z_1, x_2, z_3), (z_1, z_2, x_3)\}$.

We have $y_1 \succ^R_1 x_1 \succ^R_1 z_1$, $x_2 \succ^R_2 y_2 \succ^R_2 z_2$ and $x_3 \succ^R_3 y_3 \succ^R_3 z_3$. This shows that the partition is R-linear.

Condition *3v-graded*$^R_1$ is violated since $(x_1, y_2, y_3) \in C^4_{\geq}$, $(y_1, y_2, y_3) \in C^4_{\geq}$, $(y_1, z_2, z_3) \in C^2_{\geq}$ and $(z_1, x_2, x_3) \in C^4_{\geq}$, while $(x_1, z_2, z_3) \notin C^2_{\geq}$ and $(z_1, y_2, y_3) \notin C^4_{\geq}$.

We have $x_2 \succ_2^4 y_2 \succ_2^4 z_2$, $[x_2 \sim_2^3 y_2] \succ_2^3 z_2$ and $[x_2 \sim_2^2 y_2] \succ_2^2 z_2$. We never have $(\alpha_1, z_2, \alpha_3) \in C_{\geq}^3$. Using lemma 8, this shows that *3v-graded*$_2^R$ holds. Similarly, it is easy to check that *3v-graded*$_3^R$ holds. $\diamond$

Hence, we have shown that none of the conditions used in theorem 2 is redundant. Note that, in example 8, the weakening of condition *3v-graded*$_i^R$ obtained requiring *3v-graded*$_i^R$ only when $k = \ell$ is satisfied, for all $i \in N$. Similarly, in example 3, the weakening of *linear*$_i^R$ requiring *linear*$_i^R$ only when $\ell = k$ is satisfied, for all $i \in N$. Hence, our two conditions may not be weakened in this way.

4.4.2 Uniqueness

Let $\langle \mathscr{A}, \mathscr{U} \rangle$ be a twofold partition of X. Define let $Z_i = \{x_i \in X_i : (x_i, a_{-i}) \in \mathscr{U}$, for all $a_{-i} \in X_{-i}\}$ and $Y_i = X_i \setminus Z_i$. Let $Y = \prod_{i=1}^n Y_i$ and define $\mathscr{A}' = \mathscr{A} \cap Y$ and $\mathscr{U}' = \mathscr{U} \cap Y$. We show in Bouyssou and Marchant (2004) that the representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ is unique if and only if all attributes $i \in N$ are influent for $\langle \mathscr{A}', \mathscr{U}' \rangle$.

As was the case for the noncompensatory sorting model, the additional constraints brought by the noncompensatory sorting model with veto with more than two categories are such that this sufficient condition is no longer necessary. Since the noncompensatory sorting model is a particular case of the noncompensatory sorting model with veto, example 4 illustrates this possibility. The uniqueness of the representation in the noncompensatory sorting model with veto can be analyzed using the same lines as in section 3.5.2. Since the details are cumbersome, we do not develop this point here. It should nevertheless be clear that as soon as some attribute is degenerate for a twofold partition $\langle D_{\geq}^k, D_{<}^k \rangle$, the uniqueness of the representation in the noncompensatory sorting model with veto will be quite unlikely.

5 Conclusion

This paper has provided a characterization of the noncompensatory sorting model with and without veto, extending the results presented in Bouyssou and Marchant (2004) to an arbitrary number of ordered categories. This characterization was performed within the general framework of decomposable sorting models proposed by Goldstein (1991) that obtains for R-linear partitions. This characterization shows that the main distinctive characteristic of these models lies in the rather poor information they use on each attribute. This feature was captured using either R-2-gradedness (for the case without veto) or by R-3v-gradedness (for the case with veto). These conditions are central for the ELECTRE TRI sorting model. Hence, the reasonableness of this model is clearly linked with the reasonableness of these two conditions.

We would like to conclude emphasizing the fact that sorting models offer a widely open field for foundational research in the area of MCDM. The decomposable sorting model seems to be quite a convenient framework to analyze a large variety of sorting models. This is the subject of an ongoing research.

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