Curves and Symmetric Spaces, II

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We describe the canonical model of an algebraic curve of genus 9 over a perfect field when the Clifford index is maximal (=3) by means of linear systems of higher rank.

Let $SpG(n, 2n)$ be the *symplectic Grassmannian*, that is, the Grassmannian of Lagrangian subspaces of a $2n$-dimensional symplectic vector space, over a field k. In the case $n = 3$, $SpG(3, 6)$ is a 6-dimensional homogeneous variety and (equivariantly) embedded into the projective space $\mathbf{P}^{13}$ with homogeneous coordinate $(y : X : Y : x)$, where $x, y \in k$ are scalars and $X, Y \in Sym_3\, k$ are symmetric matrices. Then $SpG(3, 6) \subset \mathbf{P}^{13}$ is the common zero locus of the 21 (=6+6+9) quadratic equations

$$X' = yY, \quad Y' = xX \in Sym_3\, k \quad \text{and} \quad XY = xyI_3 \in Mat_3\, k. \tag{0.1}$$

In our study of Fano 3-folds, we observed that this (symmetric) projective variety has a *canonical curve section* of genus 9, that is, a transversal intersection

$$[C \subset \mathbf{P}^8] = [SpG(3, 6) \subset \mathbf{P}^{13}] \cap H_1 \cap \cdots \cap H_5$$

is a curve of genus 9 embedded in $\mathbf{P}^8$ by the ratio of the differentials of the first kind. We showed that every general curve of genus 9 was obtained in this way when $k = \mathbf{C}$ ([12], Corollary 6.3). The purpose of this article is to show the following refinement, which was partly announced in [14].

Theorem A *Let C be a curve of genus 9 over an algebraically closed field. Then C is isomorphic to a transversal linear section of the 6-dimensional symplectic Grassmannian $SpG(3, 6) \subset \mathbf{P}^{13}$ if and only if C is not pentagonal,* i.e., *C has no g^1_5.*

By Bertini's theorem we have

Corollary *C is contained in a smooth K3 surface as an ample divisor.*

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applied to our classification of Gorenstein Fano 3-folds with only canonical singularities (cf. [17]).

We prove the theorem using a certain simple vector bundle of rank 3. By its uniqueness (see below) and by a standard descent argument (§7), we have the following also:

Theorem B *Let C be a curve of genus 9 defined over a perfect field k and assume that C has no g^1_5 over the algebraic closure $\bar{k}$. Then we have*

(1) *C has an embedding into the 6-dimensional symplectic Grassmannian $SpG(3,6) \subset \mathbf{P}^{13}$ over k whose image is a transversal intersection with a k-linear subspace $P \subset \mathbf{P}^{13}$ of dimension 8, and*

(2) *such subspaces P cutting out C are unique up to the action of $PGSp(3)$. More precisely, for every isomorphism $g : C = SpG(3,6) \cap P \to C' = SpG(3,6) \cap P'$ there exists $\gamma \in PGSp(3,k)$ such that $\gamma(P) = P'$.*

Here $GSp(3)$ is the subgroup of $GL(6)$ stabilizing the 1-dimensional space generated by a symplectic form and $PGSp(3)$ is its quotient by the center. Let $G(8, \mathbf{P}^{13})$ be the Grassmannian of 8-dimensional linear subspaces P of $\mathbf{P}^{13}$ and $G(8, \mathbf{P}^{13})^t$ the open subset consisting of P's such that the intersection $P \cap SpG(3,6)$ is transversal.

Corollary *The weighted cardinality, or mass, of the non-pentagonal curves C of genus 9 over the finite field $\mathbf{F}_q$ is equal to $\#G(8, \mathbf{P}^{13})^t / \#PGSp(3, \mathbf{F}_q)$:*

$$\sum_{\text{non-pentagonal}} \frac{1}{\#\mathrm{Aut}_{\mathbf{F}} C} = \frac{\#G(8, \mathbf{P}^{13})^t(\mathbf{F}_q)}{q^9(q^6-1)(q^4-1)(q^2-1)}.$$

The key of the proof is linear systems of higher rank (§3), especially their semi-irreducibility. Let C be as in Theorem A and α a g^2_8 of C, which exists by Brill-Noether theory (cf. [1], Chap. 7). Let β be the Serre adjoint $K_C\alpha^{-1}$ and Q_β the dual of the kernel of the evaluation homomorphism $3\mathcal{O}_C \longrightarrow \beta$. Then there exists a unique nontrivial extension of α by Q_β with $h^0(E) = 6$ (Lemma 5.2 and 5.4). Moreover, such an extension E, often denoted by E_{max}, does not depend on the choice of α and is characterized by the following property (Proposition 5.6) :

$$\begin{cases} i) & \bigwedge^3 E \simeq K_C, \\ ii) & h^0(E) = 6, \text{ and} \\ iii) & |E| \text{ is free and semi-irreducible (Definition 3.3).} \end{cases} \tag{0.2}$$

the same degree (=42). As a corollary of these arguments, we have a bijection between $W^2_8(C)$ and the intersection $G(3,6) \cap \mathbf{P}^{10}$ (Remark 5.7).

Let $\Phi_E : C \longrightarrow G(H^0(E_{max}), 3)$ be the Grassmannian morphism associated with the complete linear system $|E_{max}|$.

Theorem C *Let C be a non-hyperelliptic curve of genus 9 over an algebraically closed field and assume that a rank 3 vector bundle $E = E_{max}$ on it satisfies the condition* (0.2). *Then the natural linear maps*

$$\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0(\bigwedge^2 E) \quad \text{and} \quad \lambda_3 : \bigwedge^3 H^0(E) \longrightarrow H^0(\bigwedge^3 E) \simeq H^0(K_C)$$

surjective and $\mathrm{Ker}\,\lambda_2$ *is generated by a nondegenerate bivector σ. The image of Φ_E is contained in the symplectic Grassmannian $G(H^0(E), \sigma)$ (see §2) and the commutative diagram*

$$\begin{array}{ccccc} & C & \longrightarrow & G(H^0(E), \sigma) & \\ \text{canonical} & \downarrow & & \downarrow & \text{Plücker} \\ & \mathbf{P}^8 & \longrightarrow & \mathbf{P}^* \bigwedge^3 (H^0(E), \sigma) & \\ & & \mathbf{P}^*\bar{\lambda}_3 & & \end{array} \qquad (0.3)$$

is cartesian, where $\bar{\lambda}_3$ is the linear map

$$\bigwedge^3 (H^0(E), \sigma) := \bigwedge^3 H^0(E)/(\sigma \wedge H^0(E)) \longrightarrow H^0(\bigwedge^3 E) \simeq H^0(K_C) \qquad (0.4)$$

induced by λ_3.

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Notation and conventions. For a vector space V, the second exterior product $\bigwedge^2 V$ is the quotient of $V \otimes V$ by the subspace generated by $v \otimes v$, $v \in V$. Similarly $S^2 V$ is the quotient by that generated by $u \otimes v - v \otimes u$, $u, v \in V$. An element of $\bigwedge^2 V$ is called a *bivector* of V. We denote by $G(r, V)$ and $G(V, r)$ the Grassmannians of r-dimensional subspaces and quotient spaces of V, respectively. Two projective spaces $G(1, V)$ and functor and $\mathbf{P}^*$ is contravariant. For a vector space or vector bundle V, its dual is denoted by $V^\vee$. The tensor product symbol $\otimes$ between a vector bundle and a line bundle is often omitted when there seems no fear of confusion.

All (algebraic) varieties are considered over a fixed base field k. A smooth complete irreducible curve is simply called a *curve*. By a g^r_d, we mean a line bundle L on a curve with $\deg L = d$ and $\dim H^0(L) \geq r+1$. A *saturation* of a subsheaf $F' \subset E$ is the largest subsheaf F between F' and E such that F'/F is torsion.

1 Preliminary

We prove two lemmas on the number of global sections. Let ξ be a line bundle on a curve C and η the Serre adjoint $K_C\xi^{-1}$. We denote the evaluation homomorphism $H^0(\eta) \otimes_k \mathcal{O}_C \longrightarrow \eta$ by ev_η and the dual of its kernel by Q_η. We have an exact sequence

$$0 \longrightarrow Q_\eta^\vee \longrightarrow H^0(\eta) \otimes_k \mathcal{O}_C \longrightarrow \eta. \qquad (1.1)$$

Its dual

$$0 \longrightarrow \eta^{-1} \longrightarrow H^0(\eta)^\vee \otimes_k \mathcal{O}_C \longrightarrow Q_\eta \longrightarrow 0 \qquad (1.2)$$

is also exact if η is free. The rank of Q_η is equal to $\dim |\eta| = r - 1$, where we put $r = h^0(\eta)$. The following is a variant of so called the base point free pencil trick.

Lemma 1.1 *For a vector bundle E of rank r on C, we have*

$$\dim \mathrm{Hom}\,(E, \xi) + \dim \mathrm{Hom}\,(Q_\eta, E) \geq r(h^0(E) - \deg \eta) - \chi(E).$$

Proof. Take the global section of the exact sequence (1.1) tensored with E. Then we have

$$\dim \mathrm{Hom}\,(Q_\eta, E) + h^0(E\eta) \geq rh^0(E).$$

By the Riemann-Roch theorem (and the Serre duality), we have

$$h^0(E\eta) - h^0(E^\vee\xi) = \chi(E\eta) = \chi(E) + r\deg\eta.$$

Our assertion follows immediately from these. □

If E is of canonical determinant, *i.e.*, $\bigwedge^r E \simeq K_C$, then we have

$$\dim \mathrm{Hom}\,(E, \xi) + \dim \mathrm{Hom}\,(Q_\eta, E) \geq r(h^0(E) - r - s) - 2\rho + 2, \qquad (1.3)$$

number of η, or equivalently, of ξ.

The number of global sections behaves specially if a vector bundle has a non-degenerate quadratic from with values in K_C. The following is one of such phenomena clarified in Mumford [10].

Proposition 1.2 *Let E and F be rank two vector bundles on a curve C such that $(\det E) \otimes (\det F) \simeq K_C$. Then $h^0(E \otimes F)$ is congruent to $\deg E$ modulo 2.*

Proof. Take a line subbundle L of F and put $M = F/L$. The coboundary map $\delta : H^0(E \otimes M) \longrightarrow H^1(E \otimes L)$ coming from the exact sequence

$$E \otimes [0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0]$$

is anti-self-dual, that is, $\delta + \delta^\vee = 0$, with respect to the Serre pairing. Hence $h^0(E \otimes F)$ is congruent to

$$h^0(E \otimes L) + h^0(E \otimes M) = h^0(E \otimes L) + h^1(E \otimes L)$$

modulo 2. Since $h^0(E \otimes L) - h^1(E \otimes L)$ is congruent to $\deg(E \otimes L)$, we have our assertion. $\square$

2 Symplectic Grassmannian

Let A be a k-vector space. For a subspace $B \subset A$ the linear map $\bigwedge^2 B \to \bigwedge^2 A$ is injective.

Definition 2.1 A bivector $\sigma \in \bigwedge^2 A$ is *degenerate* if σ is contained in $\bigwedge^2 B$ for a proper subspace $B \subset A$.

A bivector σ is always degenerate if $\dim A$ is odd. In the case $\dim A$ is even, σ is degenerate if and only if the value of the Pfaffian is zero. There exists a minimal subspace $B \subset A$ such that $\sigma \in \bigwedge^2 B$. This subspace B is called the *co-radical* of σ.

Definition 2.2 A *symplectic vector space* is a pair (V, σ) of a vector space V and a nondegenerate bivector $\sigma \in \bigwedge^2 V^\vee$ of the dual vector space.

a bivector σ is a coset of the subspace. When the characteristic of k is not 2, the coset has the unique anti-symmetric representative, say σ^{AS}. A subspace $U \subset V$ is a *Lagrangian* if $2\dim U = \dim V$ and the restriction $\sigma|_U : U \times U \longrightarrow k$ of σ to U is symmetric. If $char(k) \neq 2$, then the second condition is equivalent to the usual one, that is, $\sigma^{AS}|_U = 0$. We denote the set of Lagrangian subspaces of (V, σ) by $G(\sigma, V)$.

Two vectors u and $v \in V$ are *perpendicular* with respect to σ if the restriction of σ to the subspace spanned by u and v is symmetric. For a nonzero vector $v \in V$, the set of vectors $u \in V$ perpendicular to v is a subspace of codimension one. We denote this subspace by $v^\perp$. σ induces a bilinear form $\bar{\sigma}$ on the quotient space $\bar{V} := v^\perp / kv$ and $(\bar{V}, \bar{\sigma})$ becomes a symplectic vector space of dimension two less. If a Lagrangian subspace U of (V, σ) contains v, then the quotient U/kv is a Lagrangian of $(\bar{V}, \bar{\sigma})$. Conversely, if $\bar{U}$ is a Lagrangian of $(\bar{V}, \bar{\sigma})$, then its inverse image by $v^\perp \to \bar{V}$ is a Lagrangian of (V, σ) which contains v. By this correspondence we identify $G(\bar{\sigma}, \bar{V})$ with the subset of $G(\sigma, V)$ consisting of $[U]$ with $v \in U$.

For our purpose, the Grassmannian of quotient spaces is more convenient than that of subspaces. A quotient space $A \xrightarrow{f} Q$ of A is *Lagrangian* with respect to a nondegenerate bivector σ if $2\dim W = \dim A$ and if $(\bigwedge^2 f)(\sigma) = 0$. We denote the set of Lagrangian quotient spaces of the pair (A, σ) by $G(A, \sigma)$, which coincides with $G(\sigma, A^\vee)$. Let $\mathcal{U}$ be the universal quotient bundle on $G(A, n)$, $\dim A = 2n$. Then $\sigma \in \bigwedge^2 A$ determines a global section of $\bigwedge^2 \mathcal{U}$, which we denote by s. Then $G(A, \sigma)$ coincides with the zero set of $s \in H^0(G(A, n), \bigwedge^2 \mathcal{U})$. We endow $G(A, \sigma)$ with structure of scheme as the zero locus of s. Its isomorphism class is denoted by $SpG(n, 2n)$.

Proposition 2.3 *The symplectic Grassmannian $G(A, \sigma)$ is a smooth variety of dimension $n(n+1)/2$ and the anti-canonical class is $n+1$ times the the hyperplane section H of the Plücker embedding.*

Proof. Since $\bigwedge^2 A$ generates $\bigwedge^2 \mathcal{U}$, $G(A, \tilde{\sigma})$ is locally a smooth complete intersection for general $\tilde{\sigma}$ by Bertini's theorem ([13], Theorem 1.10). The normal bundle of $G(A, \tilde{\sigma})$ is the restriction of $\bigwedge^2 \mathcal{U}$. In particular, the dimension is equal to $n^2 - n(n-1)/2 = n(n+1)/2$. Since the $GL(2n)$-orbit of non-degenerate bivectors is dense in $\bigwedge^2 A$, $G(A, \sigma)$ is isomorphic to $G(A, \tilde{\sigma})$. $G(A, \sigma)$ is irreducible since the symplectic group $Sp(n)$ acts transitively. Since $c_1(G(A, n)) = 2nH$ and $c_1(\bigwedge^2 \mathcal{U}) = (n-1)H$, the anti-canonical class of $G(A, \sigma)$ is equal

The divisor class group of the Grassmannian $G(n,2n)$ is generated by the hyperplane section class H. Its Chow group of codimension 2 cycles is generated by two Schubert subvarieties:

$$Y = \{[U] \,|\, U \cap W \neq 0\} \quad \text{and} \quad Y' = \{[U] \,|\, U + W' \neq V\} \tag{2.1}$$

for a subspace W of dimension $n-1$ and W' of codimension $n-1$. It is well known that the self intersection $H \cdot H$ is (rationally) equivalent to their sum. On the symplectic Grassmannian, obviously Y and Y' are equivalent and hence we have

$$H \cdot H \sim Y + Y' \sim 2Y. \tag{2.2}$$

Let a be a nonzero vector of A. The image $\bar{\sigma}$ of σ in $\bigwedge^2(A/ka)$ is degenerate since $\dim(A/ka)$ is odd. In fact, the co-radical $\bar{A}$ of $\bar{\sigma}$ is of codimension one. Similar to the inclusion $G(\bar{\sigma},\bar{V}) \hookrightarrow G(\sigma,V)$, we have a natural inclusion $G(\bar{A},\bar{\sigma}) \hookrightarrow G(A,\sigma)$. Moreover, $G(\bar{A},\bar{\sigma})$ is the scheme of zeros of the global section of $\mathcal{E} = \mathcal{U}|_{G(A,\sigma)}$ corresponding to $a \in A$.

Let $G(A,n) \subset \mathbf{P}^*(\bigwedge^n A)$ be the Plücker embedding of the Grassmannian $G(A,n)$. The tautological line bundle $\mathcal{O}_G(1)$ is isomorphic to $\bigwedge^n \mathcal{U}$. Since σ vanishes on $G(A,\sigma)$, so do all the linear forms $\sigma \wedge (\bigwedge^{n-2} A) \subset \bigwedge^n A$. Let $\bigwedge^n(A,\sigma)$ be the quotient space of $\bigwedge^n A$ by the subspace $\sigma \wedge (\bigwedge^{n-2} A)$. Then $G(A,\sigma)$ is contained in the subspace $\mathbf{P}^*(\bigwedge^n(A,\sigma))$ and we have a commutative diagram

$$\begin{array}{ccc} G(A,\sigma) & \longrightarrow & \mathbf{P}^*(\bigwedge^n(A,\sigma)) \\ \cap & & \cap \\ G(A,n) & \longrightarrow & \mathbf{P}^*(\bigwedge^n A). \\ & \text{Plücker} & \end{array} \tag{2.3}$$

$G(A,\sigma)$ coincides with $G(A,1) = \mathbf{P}^1$ for $n=1$ and is a smooth hyperplane section of the smooth 4-dimensional quadric $G(A,2) \subset \mathbf{P}^5$ for $n=2$.

Now we set $n=3$ and investigate the conormal space of $G(A,\sigma) \subset \mathbf{P}^* \bigwedge^3(A,\sigma)$ and an important cubic cone in it. Let $A \to Q$ be a 3-dimensional quotient space and put $W = \operatorname{Ker}[A \to Q]$. Then we have a filtration of subspaces

$$F_0 = \bigwedge^3 W \subset F_1 = (\bigwedge^2 W) \wedge A \subset F_2 = W \wedge \bigwedge^2 A \subset F_3 = \bigwedge^3 A. \tag{2.4}$$

$W \otimes (\bigwedge^2 Q)$. $(F_2/F_1) \otimes \det Q^{-1} \simeq \mathrm{Hom}\,(Q, W)$ is canonically isomorphic to the cotangent space of $G(A,3)$ at $[Q]$. $F_1 \otimes \det Q^{-1}$ is canonically isomorphic to the conormal space of $G(A,3) \subset \mathbf{P}^* \bigwedge^3 A$. Hence we have an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & k \longrightarrow & \bar{F}_1 \otimes \det W^{-1} & \longrightarrow & \mathrm{Hom}\,(W,Q) & \longrightarrow 0. \\ & & || & & & \\ & & N^{\vee}_{G(A,3)/\mathbf{P}} \otimes \det Q \otimes \det W^{-1} & & & \end{array}$$

Assume that $[A \to Q] \in G(A,\sigma)$ is Lagrangian. Then σ belongs to $W \wedge A \subset \bigwedge^2 A$. Let

$$\bar{F}_0 \subset \bar{F}_1 \subset \bar{F}_2 \subset \bar{F}_3, \qquad \bar{F}_i = F_i/(F_i \cap \sigma \wedge A),$$

be the quotient filtration of (2.4) by $\sigma \wedge A \subset F_2$. Then $\bar{F}_3/\bar{F}_2 \simeq \bigwedge^3 Q$ is the Plücker coordinate of Q. The cotangent space of $G(3,\sigma)$ at $[Q]$ is $\bar{F}_2/\bar{F}_1 \otimes \det Q^{-1} \simeq S^2 W$. The conormal space is isomorphic to $\bar{F}_1 \otimes \det Q$ and we have an exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow k & \longrightarrow \bar{F}_1 \otimes \det Q & \longrightarrow & S^2 Q \longrightarrow 0. \\ & || & & \\ & N^{\vee}_{G(A,\sigma)/\mathbf{P}} \otimes (\det Q)^2 & & \end{array} \tag{2.5}$$

Let $\alpha : \mathbf{P}_*(\bigwedge^3 A) \cdots \longrightarrow \mathbf{P}_*(\bigwedge^3 (A,\sigma))$ be the projection with center $\mathbf{P}_*(\sigma \wedge A)$. Since σ is nondegenerate, $G(3,A)$ is disjoint from the center. We consider the image of the Schubert subvariety

$$S_Q = \{[U] \mid \mathrm{rk}\,[U \to A \to Q] \le 1\} \subset G(3,A)$$

by α for a Lagrangian quotient space $A \to Q$ (cf. (3.3) and (4.1)). S_Q is a 5-dimensional subvariety of $\mathbf{P}_*((\bigwedge^2 W) \wedge A) = \mathbf{P}_*(N^{\vee}_{G(A,3)/\mathbf{P},Q})$ and $\alpha(S_Q)$ is a subvariety of $\mathbf{P}_*(\bar{F}_1) = \mathbf{P}_*(N^{\vee}_{G(A,\sigma)/\mathbf{P},Q}) = \mathbf{P}^6$. By the exact sequence (2.5), $\mathbf{P}^*(N_{G(A,\sigma)/\mathbf{P},[Q]})$ has the distinguished point corresponding to $\mathrm{Ker}\,[A \to Q]$, which we denote by κ_Q, and the special projection onto $\mathbf{P}_*(S^2 Q)$. $\alpha(S_Q)$ contains the point κ_Q.

Proposition 2.4 *The image $\alpha(S_Q)$ is a cubic hypersurface of $\mathbf{P}^*(N_{G(A,\sigma)/\mathbf{P},[Q]})$. More precisely, it is the cone over the discriminant hypersurface of $\mathbf{P}_*(S^2 Q)$ with vertex κ_Q.*

Ker $[A \to Q]$ and $\sigma = v_1 \wedge v_{-1} + v_2 \wedge v_{-2} + v_3 \wedge v_{-3}$. Let $\{u_1, u_2, u_3\}$ be a basis of $U \in S_Q$ such that $u_1, u_2 \in \operatorname{Ker}[U \to Q]$. The exterior product $u_1 \wedge u_2$ is equal to

$$a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2 \in \bigwedge^2 \operatorname{Ker}[A \to Q]$$

for a_1, a_2 and $a_3 \in k$. Put $u_3 = a_4 v_1 + a_5 v_2 + a_6 v_3 + b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3}$. Then the Plücker coordinate $u_1 \wedge u_2 \wedge u_3$ of U is

$$a_0 v_1 \wedge v_2 \wedge v_3 + (a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2) \wedge (b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3})$$

$$= a_0 v_1 \wedge v_2 \wedge v_3 + (a_2 b_1 v_{12} - a_1 b_2 v_{21}) + (a_1 b_3 v_{31} - a_3 b_1 v_{13}) + (a_3 b_2 v_{23} - a_2 b_3 v_{32}) + \sum_{i=1}^{3} a_i b_i v_{ii},$$

where we put $a_0 = a_1 a_4 + a_2 a_5 + a_3 a_6$,

$$v_{11} = v_{-1} \wedge v_2 \wedge v_3, \quad v_{22} = v_1 \wedge v_{-2} \wedge v_3, \quad v_{33} = v_1 \wedge v_2 \wedge v_{-3}$$

and $v_{jk} = v_i \wedge v_j \wedge v_{-j}$ for every $\{i, j, k\} = \{1, 2, 3\}$. Since $v_{jk} + v_{kj} \in A \wedge \sigma$ for every $j \neq k$, $u_1 \wedge u_2 \wedge u_3$ is congruent to

$$a_0 v_1 \wedge v_2 \wedge v_3 - (a_1 b_2 + a_2 b_1) v_{12} - (a_1 b_3 + a_3 b_1) v_{13} + (a_2 b_3 + a_3 b_2) v_{23} + \sum_{i=1}^{3} a_i b_i v_{ii}$$

modulo $A \wedge \sigma$. Hence $\alpha(S_Q)$ consists of $\gamma_0 v_1 \wedge v_2 \wedge v_3 + \sum_{1 \le i \le j \le 3} \gamma_{ij} v_{ij}$ such that the quadratic form $\sum_{1 \le i \le j \le 4} \gamma_{ij} X_i X_j$ is of rank ≤ 2. □

Remark 2.5 The discriminant of a ternary quadratic form

$$q(x, y, z) = ax^2 + by^2 + cz^2 + dyz + exz + fxy, \quad a, b, \ldots, f \in k$$

is equal to $4abc - ad^2 - be^2 - cf^2 + def$.

3 Linear systems of higher rank

A *linear system of rank* r is a pair (E, A) of a vector bundle E of rank r and a space of global sections $A \subset H^0(E)$. The special one with $A = H^0(E)$ is called a *complete* linear system and denoted by $|E|$. A linear system (E, A) on an algebraic variety C is *free* if the evaluation homomorphism $ev_{E,A} : A \otimes_k \mathcal{O}_C \longrightarrow E$ is surjective. If this holds, we obtain a morphism $\Phi_{E,A}$ of C to the Grassmann variety $G(A, r)$ of r-dimensional quotient spaces.

quotient bundle on $G(A, r)$.

Let

$$\bigwedge^m ev_{E,A} : \bigwedge^m A \otimes_k \mathcal{O}_C \longrightarrow \bigwedge^m E$$

be the exterior product of the evaluation homomorphism $ev_{E,A}$. It induces the linear map

$$\bigwedge^m A \longrightarrow H^0(\bigwedge^m E),$$

which we denote by λ_m. The image $\lambda_m(s_1 \wedge \cdots \wedge s_m)$ of a simple m-vector $s_1 \wedge \cdots \wedge s_m$ is zero if and only if m global sections $s_1, \ldots, s_m \in A \subset H^0(E)$ are linearly dependent at the generic point of C, that is, they generate a subsheaf of rank less than m. The case $m = r$ is most important. Assume that the the linear map $\lambda_r : \bigwedge^r A \longrightarrow H^0(\det E)$, is surjective. Then the map

$$\Psi : \mathbf{P}^*(H^0(\det E)) \to \mathbf{P}^*(\bigwedge^r A). \tag{3.1}$$

induced by λ_r is a linear embedding and the following diagram is commutative:

$$\begin{array}{ccccc} C & \stackrel{\Phi_E}{\longrightarrow} & G(A, r) & & \\ \cap & & \cap & \text{Plücker} & \\ \mathbf{P}^*(H^0(\det E)) & \stackrel{\Psi}{\longrightarrow} & \mathbf{P}^*(\bigwedge^r A). & & \end{array} \tag{3.2}$$

Even when λ_r is not surjective, the above is still commutative though $\Psi = \mathbf{P}^*\lambda_r$ is only a rational map. The linear map λ_r is important in analyzing E itself also.

Now we assume that the base field k is algebraically closed (until the end of §6). The dual Grassmannian $G(r, A) \subset \mathbf{P}_*(\bigwedge^r A)$ is also important for (E, A).

Definition 3.1 A linear system (E, A) of rank r is *irreducible* if it satisfies the following equivalent conditions:

i) for every r-dimensional linear subspace U of A the image of $U \otimes_k \mathcal{O}_C \longrightarrow E$ is of rank r, and

ii) the kernel of the natural linear map $\lambda_r : \bigwedge^r A \longrightarrow H^0(C, \det E)$ contains no (non-zero) simple r-vectors, that is, $G(r, A) \cap \mathbf{P}_*(\operatorname{Ker} \lambda_r) = \emptyset$.

The following is known as Castelnuovo's trick (cf. [2], Chap. 10):

Proposition 3.2 *If* $r(\dim A - r) \geq h^0(\det E)$, *then* (E, A) *is reducible,* sion of $\mathbf{P}_*(\mathrm{Ker}\,\lambda_r) \subset \mathbf{P}_*(\bigwedge^r H^0(E))$ is at most $h^0(\det E)$. Hence, if the inequality holds, then the intersection $G(r, A) \cap \mathbf{P}_*\mathrm{Ker}\,\lambda_r$ is not empty. □

A line bundle is irreducible. But the irreducibility seems a strong condition in general. Irreducible ones of rank ≥ 2 will not appear in the sequel. Instead the following concept plays a crucial role in our proof.

Definition 3.3 A linear system (E, A) of rank r on a (smooth complete) curve C is *semi-irreducible* if the evaluation homomorphism $ev_U : U \otimes_k \mathcal{O}_C \longrightarrow E$ is either injective or everywhere of rank $r-1$ for every r-dimensional subspace U of A.

For an r-dimensional quotient space $A \to Q$, we denote by S_Q the Schubert subvariety

$$\{[U] \mid \mathrm{rk}\,[U \to A \to Q] \leq r-2\} \subset G(r, A) \tag{3.3}$$

associated to Q. S_Q is contained in the projective space $\mathbf{P}_*((\bigwedge^2 W) \wedge (\bigwedge^{r-2} A))$, which is the projectivisation $\mathbf{P}_*(N^\vee_{G(A,r)/\mathbf{P},[Q]})$ of the conormal space of $G(A, r) \subset \mathbf{P}_*(\bigwedge^r A)$ at $[Q]$. The following is obvious:

Lemma 3.4 *(E, A) is semi-irreducible if and only if $S_{E_p} \cap \mathbf{P}_*\mathrm{Ker}\,\lambda_r = \emptyset$ for every fiber E_p of E, $p \in C$.*

Now we restrict ourselves to complete linear systems for simplicity.

Proposition 3.5 *Assume that a complete linear system $|E|$ of rank r is free and semi-irreducible.*

(1) *If F is a proper nonzero subbundle, then $h^0(F) \leq r(F)+1$, where $r(F)$ is the rank of F.*

(2) *If $h^0(E) \geq r+3$, then E is simple,* i.e., $\mathrm{End}\,E = k$.

Proof. (1) Assume that F is of rank $r-1$ and $h^0(F) \geq r$. Then the evaluation homomorphism $B \otimes_k \mathcal{O}_C \to F$ is surjective for every r-dimensional subspace $B \subset H^0(F)$ by semi-irreducibility. Hence we have $h^0(F) \leq r$. General case follows from this since, for every proper subbundle F, there exists a subsheaf $F' \subset E$ of rank $r-1$ which contains F and $h^0(F') \geq h^0(F) + r(F') - r(F)$.

isomorphism. Assume that ϕ is neither. Then both the kernel and the image are proper subsheaves and we have

$$h^0(E) \le h^0(\operatorname{Ker}\phi) + h^0(\operatorname{Im}\phi) \le r(\operatorname{Ker}\phi) + 1 + r(\operatorname{Im}\phi) + 1 = r + 2$$

by (1), which is a contradiction. □

The following is proved similarly.

Lemma 3.6 *Assume that two complete linear systems $|E|$ and $|E'|$ are free, semi-irreducible and of the same rank r and assume further that $h^0(E') \ge r+3$. Then every nonzero homomorphism $E \to E'$ is injective.*

4 Linear sections of the symplectic Grassmannian

Throughout this section $C \subset \mathbf{P}^8$ is a transversal linear section $SpG(3,6) \cap H_1 \cap \cdots \cap H_5$ of the 6-dimensional symplectic Grassmannian.

Lemma 4.1 *C is of genus 9 and the restriction of tautological line bundle $\mathcal{O}(1)$ is isomorphic to the canonical bundle K_C of C.*

Proof. By Proposition 2.3 and by adjunction, we have $K_C \simeq \mathcal{O}_C(K_{SpG} + H_1 + \cdots + H_5) \simeq \mathcal{O}_C(1)$. The Chern class of the universal quotient bundle $\mathcal{U}$ on $G(3,6)$ is the sum $1 + \sigma_1 + \sigma_2 + \sigma_3$ of the special Schubert cycles ([8], Chap. 1). By Pieri's formula, we have

$$2g(C) - 2 = \deg[SpG(3,6) \subset \mathbf{P}^{13}] = (c_3(\bigwedge^2 \mathcal{U}).c_1(\mathcal{U})^6) = (\sigma_1\sigma_2 - \sigma_3.\sigma_1^6) = 21 - 5 = 16,$$

since $SpG(3,6)$ is the zero locus of a global section of $\bigwedge^2 \mathcal{U}$. Hence C is of genus 9. □

Let $G(A,\sigma)$, $\dim A = 6$, be a representative of $SpG(3,6)$.

Lemma 4.2 *The linear map $\bigwedge^3(A,\sigma) \to H^0(K_C)$ is surjective and its kernel is generated by the linear forms $f_1, \ldots, f_5 \in \bigwedge^3(A,\sigma)$ defining the five hyperplanes $H_1, \ldots, H_5$.*

Proof. Let X_i be the common zero locus of the first i linear forms $f_1, \ldots, f_i$ for $1 \le i \le 5$. Then we obtain a ladder

$$C = X_5 \subset X_4 \subset X_3 \subset X_2 \subset X_1 \subset X_0 := G(A,\sigma).$$

$H^0(X_{i+1}, \mathcal{O}_X(1))$ is generated by f_{i+1}, for every $1 \le i \le 4$. Hence $\bigwedge^3(A,\sigma)/\langle f_1,\dots,f_5\rangle \longrightarrow H^0(K_C)$ is injective. This map is also surjective because the source and the target have the same dimension. □

Let $\mathcal{E}$ be the restriction of $\mathcal{U}$ to $G(A,\sigma)$ and E that to C.

Lemma 4.3 *The restriction map $A \to H^0(E)$ is injective.*

Proof. Assume the contrary. Then for each of the Lagrangian quotient spaces $A \to Q$ parameterized by C, $\mathrm{Ker}\,[A \to Q]$ contains a nonzero common vector a. Hence C is contained in the symplectic Grassmannian $G(\bar{A},\bar{\sigma})$, where $\bar{A}$ is the co-radical of A/ka. This contradicts the preceding lemma since $G(\bar{A},\bar{\sigma})$ lies in a 4-dimensional linear subspace.
□

By this lemma we identify A with its image in $H^0(E)$.

Lemma 4.4 (1) *A nonzero global section $s \in A$ of E has at most two zeros (counted with multiplicity), that is, $A \cap H^0(E(-D)) = 0$ for every effective divisor D of degree 3 on C.*

(2) *If $A' \subset A$ is a subspace of codimension one, then the cokernel of the evaluation homomorphism $A' \otimes_k \mathcal{O}_C \longrightarrow E$ is of length ≤ 2.*

Proof. Assume that s has at least three zeros. Then we have an exact sequence $E^\vee \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$ for an effective divisor D of degree ≥ 3. Let $G(\bar{A},\bar{\sigma}) \subset G(A,\sigma)$ be the 3-dimensional symplectic Grassmannian determined by $s \in A$. Then the intersection $G(\bar{A},\bar{\sigma}) \cap C$ contains D. Since $G(\bar{A},\bar{\sigma})$ is a quadric, its intersection with the linear span $\langle D\rangle$ is of positive dimension, which is a contradiction. This shows (1). The proof of (2) is similar. □

Let $U \subset A$ be a 3-dimensional subspace and $H_U \subset \mathbf{P}^* \bigwedge^3 A$ the hyperplane corresponding to it. Then the intersection $H_U \cap G(A,r)$ consists of the r-dimensional quotient spaces $A \to Q$ such that the composite $U \hookrightarrow A \to Q$ is not an isomorphism. It is singular along the Schubert subvariety

$$\{[A \to Q] \mid \mathrm{rank}\,[U \hookrightarrow A \to Q] \le r-2\}. \tag{4.1}$$

If $H_U \not\supset C$, then the evaluation homomorphism $ev_U : U \otimes \mathcal{O}_C \longrightarrow E$ is of rank 3 at the generic point. Hence it is injective. If $H_U \supset C$, then H_U belongs to $\langle [H_1],\dots,[H_5]\rangle$.

smooth along C. Hence ev_U is of rank 2 everywhere. So we have proved the following, which indicates that the semi-irreducibility is a key concept in for canonical curves of genus 9.

Proposition 4.5 *The induced rank three linear system (A, E) on $C = G(A, \sigma) \cap H_1 \cap \cdots \cap H_5$ is semi-irreducible.*

By Proposition 3.2, there exists a 3-dimensional subspace U of A such that $H_U \supset C$. Let F and α be the image and the cokernel of ev_U. Then α is a line bundle, $\det F$ is isomorphic to $\beta := K_C \alpha^{-1}$ and we have exact sequences

$$0 \longrightarrow \beta^{-1} \longrightarrow 3\mathcal{O}_C \longrightarrow F \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \longrightarrow E \longrightarrow \alpha \longrightarrow 0. \tag{4.2}$$

By (2.2), the line bundles α and β are both of degree 8.

Proposition 4.6 *C is non-pentagonal.*

Proof. It is obvious that C is non-hyperelliptic. Since $SpG(3, 6) \subset \mathbf{P}^{13}$ is an intersection of quadrics (see (0.1)), so is $C \subset \mathbf{P}^8$. Hence C has no g^1_3 or g^2_5. Let ξ be a g^1_5 on C. Then we have $h^0(\xi) = 2$. Taking the global section of the exact sequence

$$[0 \longrightarrow F^\vee \longrightarrow 3\mathcal{O}_C \longrightarrow \beta \longrightarrow 0] \otimes \xi,$$

we have

$$6 \leq 3h^0(\xi) \leq \dim \mathrm{Hom}\,(F, \xi) + h^0(\xi\beta) = \dim \mathrm{Hom}\,(F, \xi) + 5 + h^1(\xi\beta).$$

Hence we have

$$\dim \mathrm{Hom}\,(F, \xi) + \dim \mathrm{Hom}\,(\xi, \alpha) \geq 1. \tag{4.3}$$

Assume that there exists a nonzero homomorphism $F \to \xi$ and let s be a nonzero global section in the kernel of $B \hookrightarrow H^0(F) \to H^0(\xi)$. Then s has at least three zeros since $\deg F - \deg \xi = 3$. If $\mathrm{Hom}\,(F, \xi)$ is zero, then $\mathrm{Hom}\,(\xi, \alpha)$ is not by (4.3). Hence α contains a subsheaf isomorphic to ξ. Let A' be the inverse image of $H^0(\xi)$ by $A \to H^0(\alpha)$. Then the cokernel of the evaluation homomorphism $A' \otimes_k \mathcal{O}_C \to E$ is of length 3. Both contradict Lemma 4.4. $\square$

index equals to three (Martens[9], Beispiel 9).

(2) The Green's property (N_p) ([6]) gives another proof of the proposition: First a general curve of genus 9 satisfies (N_3) by Ein[3]. Hence $SpG(3,6) \subset \mathbf{P}^{13}$ and its complete linear sections do so. By the converse of Green's conjecture (Green-Lazarsfeld[7]), they are non-pentagonal.

By the proposition and (1) of the remark, C has no g^3_8. Hence we have $h^0(\alpha) = h^0(\beta) = 3$. By Lemma 5.1 below, we have $h^0(E) \le h^0(\alpha) + H^0(Q_\beta) \le 6$. Combining with Lemma 4.3, we have

Proposition 4.8 *The restriction map $A \to H^0(E)$ is an isomorphism.*

In the following sections we aim at a kind of converse of Proposition 4.5.

5 Rank 3 linear systems on a non-pentagonal curve

Throughout this section we assume that C is a non-pentagonal curve of genus 9. In particular, C has no g^2_7. Let α be a g^2_8, β its Serre adjoint and Q_β the cokernel of ev_β as in the introduction and in (1.1). The image of $\Phi_\beta : C \longrightarrow \mathbf{P}^2$ is a singular plane curve of degree 8. Hence there exists a pair (p,q) of points such that $h^0(\beta(-p-q)) = 2$. Since C is non-pentagonal, $\xi := \beta(-p-q)$ is a free g^1_6. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \beta^{-1} & \longrightarrow & 3\mathcal{O}_C & \longrightarrow & Q_\beta & \longrightarrow & 0 \\ & & \cap & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \xi^{-1} & \longrightarrow & 2\mathcal{O}_C & \longrightarrow & \xi & \longrightarrow & 0 \end{array} \tag{5.1}$$

and an exact sequence

$$0 \longrightarrow \mathcal{O}_C(p+q) \longrightarrow Q_\beta \longrightarrow \xi \longrightarrow 0. \tag{5.2}$$

Lemma 5.1 (1) $h^0(Q_\beta) = 3$.

(2) $H^0(\alpha^{-1}Q_\beta) = 0$.

Proof. (1) $h^0(Q_\beta) \ge 3$ is obvious from the defining exact sequence of Q_β. The opposite inequality $h^0(Q_\beta) \le 3$ follows from (5.2).

$\alpha \not\simeq \beta$, then $H^0(\alpha^{-1}\beta) = 0$ and hence $H^0(\alpha^{-1}Q_\beta) = 0$. If $\alpha \simeq \beta$, then $H^0(\alpha^{-1}Q_\beta) \simeq H^0(Q_\beta^\vee) = 0$ by the exact sequence (1.1). □

We consider Γ-split extensions

$$0 \longrightarrow Q_\beta \longrightarrow E \longrightarrow \alpha \longrightarrow 0. \tag{5.3}$$

Lemma 5.2 *There exists a nontrivial extension E of α by Q_β with $h^0(E) = 6$.*

Proof. The extensions with $h^0(E) = 6$ are parameterized by the kernel of the natural linear map $\varphi : \mathrm{Ext}^1(\alpha, Q_\beta) \longrightarrow H^0(\alpha)^\vee \otimes H^1(Q_\beta)$, which is equal to the first cohomology H^1 of the homomorphism

$$[\alpha^{-1} \xrightarrow{ev^\vee} H^0(\alpha)^\vee \otimes \mathcal{O}_C] \otimes Q_\beta.$$

Since its cokernel is $Q_\alpha \otimes Q_\beta$, we have an exact sequence

$$H^0(\alpha)^\vee \otimes H^0(Q_\beta) \xrightarrow{\psi} H^0(Q_\alpha \otimes Q_\beta) \longrightarrow H^1(\alpha^{-1}Q_\beta) \xrightarrow{\varphi} H^0(\alpha)^\vee \otimes H^1(Q_\beta) \tag{5.4}$$

The first map ψ is injective by (2) of Lemma 5.1 and $h^0(Q_\alpha \otimes Q_\beta)$ is even by Proposition 1.2. Since $h^0(\alpha)h^0(Q_\beta) = 9$, φ is not injective. □

Proposition 5.3 *Let E be as in the preceding lemma. Then the complete linear system $|E|$ is free and semi-irreducible.*

Proof. $|E|$ is free since both $|Q_\beta|$ and $|\alpha|$ are so. Let $U \subset H^0(E)$ be a 3-dimensional subspace and $F \subset E$ the saturation of the subsheaf F' generated by U. Obviously $h^0(F) \geq 3$. If F is of rank one, then $\deg F \geq 8$ by our assumption. Since $F \not\subset Q_\beta$, the extension (5.3) splits, which is a contradiction. Hence F is of rank two. Let ξ be the quotient line bundle E/F. Since $|E|$ is free, so is ξ. Since $\mathrm{Hom}(E, \mathcal{O}_C) = 0$, we have $h^0(\xi) \geq 2$, which implies $\deg \xi \geq 6$ by our assumption. By duality, we have $h^0(\det F) = h^1(\xi) \leq 4$.

Assume that $h^0(F) \geq 4$. Then F contains a line subbundle ζ with $h^0(\zeta) \geq 2$ by Proposition 3.2. Since $\zeta \not\subset Q_\beta$, ζ is isomorphic to a proper subsheaf of α. Hence we have $h^0(\zeta) = 2$. Let η be the quotient line bundle F/ζ. Then we have $h^0(\eta) \geq h^0(F) - h^0(\zeta) = 2$. Since $\deg \zeta + \deg \eta + \deg \xi = 16$, one of the three line bundles is of degree ≤ 5, which is a contradiction. Hence we have $h^0(F) = 3$ and $h^0(\xi) \geq h^0(E) - h^0(F) = 3$. Since

F' is a subbundle. □

Now conversely we study a uniqueness.

Lemma 5.4 *Nontrivial extensions E of α by Q_β with $h^0(E) = 6$ are unique.*

Proof. The assertion is equivalent to $h^0(Q_\alpha \otimes Q_\beta) \le 10$ by the exact sequence (5.4). Take the global section of the exact sequence

$$(5.2) \otimes Q_\alpha : 0 \longrightarrow Q_\alpha(p+q) \longrightarrow Q_\alpha \otimes Q_\beta \longrightarrow Q_\alpha \xi \longrightarrow 0$$

and we have

$$\begin{aligned} h^0(Q_\alpha \otimes Q_\beta) &\le h^0(Q_\alpha(p+q)) + h^0(Q_\alpha \xi) = h^0(Q_\alpha(p+q)) + h^1(Q_\alpha(p+q)) \\ &= 2h^0(Q_\alpha(p+q)) - \chi(Q_\alpha(p+q)). \end{aligned}$$

Since $\chi(Q_\alpha(p+q)) = -4$, it suffices to show $h^0(Q_\alpha(p+q)) \le 3$. Assume the contrary.

Case where $h^0(Q_\alpha(p+q)) = 4$. Let $\{s_1, s_2, s_3, s_4\}$ be a basis of $H^0(Q_\alpha(p+q))$ such that $s_1, s_2, s_3 \in H^0(Q_\alpha)$ and F the image of the evaluation homomorphism $H^0(Q_\alpha(p+q)) \otimes_k \mathcal{O}_C \longrightarrow Q_\alpha(p+q)$. Then the quotient F/Q_α is generated by the image of s_4. Hence we have $\deg F \le \deg Q_\alpha + 2 = 10$. We have $h^0(\det F) \le 4$ by the non-existence of g^2_6. Since $h^0(F) \ge 4$, there exists a 2-dimensional subspace of $H^0(F)$ which generates a rank one subsheaf by Proposition 3.2. This contradicts the non-existence of g^1_5.

Case where $h^0(Q_\alpha(p+q)) \ge 5$. Since $\deg Q_\alpha(p+q) = 12$ and since C has no g^1_4, we have $h^0(\det(Q_\alpha(p+q))) \le 5$. By Proposition 3.2, there exists an exact sequence

$$0 \longrightarrow \zeta \longrightarrow Q_\alpha(p+q) \longrightarrow \eta \longrightarrow 0$$

such that $h^0(\zeta) \ge 2$. Since $\eta(-p-q)$ is a quotient of Q_α, we have $h^0(\eta(-p-q)) \ge 2$ and $\deg \eta(-p-q) \ge 6$, which implies $\deg \zeta \le 4$. This is a contradiction. □

We strengthen this lemma.

Lemma 5.5 *A rank 3 vector bundle E on C which satisfies*

i) $\bigwedge^3 E \simeq K_C$, ii) $h^0(E) \ge 6$ and iii) $|E|$ is semi-irreducible

is an extension of α by Q_β.

$$\dim \mathrm{Hom}\,(Q_\beta, E) + \dim \mathrm{Hom}\,(E, \alpha) \geq 2.$$

($h^0(E) = r+s$ and the Brill-Noether number ρ is equal to 0.) Hence there exists a nonzero homomorphism either $f : Q_\beta \longrightarrow E$ or $g : E \longrightarrow \alpha$.

If the image of f is a line bundle L, then $h^0(L) \geq 2$ since $\mathrm{Hom}\,(Q_\beta, \mathcal{O}_C) = 0$. This contradicts (1) of Proposition 3.5. Hence f is injective. By semi-irreducibility, the cokernel is a line bundle and isomorphic to α.

If $g : E \longrightarrow \alpha$ is not surjective, then the kernel of $H^0(E) \longrightarrow H^0(\alpha)$ is of dimension $\geq$ 4, which contradicts semi-irreducibility. Hence g is surjective and its kernel is isomorphic to Q_β. □

By the two lemmas above, we have the following

Proposition 5.6 *Vector bundles E on C which satisfy the condition of the lemma are unique up to isomorphism.*

This vector bundle is often denoted by E_{max}.

Corollary *If E is a rank 3 vector bundle of canonical determinant on C and if $|E|$ is semi-irreducible, then $h^0(E) \leq 6$.*

Remark 5.7 (1) By the proposition and its proof, we obtain an explicit bijection between two sets: $W^2_8(C)$, the set of g^2_8's of C, and the intersection $G(3, H^0(E_{max})) \cap \mathbf{P}^{10}$. It is known that the cardinality of $W^{r-1}_d(C)$ of a general curve C of genus g is equal to the degree of a g-dimensional Grassmannian when the Brill-Noether number ρ is zero (cf. [1] Chap. VII and [4] Example 14.4.5).

(2) By (1) of Proposition 3.5, it is easy to show that E_{max} is stable. It is also easy to show a converse: if E is stable, $\bigwedge^3 E \simeq K_C$ and $h^0(E) = 6$, then $|E|$ is semi-irreducible.

6 Linear section theorems

We prove Theorem C in several steps. Assume that E satisfies the condition (0.2). Since E is a rank 3 vector bundle of canonical determinant, $K_C E^\vee$ is isomorphic to $\bigwedge^2 E$. Hence, by the Riemann-Roch theorem, we have

$$h^0(E) - h^0(\bigwedge^2 E) = \deg E + 3(1-9) = -8.$$

and $h^0(\bigwedge^2 E) = 14$. Since $\dim \bigwedge^2 H^0(E) = 15$, the linear map $\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0(\bigwedge^2 E)$ is not injective.

Step 1. Every nonzero bivector σ in $\operatorname{Ker} \lambda_2$ is non-degenerate.

Proof. The rank of σ is either 2, 4 or 6. If σ is of rank 2, then σ is equal to $s_1 \wedge s_2$ for a pair of global sections s_1 and s_2 which are linearly independent in $H^0(E)$ and generate a rank one subsheaf in E. This contradicts Proposition 3.5. Assume that σ is of rank 4. Then σ is equal to $s_1 \wedge s_2 - s_3 \wedge s_4$ for s_1, s_2, s_3 and $s_4 \in H^0(E)$. By semi-irreducibility, s_1 and s_2 generate a rank two subsheaf in E. Let F be its saturation. Since $\lambda_2(s_1 \wedge s_2) = \lambda_2(s_3 \wedge s_4)$, we have $\lambda_3(s_1 \wedge s_2 \wedge s_i) = \lambda_3(s_3 \wedge s_4 \wedge s_i) = 0$ for $i = 3, 4$. Hence s_3 and s_4 are contained in $H^0(F)$ and we have $h^0(F) \geq 4$. This contradicts the semi-irreducibility of $|E|$. □

The nondegeneracy of σ is equivalent to the non-vanishing of Pfaffian. Hence $\operatorname{Ker} \lambda_2$ is of dimension one and λ_2 is surjective. Since $|E|$ is free, we obtain the Grassmannian morphism $\Phi_E : C \longrightarrow G(A, 3)$, were we put $A = H^0(E)$. Its image is contained in the symplectic Grassmannian $G(A, \sigma)$ and we obtain the commutative diagram (0.3), where σ is a generator of $\operatorname{Ker} \lambda_2$. Since $\bigwedge^3(A, \sigma)$ is of dimension 14, the kernel of $\bar{\lambda}_3 : \bigwedge^3(A, \sigma) \longrightarrow H^0(K_C)$ is of dimension $\geq 14 - 9 = 5$. Let $f_1, \dots, f_k$, $k \geq 5$, be its basis and $H_1, \dots, H_k$ the hyperplanes corresponding to them. Since $|E|$ is semi-irreducible, the intersection $S_{E_p} \cap \mathbf{P}_* \operatorname{Ker} \lambda_3$ is empty for every $p \in C$ by Lemma 3.4. Hence so is $\alpha(S_{E_p}) \cap \mathbf{P}_* \operatorname{Ker} \bar{\lambda}_3$.

Step 2. There exists a point $p \in C$ such that the intersection $G(A, \sigma) \cap H_1 \cap \cdots \cap H_k$ is transversal at $\Phi_E(p)$.

Proof. Assume the contrary. Then, for every $p \in C$, there exists a member H_p of $\langle [H_1], \dots, [H_k] \rangle = \mathbf{P}_* \operatorname{Ker} \bar{\lambda}_3$ such that the intersection $G(A, \sigma) \cap H_p$ is singular at $\Phi_E(p)$. Hence the intersection $\mathbf{P}_*(N^\vee_{G(A,\sigma)/\mathbf{P},[E_p]}) \cap \mathbf{P}_* \operatorname{Ker} \bar{\lambda}_3$ is a point by Proposition 2.4. Therefore, we obtain a section of the $\mathbf{P}^6$-bundle $\mathbf{P}^*(\Phi_E^* N_{G(A,\sigma)/\mathbf{P}})$ over C which is disjoint from $\coprod_{p \in C} \alpha(S_{E_p})$. By projecting from $\coprod_{p \in C} \kappa_p$, we obtain a section of $\mathbf{P}_*(S^2 E)$ over which discriminant form $\delta \in H^0(S^3(S^2 \mathcal{E})^\vee \otimes (\det \mathcal{E})^{\otimes 2})$ has no zeros. Let $\xi \subset S^2 E$ be the line subbundle corresponding to the section. Then δ induces a nowhere vanishing global section of $\xi^{-3} \otimes (\det \mathcal{E})^{\otimes 2}$. This implies $3 \deg \xi = 2 \deg E = 32$, which is absurd. □

$\mathbf{P}^*\bar{\lambda}_3$ is a linear embedding. Since the canonical morphism Φ_K is an embedding, so is Φ_E by the commutative diagram (0.3). We identify C with its image $\Phi_E(C)$.

By Step 2, the intersection $G(A,\sigma) \cap H_1 \cap \cdots \cap H_5$ is complete on a non-empty open subset C_0 of C. Hence the twisted normal bundle $N_{C/G(A,\sigma)}(-1)$ is generated by the five global sections induced from $f_1, \ldots, f_5$ over C_0. Since $N_{C/G(A,\sigma)}(-1)$ is of trivial determinant, it is generated over C. Therefore, the intersection is complete along C and contains it as a connected component. By the connectedness of linear sections (Fulton-Lazarsfeld [5], Theorem 2.1), the intersection coincides with C, which completes the proof of Theorem C. (If we use the refined Bézout theorem (Fulton[4], Theorem 12.3), the proof finishes at the last paragraph.)

Theorem A is an immediate consequence of Theorem C, Proposition 5.3 and Proposition 4.6.

7 Proof of Theorem B

We do not assume that k is algebraically closed any more. Let $C \simeq G(A', \sigma') \cap P'$ be another expression of $C = G(A, \sigma) \cap P$ as a complete linear section of a 6-dimensional symplectic Grassmannian and $\mathcal{E}'|_C$ the restriction of the universal quotient bundle. Both $|\mathcal{E}|_C|$ and $|\mathcal{E}'|_C|$ are semi-irreducible (over $\bar{k}$) by Proposition 4.5. Hence they are isomorphic to each other over $\bar{k}$ by Proposition 5.6 and there exists a nonzero homomorphism $f : \mathcal{E}|_C \longrightarrow \mathcal{E}'|_C$ over k. This is an isomorphism by Lemma 3.6. Since the diagram

$$\begin{array}{ccc} & \bigwedge^2 H^0(f) & \\ \bigwedge^2 A = \bigwedge^2 H^0(\mathcal{E}|_C) & \longrightarrow & \bigwedge^2 H^0(\mathcal{E}'|_C) = \bigwedge^2 A' \\ \downarrow & & \downarrow \\ H^0(\bigwedge^2 \mathcal{E}|_C) & \longrightarrow & H^0(\bigwedge^2 \mathcal{E}'|_C) \\ & H^0(\bigwedge^2 f) & \end{array}$$

is commutative, the isomorphism $H^0(f)$ maps $k\sigma$ onto $k\sigma'$. Thus we have proved (2) of Theorem B.

Assume that k is perfect and let $\bar{E}$ be a vector bundle on $\bar{C} = C \otimes_k \bar{k}$. We consider a descent problem of $\bar{E}$ under the following condition:

($*$) $\bar{E}$ is simple and $\sigma^*\bar{E} \simeq \bar{E}$ for every element σ of the Galois group $\mathrm{Gal}\, k$ of $\bar{k}/k$.

of the second Galois cohomology group $H^2(\mathrm{Gal}\, k, \mathrm{Aut}\, \bar{E})$. Choose an isomorphism $f_\sigma : \bar{E} \xrightarrow{\sim} \sigma^* \bar{E}$ for each $\sigma \in \mathrm{Gal}\, k$. Then $ob(\bar{E})$ is the cohomology class of the cocycle $\{c_{\sigma,\tau}\}_{\sigma,\tau \in \mathrm{Gal}\, k}$ defined by $c_{\sigma,\tau} = f_{\sigma\tau}^{-1} \circ \tau^*(f_\sigma) \circ f_\tau \in \mathrm{Aut}_{\bar{k}}\, \bar{E}$. In other words, $ob(\bar{E})$ is the factor set of the extension

$$1 \longrightarrow \mathrm{Aut}_{\bar{k}}\, \bar{E} \longrightarrow \mathrm{Aut}_k\, \bar{E} \longrightarrow \mathrm{Gal}\, k \longrightarrow 1.$$

Lemma 7.1 *If* $\dim H^i(\bar{C}, \bar{E}) = n > 0$, *then the obstruction* $ob(\bar{E})$ *is an* n*-torsion.*

Proof. Let $\{s_1, \ldots, s_n\}$ be a basis of $H^i(\bar{C}, \bar{E})$ and $A_\sigma \in M_n(\bar{k})$ the matrix representing

$$H^i(f_\sigma) : H^i(\bar{C}, \bar{E}) \longrightarrow H^i(\bar{C}, \sigma^* \bar{E})$$

with respect to the bases $\{s_1, \ldots, s_n\}$ and $\{\sigma^* s_1, \ldots, \sigma^* s_n\}$. Then we have

$$\det H^i(c_{\sigma,\tau}) = (\det A_{\sigma\tau})^{-1} \tau(\det A_\sigma) \det A_\tau$$

in $\bar{k}^\times$. Therefore, $\{\det H^i(c_{\sigma,\tau})\}_{\sigma,\tau \in \mathrm{Gal}\, k}$ is cohomologous to zero. Since $c_{\sigma,\tau}$ are all constant multiplications, $\det H^i(c_{\sigma,\tau})$ are equal to $c_{\sigma,\tau}^n$. Hence $ob(\bar{E})$ is an n-torsion. □

Now we prove (1) of Theorem B. Let C be a non-pentagonal curve of genus 9 defined over k. It suffices to show the following:

Proposition 7.2 *Assume that* C *has no* g_5^1 *over* $\bar{k}$. *Then there exists a vector bundle* E *on* C *such that* $E \otimes_k \bar{k}$ *is isomorphic to the vector bundle* E_{max} *on* $C \otimes_k \bar{k}$ *in Theorem C.*

Proof. By Propositions 3.5 and 5.6, E_{max} satisfies $(*)$. Hence the obstruction $ob(E_{max})$ belongs to $H^2(\mathrm{Gal}\, k, \mathrm{Aut}_{\bar{k}} E_{max}) = H^2(\mathrm{Gal}\, k, \bar{k}^\times)$. Let

$$Det : H^2(\mathrm{Gal}\, k, \mathrm{Aut}_{\bar{k}} E_{max}) \longrightarrow H^2(\mathrm{Gal}\, k, \mathrm{Aut}_{\bar{k}} \det E_{max})$$

be the determinant homomorphism. Since $\det E_{max}$ is the canonical bundle, it descends to C. Hence $ob(E_{max})$ belongs to the kernel and is a 3-torsion. On the other hand, $ob(E_{max})$ is a 14-torsion by the preceding lemma since $\dim H^1(E_{max}) = 14$. Therefore, $ob(E_{max})$ vanishes and E_{max} descends to C. (This is a Galois group variant of an argument of Mumford-Newstead [11].) □

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