Frequency Domain Fractional Delay Estimation for Noisy Channels

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Abstract—**A channel delay can be estimated by evaluating the channel response to a deterministic signal. In the frequency domain, a maximum likelihood delay estimator which accounts for colored Gaussian measurement noise can be formulated. A procedure that allows fast estimation of the channel delay is proposed, in case a low-pass deterministic signal is applied to the channel.**

Keywords—**Fractional delay; delay estimation**

I. Introduction

In some applications, the delay of a transmission channel is estimated by applying a deterministic pulse to the channel, measuring the channel response and calculating the time difference between the pulses. Since we require digital signal processing techniques to calculate the time difference, such a delay generally can not be expressed as an exact multiple of the sampling period, which explains the need for fractional digital delays.

By formulating the maximum likelihood estimator (MLE) of the fractional delay in frequency domain, the computational complexity is reduced in case the measurements are corrupted by colored Gaussian noise [2]. Nevertheless, a very fine grid search must be performed in order to obtain the most likely delay estimate, resulting in tedious calculations. In this paper, a procedure is proposed that provides an approximation of the delay estimate in case the deterministic pulse is a low-pass signal.

II. Problem Formulation

Consider a known deterministic signal $s(t)$ corrupted by additive noise $n(t)$, which arrives at a sensor with time-delay τ. The sensor signal $x(t)$ is expressed as

$$x(t) = s(t-\tau) + n(t)\ . \qquad (1)$$

A classical approach to determining the time-delay τ is to identify the absolute maximum of the cross-correlation of $s(t)$ and $x(t)$. If the additive noise is white, then the conventional method is correlation by matched filtering; if the noise is non-white an additional whitening filter is used to pre-whiten the noise, resulting in a whitening matched filter [3][4].

In a discrete-time sampled system, the problem reduces to estimating the continuous-time delay τ using a finite set of samples. However, since the delay is not necessarily an integer multiple of the sampling period T_s, a fractional delay must be estimated and some kind of interpolation process is required.

III. Frequency Domain Delay MLE

Let $\underline{x} = (x(0), x(T_s), \ldots, x((B-1)T_s))^t$ contain B subsequent samples of the sensor signal, such that the delayed version of $s(t)$ is observed entirely. Assume that the signals are bandlimited, therefore the sinc function can be used as an interpolation filter, as follows from the sampling theorem. The discrete Fourier transform (DFT) coefficient $X(i)$, with $i \in \{0, 1, \ldots, B-1\}$, equals

$$\begin{aligned} X(i) &= \sum_{k=0}^{B-1} x(kT_s)\exp\{\text{-}j2\pi ik/B\} \\ &= S(i)\exp\{\text{-}j2\pi i\tau/BT_s\} + N(i)\ , \end{aligned} \qquad (2)$$

where $S(i)$ and $N(i)$ are DFT coefficients of $s(kT_s)$ and $n(kT_s)$, respectively. The additive noise is a zero-mean wide sense stationary Gaussian process with sampled noise power spectral density function (psdf) given by $P_N(i) = \frac{1}{B}E\{|N(i)|^2\}$. The probability distribution function (pdf) of $N(i)$ equals

$$p\left(N(i)\right) = \frac{1}{\sqrt{2\pi BP_N(i)}}\exp\left\{\text{-}\frac{|N(i)|^2}{2BP_N(i)}\right\}. \qquad (3)$$

Let $\underline{X} = (X(0), X(1), \ldots, X(B-1))^t$ contain all DFT coefficients (2) of the observation vector $\underline{x}$. Since the DFT coefficients are uncorrelated, the scaled conditional a-posteriori likelihood function of $\underline{X}$ equals

$$L(\underline{X}|\tau) = \exp\left\{\text{-}\sum_{i=0}^{B-1}\frac{|X(i) - S(i)\ \exp\{\text{-}j2\pi i\tau/BT_s\}|^2}{2BP_N(i)}\right\}. \qquad (4)$$

The maximum likelihood estimator (MLE) of the time-delay τ is obtained by evaluating expression (4) on the observation interval $[0, B\,T_s\rangle$ and selecting the value of the delay parameter which maximizes $L(\underline{X}|\tau)$. Since $\ln L(\underline{X}|\tau)$ is a monotonically increasing function of $L(\underline{X}|\tau)$ we can also evaluate $\ln L(\underline{X}|\tau)$. After eliminating all terms in $\ln L(\underline{X}|\tau)$ that are independent of τ, the MLE is expressed as

$$\hat{\tau} = \arg\max_{\hat{\tau}} \sum_{i=0}^{B-1} \frac{\mathrm{Re}[X^*(i)S(i)\exp\{\text{-}j2\pi i\hat{\tau}/BT_s\}]}{BP_N(i)} \ . \tag{5}$$

In a practical measuring situation, the noise psdf is unknown and needs to be estimated before calculating $\hat{\tau}$. Assuming that R independent observations of the noise process are available, the MLE of $P_N(i)$ equals

$$\hat{P}_N(i) = \frac{1}{BR}\sum_{k=1}^{R}|N_k(i)|^2 \ , \tag{6}$$

where $N_k(i)$ equals the i-th DFT coefficient of the k-th observation. From now on it is assumed that $P_N(i)$ in expression (5) can be replaced by $\hat{P}_N(i)$.

The variance of any estimator is lower bounded by the Cramer-Rao bound (CRB) [3][4]. The CRB of $\hat{\tau}$ equals (see the appendix)

$$\mathrm{var}(\hat{\tau}-\tau) \geq \left(\sum_{i=0}^{B-1}\left(\frac{2\pi i}{BT_s}\right)^2 \frac{P_S(i)}{P_N(i)}\right)^{-1} \ , \tag{7}$$

where $P_S(i) = \frac{1}{B}|S(i)|^2$ equals the signal power in the i-th frequency bin.
Evidently, the estimation variance is inversely proportional to a weighted sum of the signal-to-noise ratio's (SNR) in each frequency bin. Consequently, frequency bins with low SNR contribute to the estimation variance to a large extent, while frequency bins with high SNR do not. Moreover, the SNR of each bin is weighted proportional to the squared center-frequency of that bin, therefore relatively high frequency bins with high SNR contribute largely to the quality of the estimation, which is intuitively valid.

As mentioned, the MLE can be numerically determined by evaluating the likelihood function on the observation interval. Seeing that an accurate estimate of the time-delay is required, a very fine grid search must be performed, resulting in rather tedious calculations. Alternative search procedures based on iterative maximization are obviously more efficient, however, since the likelihood function generally has local maxima, iterative maximization does not necessarily result in convergence to the MLE.

IV. Alternative Time-Delay Estimator

A. Single Frequency Bin MLE

Instead of constructing the a-posteriori likelihood function of the vector of DFT coefficients, we can also consider the likelihood function of a single coefficient and formulate its MLE. The single bin MLE equals

$$\begin{aligned}\hat{\tau}_i &= \arg\max_{\hat{\tau}_i} \mathrm{Re}[X^*(i)S(i)\exp\{\text{-}j2\pi i\hat{\tau}_i/BT_s\}] \\ &= \arg\max_{\hat{\tau}_i} \|X(i)\|\|S(i)\|\cos(\angle S(i) \text{ - } \angle X(i) \text{ - } \frac{2\pi i\hat{\tau}_i}{BT_s}) \ ,\end{aligned} \tag{8}$$

for $i = 1, \ldots, B/2+1$, and for simplicity, it is assumed that the number of samples B is restricted to a power of 2. Obviously, the DC component of the sensor signal provides no information on the time-delay, therefore the estimator for $i = 0$ is ignored.
On the observation interval $[0, BT_s\rangle$ the single bin MLE has exactly $i-1$ solutions that satisfy the equation

$$\angle S(i) \text{ - } \angle X(i) \text{ - } \frac{2\pi i\hat{\tau}_i}{BT_s} \text{ - } k_i 2\pi = 0 \ , \tag{9}$$

with $k_i = 0, \ldots, i-1$.
Evidently, expression (9) has a unique solution for $i = 1$. By selecting k_i for $i > 1$ such that the resulting delay estimate $\hat{\tau}_i$ is closest to $\hat{\tau}_{i-1}$, a procedure is obtained that iteratively finds the maximum likelihood estimates for each frequency bin. Subsequently, the best single bin delay estimate is obtained by selecting the estimator with lowest variance, where the CRB of the single bin MLE equals

$$\mathrm{var}(\hat{\tau}_i-\tau_i) \geq \left(\left(\frac{2\pi i}{BT_s}\right)^2 \frac{P_S(i)}{P_N(i)}\right)^{-1} \ . \tag{10}$$

Note that the procedure relies on fairly good estimates in the lower frequency regions, therefore the method only applies to low-pass signals.

B. Weighted Single Frequency Bin MLEs

We can also construct an estimator by defining a weighted mean of the single bin MLEs

$$\hat{\tau} = \left(\sum_{i=1}^{B/2+1} w_i\right)^{-1} \sum_{i=1}^{B/2+1} w_i\hat{\tau}_i \ , \tag{11}$$

where the weights w_i express our confidence in the delay estimate $\hat{\tau}_i$ based on the estimator CRB, *i.e.*

$$w_i = \left(\frac{2\pi i}{BT_s}\right)^2 \frac{P_S(i)}{P_N(i)} \ . \tag{12}$$

In addition, we can select frequency bins of which the weighting function exceeds a certain threshold, thus ignoring estimates we do not trust at all.

V. Numerical Determination of the MLE

Evidently, the estimators of the previous section are not maximum likelihood. However, since the single bin MLEs (8) provide estimates that are close to the maximum likelihood delay estimate, the evaluation interval can be decreased. Consequently, the MLE (5) can be numerically determined by a fine grid search on a small search space, thus reducing computation time to a large extent.

Moreover, on the evaluation interval as provided by the single bin estimates, the likelihood function (4) can be approximated by a second order polynomial. The coefficients of the polynomial are obtained by solving the following system of equations in the least-squares sense

$$\begin{pmatrix} \hat{\tau}_1^2 & \hat{\tau}_1 & 1 \\ \hat{\tau}_2^2 & \hat{\tau}_2 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} L(\underline{X}|\hat{\tau}_1) \\ L(\underline{X}|\hat{\tau}_2) \\ \vdots \end{pmatrix}, \quad (13)$$

which is done by multiplying the pseudo-inverse of the data matrix of single bin delay estimates by the vector of likelihoods [5]. Since the maximum of the polynomial approximates the maximum of the likelihood function, an approximation of the MLE is obtained by

$$\hat{\tau} = -\frac{c_1}{2c_2}. \quad (14)$$

Consequently, in order to find the maximum likelihood delay estimate, only the likelihoods of a small set of delays have to be calculated.

VI. Simulations

The concepts of the previous sections are illustrated by means of Monte-Carlo simulations. A pulse-shaped deterministic signal $s(t)$ is a sampled with $T_s = 10$ns. The signal is delayed by $\tau = 3.1\mu$s and downsampled such that $T_s = 160$ns, the time-delay is now a fraction of the sampling period. Subsequently, 500 realizations of colored Gaussian noise are added to the signal in order to obtain 500 sensor signals $x(t)$, where the mean SNR of the sensor signals equals -6dB. In Figure 1 a single sensor signal is plotted as an example. Furthermore, in Figure 2 the power spectrum of the downsampled signal and the psdf of the noise are shown.

For each sensor signal the maximum likelihood estimate of the pulse time-delay is determined as a reference. Then the single bin maximum likelihood delay estimates are computed, and for each trial, the single bin estimator with

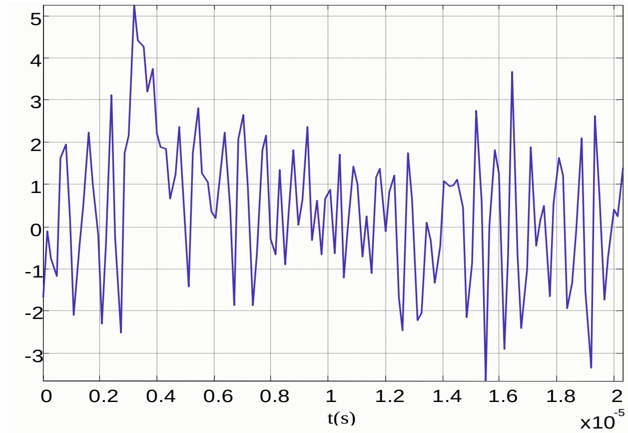

Fig. 1. Example of a sampled sensor signal $x(t)$.

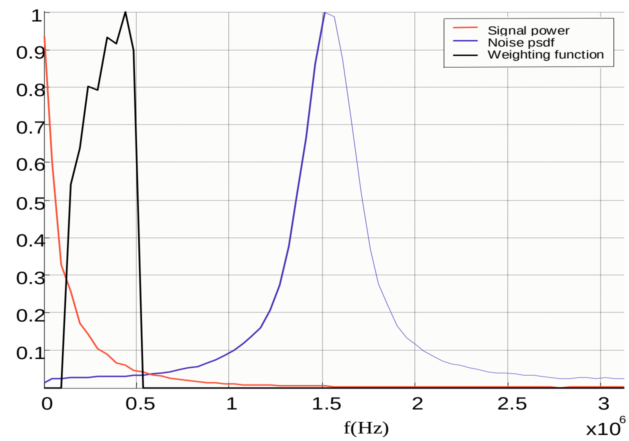

Fig. 2. Signal power, noise psdf, and weighting function.

lowest CRB (best single bin MLE) is selected. A weighting function is constructed using the signal power spectrum and the noise psdf (see Figure 2), thus only reliable estimates are considered. Subsequently, the weighted mean of the single bin delay estimates is computed for every trial. Finally, a second order polynomial is fitted through the most reliable single bin estimates of each trial in order to obtain an approximation of the maximum likelihood estimate.

In Figure 3 the resulting estimates for the first number of trials are plotted. In Table I the mean and standard deviation of the estimates for all trials are given. Clearly, the means of the estimates are equal to the mean of the maximum likelihood estimate, and the true time-delay. The standard deviation of the single bin estimates is much higher than the standard deviation of the maximum likelihood estimate, as is expected. However, the standard deviation of the estimates obtained by the approximated MLE is very close to the standard deviation of the estimates obtained by the true MLE. Moreover, the

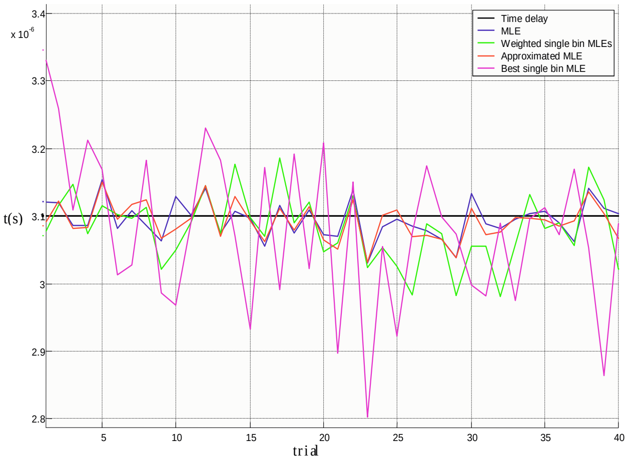

Fig. 3. Delay estimates obtained by different estimators for a number of trials.

TABLE I

STATISTICAL AVERAGES OF ESTIMATES (500 TRIALS).

Estimator	mean(μs)	std(μs)
MLE	3.10	.034
Best single bin MLE	3.10	.145
Weighted bin MLE	3.09	.053
Approximated MLE	3.10	.035

standard deviation of the weighted single bin estimates is slightly higher than the standard deviation of the maximum likelihood estimate.

VII. CONCLUSIONS

A channel delay can be estimated by evaluating the channel response to a deterministic signal. In the frequency domain, a maximum likelihood fractional delay estimator which accounts for colored Gaussian measurement noise can be formulated. The MLE can be numerically determined by evaluating the likelihood function on the observation interval. However, since an accurate estimate of the time-delay is required, a fine grid search must be performed, resulting in tedious calculations.

In case the applied deterministic signal is low-pass, an iterative procedure can be applied to obtain a MLE for each individual frequency bin. By constructing a weighting function based on the CRB of each single bin MLE, we can select the most reliable estimates and formulate an alternative estimator by constructing the weighted mean of the single bin estimators. Moreover, the single bin estimates provide a decreased evaluation interval, thus a fine grid search is only required on a small search space in order to obtain the maximum likelihood estimate.

Since the reliable single bin estimates are close to the maximum likelihood estimate a second order polynomial can be fitted through the likelihoods of the single bin estimates. Consequently, the delay that corresponds to the maximum of the polynomial is a close approximation of the maximum likelihood estimate. In addition, the approximation can be obtained by computing only a small number of likelihoods.

VIII. ACKNOWLEDGEMENT

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APPENDIX

The Cramer-Rao bound is given by [3][4]

$$\text{var}\,(\hat{\tau}-\tau) \geq \left(\text{-}E\left\{ \frac{\partial^2 \ln L(\underline{X})}{\partial \tau^2} \right\} \right)^{-1} . \tag{15}$$

Since

$$\frac{\partial^2 \ln L(\underline{X})}{\partial \tau^2} = \sum_{i=0}^{B-1} \frac{\partial^2}{\partial \tau^2} \frac{\text{Re}[X^*(i)S(i)\exp\{\text{-}j2\pi i\tau/BT_s\}]}{BP_N(i)}$$

$$= \text{-} \sum_{i=0}^{B-1} \left(\frac{2\pi i}{BT_s} \right)^2 \frac{\text{Re}[X^*(i)S(i)\exp\{\text{-}j2\pi i\tau/BT_s\}]}{BP_N(i)} , \tag{16}$$

it follows that

$$E\left\{ \frac{\partial^2 \ln L(\underline{X})}{\partial \tau^2} \right\} = -\sum_{i=0}^{B-1} \left(\frac{2\pi i}{BT_s} \right)^2 \frac{|S(i)|^2}{BP_N(i)} , \tag{17}$$

since $E\{X(i)\} = S(i)\exp\{-j2\pi i\tau/BT_s\}$.
By combining expressions (15) and (17), the estimator variance as given by equation (7) is obtained.

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