A Robust Optimal Control for Constrained Robot Manipulators

Faiçal Mnif

***Abstract*— A mixed optimal/robust control is proposed in this paper for the tracking constrained robotic systems under parametric uncertainties and external perturbations. The dynamic model of the constrained system is modified to contain two sets of state variables, where one describes the constrained motion and the other describes the unconstrained one. The design objective is that under a prescribed disturbance norm level, an optimal control system is to be designed as well as a robust control to overcome the effect of uncertainties. The optimal control is based on the solution of a nonlinear Ricatti equation, which by virtue of the skew symmetry property of the reduced dynamics of the constrained manipulators and an adequate choice of state variables becomes an algebraic equation easy to solve. We then investigate the design of the robust control of the uncertain by a continuous state feedback function. It will be shown by using Lyapunov stability theory that our approach globally asymptotically stabilizes the uncertain constrained robotic system. We illustrate our approach by applying our control approach to a 2-DOF constrained manipulator.** *Copyright © 2003-2005 Yang's Scientific Research Institute, LLC. All rights reserved.*

***Index Terms*— Constrained manipulators, optimal control, stabilization, robust control, uncertainties.**

I. INTRODUCTION

ROBOTIC motions can be classified into two categories: unconstrained and constrained. Constraints imposed on a dynamic system may be holonomic or nonholonomic. The subject matter of this paper deals with robotic systems that are subject to holonomic constraints. Holonomic constraints are characterized by equality equations in terms of position variables. A General framework of constrained motion has been rigorously developed by Mc-Clamroch and Wang [1]. They decomposed the robotic dynamic model into two sets of equations; the first describes the end-effector motion on the surface of contact while the second is used to determine the interacting force between the robot and its work environment.

Based on this framework many authors proposed several control schemes for constrained robotic systems [2], [3]. Robust control problem for constrained manipulators has also received considerable interest by many researchers the last decade. In fact, as for unconstrained systems, constrained systems are subject to many type of uncertainties such as the rigid body parameters, environment parameters such as stiffness which usually vary from one task to another and can not be precisely known in advance. To deal with such a stability problem Yao el al. [20] proposed the variable structure control and adaptive control, Mnif in [9] proposed a continuous state feedback control. In this paper we address the problem differently from what is presented in [9], we investigate first the nonlinear optimal control to ensure both zero-error convergence of the nominal system and to minimize the torque input of to manipulator. We then investigate the robust control problem for the uncertain system. The first control is based on Johansson's work [19], where he addressed the optimal control problem for unconstrained manipulators which minimizes a quadratic performance index involving system error and torque input, as an explicit solution of the Hamilton-Jacobi equation. The proposed robust control consists of a state feedback of a continuous time function. A stability analysis based on Lyapunov theory is investigated to prove the global asymptotic stability of the uncertain system for both position and force variables. The continuity of the solution will be also investigated and proved.

This paper is organized as follows. In Section 2, we consider the algebraic-differential equations describing the dynamics of the constrained manipulator and show how the uncertainties discussed above can be incorporated in the equation. We then formulate the robust control problem for the constrained robot manipulator. In Section 3, an exhaustive analysis of the perturbed system is provided on the basis of which the force control problem is addressed. In Section 4 we investigate the robust optimal control problem for the reduced dynamical system. In Section 5 simulations results of a 2-DOF constrained robot manipulator are provided to illustrate the validity of our approach.

II. MODEL DESCRIPTION AND PROBLEM STATEMENT

A. System Description

The motion dynamic equation, as given in Mnif et al. **[8]**, of a constrained mechanical system with geometrical constraints can be expressed as:

$$\begin{cases} \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{u} + \mathbf{J}^\top(\mathbf{q})\lambda \\ \Phi(\mathbf{q}) = \mathbf{0} \end{cases} \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of joint coordinates of the mechanical system. $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the generalized mass matrix of the system, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n\times n}$ is the matrix of centrifugal and corriolis terms which can also include unmodelled joint friction terms , $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^n$ is the vector of the gravitational

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forces, $\mathbf{u} \in \mathbb{R}^n$ denotes the vector of generalized input forces (torques).

The second equation of (1) represents the set of $m(m¡n)$ geometrical constraint equations which is supposed to be a mapping at least twice differentiable and $\mathbf{J}(\mathbf{q}) := \frac{\partial \Phi(\mathbf{q})}{\partial \mathbf{q}}$is the Jacobian force matrix which is assumed to be of full rank. Such assumption is always true as long as the system is not redundant. $\lambda \in \mathbb{R}^m$ is the vector of generalized Lagrange multipliers associated with the constraints such that the generalized constraint forces are associated with this vector by $\mathbf{f} = \mathbf{J}^\top(\mathbf{q})\lambda$.

Assumption 1

We suppose that the constraint equations are independent in the sense that the Jacobian matrix **J(q)** of $m \times n$ dimension is of full rank for any **q** satisfying the second equation in (1).

By defining the vector partition $\mathbf{q}^\top = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}^\top$ where $\mathbf{q}_1 \in \mathbb{R}^m$ and $\mathbf{q}_2 \in \mathbb{R}^{n-m}$ representing the force and position coordinate vectors respectively, the constraint equation can be written as

$$\mathbf{q}_1 = \Omega(\mathbf{q}_2) \tag{2}$$

where $\Omega \subseteq \mathbb{R}^m$.

Thus, equation (2) can be written as

$$\Phi(\Omega(\mathbf{q}_2), \mathbf{q}_2) = 0 \tag{3}$$

This partition should be made so that the Jacobian matrix **J(q)** can be divided into two terms

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \mathbf{J}_1(\mathbf{q}) & \mathbf{J}_2(\mathbf{q}) \end{bmatrix}^\top \tag{4}$$

with $\mathbf{J}_1(\mathbf{q}) \in \mathbb{R}^{m\times m}$ and $\mathbf{J}_2(\mathbf{q}) \in \mathbb{R}^{m\times(n-m)}$ such that $\mathbf{J}_1(\mathbf{q})$ is of full rank $\forall t \in T_d$ where T_d is a finite interval within the overall duration of the planned motion **[13]**.

Let's define the vector partition $\mathbf{x}^\top = (\mathbf{x}_1, \mathbf{x}_2)^\top$, $\mathbf{x}_1 \in \mathbb{R}^m, \mathbf{x}_2 \in \mathbb{R}^{n-m}$ and the transformation

$$\mathbf{x} = \mathbf{X}(\mathbf{q}) = \begin{bmatrix} \mathbf{q}_1 - \Omega(\mathbf{q}_2) \\ \mathbf{q}_2 \end{bmatrix} \tag{5}$$

which is necessarily nonsingular and its inverse is defined as

$$\mathbf{q} = \mathbf{Q}(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_1 + \Omega(\mathbf{x}_2) \\ \mathbf{x}_2 \end{bmatrix} \tag{6}$$

Define also the Jacobian of the inverse transformation

$$\mathbf{T}(x) = \frac{\partial}{\partial \mathbf{x}}\mathbf{Q}(\mathbf{x}) = \begin{bmatrix} \mathbf{I}_m & \frac{\partial}{\partial \mathbf{x}_2}\Omega(\mathbf{x}_2) \\ 0 & \mathbf{I}_{n-m} \end{bmatrix} \tag{7}$$

and define the identity matrix partition $\mathbf{I}_n = \begin{bmatrix} \mathbf{E}_1^\top & \mathbf{E}_2^\top \end{bmatrix}$ with $\mathbf{E}_1^\top = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \end{bmatrix}^\top \in \mathbb{R}^{n\times m}$ and $\mathbf{E}_2^\top = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n-m} \end{bmatrix}^\top \in \mathbb{R}^{n\times(n-m)}$. By using the properties of the transformation listed above the transformed dynamics of the mechanical system (1) can be written as in [8]:

$$\mathbf{M}^*(\mathbf{x}_2)\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 + \mathbf{C}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)\mathbf{E}_2^\top\dot{\mathbf{x}}_2 + \mathbf{G}^*(\mathbf{x}_2) = \mathbf{u}* + \mathbf{J}^{*T}(\mathbf{x}_2)\lambda \tag{8}$$

where

$$\begin{aligned}
\mathbf{M}^*(\mathbf{x}_2) &= \mathbf{T}^\top(\mathbf{E}_2^\top\mathbf{x}_2)\mathbf{M}(\mathbf{Q}(\mathbf{E}_2^\top\mathbf{x}_2))\mathbf{T}(\mathbf{E}_2^\top\mathbf{x}_2) \\
\mathbf{C}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2) &= \mathbf{T}^\top(\mathbf{E}_2^\top\mathbf{x}_2)\Big[\mathbf{M}(\mathbf{Q}(\mathbf{E}_2^\top\mathbf{x}_2)\dot{\mathbf{T}}(\mathbf{E}_2^\top\mathbf{x}_2)) \\
&\quad + \mathbf{C}(\mathbf{Q}(\mathbf{E}_2^\top\mathbf{x}_2), \mathbf{T}(\mathbf{E}_2^\top\mathbf{x}_2))\mathbf{E}_2^\top\dot{\mathbf{x}}_2\mathbf{T}(\mathbf{E}_2^\top\mathbf{x}_2)\Big] \\
\mathbf{G}^*(\mathbf{x}_2) &= \mathbf{T}^\top(\mathbf{E}_2^\top\mathbf{x}_2)\mathbf{G}(\mathbf{Q}(\mathbf{E}_2^\top\mathbf{x}_2)) \\
\mathbf{J}^*(\mathbf{x}_2) &= \mathbf{T}^\top(\mathbf{E}_2^\top\mathbf{x}_2)\mathbf{J}^\top(\mathbf{Q}(\mathbf{E}_2^\top\mathbf{x}_2)) \\
\mathbf{u}^* &= \mathbf{T}^\top(\mathbf{E}_2\mathbf{x}_2)\mathbf{u}
\end{aligned}$$

Property 1

We can easily show that $\mathbf{E}_1^\top\mathbf{J}^{*T}(\mathbf{x}_2)$ *is nonsingular and that* $\mathbf{E}_2\mathbf{J}^{*T}(\mathbf{x}_2) = \mathbf{0}$.

With this crucial property, a so-called reduced form of the dynamic equation with holonomic constraints is derived as follows

$$\mathbf{M}_1^*(\mathbf{x}_2)\ddot{\mathbf{x}}_2 + \mathbf{C}_1^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 + \mathbf{G}_1^*(\mathbf{x}_2) = \mathbf{u}_1^* + \mathbf{J}_1^{*T}(\mathbf{x}_2)\lambda \tag{9}$$

$$\mathbf{M}_2^*(\mathbf{x}_2)\ddot{\mathbf{x}}_2 + \mathbf{C}_2^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 + \mathbf{G}_2^*(\mathbf{x}_2) = \mathbf{u}_2^* \tag{10}$$

where

$$\begin{aligned}
&\mathbf{M}_1^* = \mathbf{E}_1\mathbf{M}^*\mathbf{E}_2^\top; \mathbf{C}_1^* = \mathbf{E}_1\mathbf{C}^*\mathbf{E}_2^\top; \mathbf{G}_1^* = \mathbf{E}_1\mathbf{G}^*, \\
&\mathbf{M}_2^* = \mathbf{E}_2\mathbf{M}^*\mathbf{E}_2^\top; \mathbf{C}_2^* = \mathbf{E}_2\mathbf{C}^*\mathbf{E}_2^\top; \mathbf{G}_2^* = \mathbf{E}_2\mathbf{G}^* \\
&\mathbf{u}_1^* = \mathbf{E}_1\mathbf{u}^*; \mathbf{u}_2^* = \mathbf{E}_2\mathbf{u}^*; \mathbf{J}_1^{*T} = \mathbf{E}_1\mathbf{J}^{*T}.
\end{aligned}$$

The ordinary differential equation (10) characterizes the motion of the mechanical system on the constrained manifold, whereas equation (9) can be viewed as an algebraic equation for the constraint force expressed in terms of the motion on the constraint manifold.

B. The control problem statement

The objective of the control problem can be stated as follows: Given a constrained mechanical system described by the dynamic model

$$\begin{aligned}
&\mathbf{M}^*(\mathbf{x}_2)\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 + \mathbf{C}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)\mathbf{E}_2^\top\dot{\mathbf{x}}_2 + \mathbf{G}^*(\mathbf{x}_2) \\
= \;&\mathbf{T}^\top(\mathbf{x}_2)\mathbf{u} + \mathbf{J}^{*T}(\mathbf{x}_2)\lambda + \mathbf{p}^*
\end{aligned} \tag{11}$$

where $\mathbf{M}^*, \mathbf{C}^*$ and $\mathbf{G}^*$ are uncertain. $\mathbf{p}^*$ represents a vector of finite energy (square integrable) exogenous disturbance such as constant friction torques due to the contact of the end effector with the work environment. $\mathbf{M}^*, \mathbf{C}^*$, $\mathbf{G}^*$ and $\mathbf{p}^*$ can not be a priori estimated precisely. Given also a desired trajectory $\mathbf{x}_{2d}(t)$ and a desired force represented by the Lagrange multiplier, our aim is to select the control law $\mathbf{u}^*(t)$ such that the uncertain nonlinear system follows the desired trajectory and ensures the desired contact force applied by the end-effector on the constrained manifold.

III. PERTURBED SYSTEM ANALYSIS AND FORCE CONTROL DESIGN

A. State Space Model of the Closed Loop Uncertain System

We suppose now that the system is uncertain and that the estimated parameters of the system are to be

$$\mathbf{M}^* = \mathbf{M}_0 + \Delta\mathbf{M}$$
$$\mathbf{C}^* = \mathbf{C}_0 + \Delta\mathbf{C}$$
$$\mathbf{G}^* = \mathbf{G}_0 + \Delta\mathbf{G}$$

Then the dynamic model (11) can be rewritten as

$$\begin{aligned} & (\mathbf{M}_0^*(\mathbf{x}_2) + \Delta\mathbf{M}^*(\mathbf{x}_2))\,\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 \\ & + (\mathbf{C}_0^*(\mathbf{x}_2,\dot{\mathbf{x}}_2) + \Delta\mathbf{C}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2))\,\mathbf{E}_2^\top\dot{\mathbf{x}}_2 \\ & + (\mathbf{G}_0^*(\mathbf{x}_2) + \Delta\mathbf{G}^*(\mathbf{x}_2)) \\ = \ & \mathbf{T}^\top(\mathbf{x}_2)\mathbf{u} + \mathbf{J}^{*T}(\mathbf{x}_2)\lambda + \mathbf{p}^*. \end{aligned} \tag{12}$$

Assumption 2

Assume that we can define some continuous functions $\varphi_i(.,.)$defining the upper bound limits of $\Delta\mathbf{M}^*, \Delta\mathbf{C}^*, \Delta\mathbf{G}^*$ and $\mathbf{p}^*$ such that

$$\begin{aligned} &\|\Delta\mathbf{M}^*(\mathbf{x}_2)\| \le \varphi_1(\mathbf{x}_2,t) \\ &\|\Delta\mathbf{C}^*(\mathbf{x}_2)\| \le \varphi_2(\mathbf{x}_2,t) \\ &\|\Delta\mathbf{G}^*(\mathbf{x}_2)\| \le \varphi_3(\mathbf{x}_2,t) \\ &\|\mathbf{p}^*\| \le \rho. \end{aligned} \tag{13}$$

Note that the exact knowledge of $\varphi_i, (i = 1,2,3)$ and ρ is not necessary for the construction of the controller. ρ is supposed here to be the upper bound limit of constant disturbances that can affect the system.

Equation (12) can be rewritten as

$$\begin{aligned} & \mathbf{M}_0^*(\mathbf{x}_2)\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 + \mathbf{C}_0^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\mathbf{E}_2^\top\dot{\mathbf{x}}_2 + \mathbf{G}_0^*(\mathbf{x}_2) \\ = \ & \mathbf{u}^* + \mathbf{J}^{*T}(\mathbf{x}_2)\lambda \\ & + (\mathbf{p}^* - \Delta\mathbf{M}^*(\mathbf{x}_2) - \Delta\mathbf{C}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2) - \Delta\mathbf{G}^*(\mathbf{x}_2)\,. \end{aligned} \tag{14}$$

Denoting

$$\begin{aligned} \delta^* \ = \ & -[-\mathbf{p}^* + \Delta\mathbf{M}^*(\mathbf{x}_2)\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 + \Delta\mathbf{C}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\mathbf{E}_2^\top\dot{\mathbf{x}}_2 \\ & + \Delta\mathbf{G}^*(\mathbf{x}_2)]. \end{aligned} \tag{15}$$

Assumption 3

We suppose that the vector representing the global uncertainties $\delta*$is always bounded by a certain known function,ε which represents the worst case bound on δ^* such that

$$\|\delta_1^*(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2)\| \le \pi(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2) \tag{16}$$

Then (13) becomes

$$\begin{aligned} & \mathbf{M}_0^*(\mathbf{x}_2)\mathbf{E}_2^\top\ddot{\mathbf{x}}_2 + \mathbf{C}_0^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\mathbf{E}_2^\top\dot{\mathbf{x}}_2 + \mathbf{G}_0^*(\mathbf{x}_2) \\ = \ & \mathbf{u}^* + \mathbf{J}^{*T}(\mathbf{x}_2)\lambda + \delta^* \end{aligned} \tag{17}$$

and the reduced model as

$$\mathbf{M}_{10}^*(\mathbf{x}_2)\ddot{\mathbf{x}}_2 + \mathbf{C}_{10}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 + \mathbf{G}_{10}^*(\mathbf{x}_2) = \mathbf{u}_1^* + \mathbf{J}_1^{*T}(\mathbf{x}_2)\lambda + \delta_1^* \tag{18}$$

$$\mathbf{M}_{20}^*(\mathbf{x}_2)\ddot{\mathbf{x}}_2 + \mathbf{C}_{20}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 + \mathbf{G}_{20}^*(\mathbf{x}_2) = \mathbf{u}_2^* + \delta_2^* \tag{19}$$

where $\delta_1^* = \mathbf{E}_1\delta^*$ and $\delta_2^* = \mathbf{E}_2\delta^*$ and satisfying $\|\delta_1^*(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2)\| \le \pi(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2)$ and $\|\delta_2^*(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2)\| \le \varepsilon(\mathbf{x}_2,\dot{\mathbf{x}}_2,\ddot{\mathbf{x}}_2)$.

B. Force Control Loop Design

Proposition 1

For the constrained uncertain mechanical system (18)-(19), *if a robust control $\mathbf{u}_2^*$ is designed to achieve the convergence of the reduced dynamics* (19) so that $x_2(t)_{t\to\infty} \to x_{2d}(t)$, a *PI-like control law in* (18) *of the form*

$$\begin{aligned} u_1^* \ = \ & J_1^{*T}(x_2)\left[-\lambda_d - \mathbf{K}_\lambda \int \tilde{\lambda} dt\right] \\ & + \mathbf{M}_{10}^*(\mathbf{x}_2)\ddot{\mathbf{x}}_{2d} + \mathbf{C}_{10}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\dot{\mathbf{x}}_{2d} \\ & + \mathbf{G}_{10}^*(\mathbf{x}_{2d}) \end{aligned} \tag{20}$$

guarantees the convergence of the force error to zero.

Proof: The proof is straightforward by substituting (20) into (18). We get the

$$\tilde{\lambda} + \mathbf{K}_\lambda \int \tilde{\lambda} dt = -\mathbf{J}_1^{*-T}\delta_1^* \tag{21}$$

and then $\tilde{\lambda}_{t\to\infty} = 0$ since $\delta_1^*$is bounded.

According to Assumption 3, a robust control loop is required only for the uncertain reduced dynamics (19).

IV. ROBUST OPTIMAL CONTROL FOR THE REDUCED DYNAMICS

The motion equations of the reduced dynamics of the constrained manipulator with external disturbances can be expressed as:

$$\mathbf{M}_{20}^*(\mathbf{x}_2)\ddot{\mathbf{x}}_2 + \mathbf{C}_{20}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 + \mathbf{G}_{20}^*(\mathbf{x}_2) = \mathbf{u}_2^* + \delta_2^* \tag{22}$$

Two key properties related to dynamics characterize dynamics:

Property 2: For all $\mathbf{x}_2 \in \mathbb{R}^{n-m}$ the matrix $\mathbf{M}_{20}^*(\mathbf{x}_2)$ is symmetric and uniformly positive definite.

Property 3: A suitable definition of $\mathbf{C}_{20}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2)$ makes the matrix $\dot{\mathbf{M}}_{20}^* - 2\mathbf{C}_{20}^*$ skew-symmetric, so that

$$\xi^\top(\dot{\mathbf{M}}_{20}^* - 2\mathbf{C}_{20}^*)\xi = 0, \ \forall\xi \in \mathbb{R}^{n-m}.$$

The desired reference trajectory to be followed is assumed to be bounded functions of time in terms of the generalized positions $\mathbf{x}_2^d \in C^2$ where C^2 is the class of functions at least twice continuously differentiable.

Define the state tracking error as

$$\mathbf{e} := \begin{bmatrix} \dot{\tilde{\mathbf{x}}}_2 \\ \tilde{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_2^d \\ \mathbf{x}_2 - \mathbf{x}_2^d \end{bmatrix} \tag{23}$$

Then the tracking problem of the generalized position $\mathbf{x}_2$and velocity is reduced to the regulation problem of the state error **e**. Combining (22) and (23), the dynamic equation for the state tracking error **e** is obtained as

$$\begin{aligned}\dot{\mathbf{e}} &= \begin{bmatrix} \ddot{\tilde{\mathbf{x}}}_2 \\ \dot{\tilde{\mathbf{x}}}_2 \end{bmatrix} = \mathbf{A}(\mathbf{x}_2, \dot{\mathbf{x}}_2)\mathbf{e} + \mathbf{B}_0(\ddot{\mathbf{x}}_2^d, \dot{\mathbf{x}}_2^d, \dot{\mathbf{x}}_2, \mathbf{x}_2) \\ &\quad + \mathbf{B}\mathbf{M}_{20}^{*-1}(\mathbf{x}_2)(\mathbf{u}_2^* + \delta_2^*) \end{aligned} \tag{24}$$

where

$$\mathbf{A}(\mathbf{x}_2, \dot{\mathbf{x}}_2) = \begin{bmatrix} -\mathbf{M}_{20}^{*-1}(\mathbf{x}_2)\mathbf{C}_{20}^*(\mathbf{x_2}, \dot{\mathbf{x}}_2) & \mathbf{0}_{(n-m)\times(n-m)} \\ \mathbf{I}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix},$$

$$\mathbf{B}_0(\ddot{\mathbf{x}}_2^d, \dot{\mathbf{x}}_2^d, \dot{\mathbf{x}}_2, \mathbf{x}_2) = \begin{bmatrix} -\ddot{\mathbf{x}}_2^{\mathbf{d}} - \mathbf{M_{20}^{*-1}}(\mathbf{G_{20}^*}(\mathbf{x_2}) + \mathbf{C_{20}^*}(\mathbf{x_2}, \dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2^{\mathbf{d}}) \\ \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_{(n-m)\times(n-m)} \\ \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}.$$

Define the filtered link tracking error $r(t)$, [14] as

$$r(t) = \alpha\dot{\tilde{\mathbf{x}}}_2(t) + \mathbf{\Gamma}\tilde{\mathbf{x}}_2(t) \tag{25}$$

for some constant scale $\alpha > 0$and constant positive matrix $\mathbf{\Gamma} \in \mathbb{R}^{(n-m)\times(n-m)}$ which should be adequately determined.

Define

$$\begin{aligned}\mathbf{T} &= \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{I}_{(n-m)\times(n-m)} \end{bmatrix} \\ &= \begin{bmatrix} \alpha\mathbf{I}_{(n-m)\times(n-m)} & \mathbf{\Gamma} \\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{I}_{(n-m)\times(n-m)} \end{bmatrix} \end{aligned} \tag{26}$$

then the error dynamics equation can be modified as the compact form

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{T}^{-1}\begin{bmatrix} \dot{r}(t) \\ \dot{\tilde{\mathbf{x}}}_2(t) \end{bmatrix} \\ &= \mathbf{A}_T(\mathbf{e}, t)\mathbf{e} + \mathbf{B}_T(\mathbf{e}, t)\mathbf{T}_{11}(-f(\mathbf{e}, t) + \mathbf{u}_2^*) + \mathbf{B}_T(\mathbf{e}, t)\mathbf{w} \end{aligned} \tag{27}$$

where

$$\begin{aligned} &\mathbf{A}_T(\mathbf{e}, t) = \\ &\mathbf{T}^{-1}\begin{bmatrix} -\mathbf{M}_{20}^{*-1}(\mathbf{x}_2)\mathbf{C}_{20}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2) & \mathbf{0}_{(n-m)\times(n-m)} \\ \mathbf{T}_{11}^{-1} & -\mathbf{T}_{11}^{-1}\mathbf{T}_{12} \end{bmatrix}\mathbf{T}, \\ &\mathbf{B}_T(\mathbf{e}, t) = \mathbf{T}^{-1}\begin{bmatrix} \mathbf{I}_{(n-m)\times(n-m)} \\ \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}\mathbf{M}_{20}^{*-1}(\mathbf{x}_2) \\ &f(\mathbf{e}, t) = \mathbf{M}_{20}^*(\mathbf{x}_2)(\ddot{\mathbf{x}}_2^d - \mathbf{T}_{11}^{-1}\mathbf{T}_{12}\dot{\tilde{\mathbf{x}}}_2) \\ &+ \mathbf{C}_{20}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)(\dot{\mathbf{x}}_2^d - \mathbf{T}_{11}^{-1}\mathbf{T}_{12}\tilde{\mathbf{x}}_2) + \mathbf{G}(\mathbf{x}_2), \\ &\mathbf{w} = \alpha\delta_2^*. \end{aligned}$$

Assumption 4: For any $\tilde{\mathbf{e}} \in \mathbb{R}^{2(n-m)}, \mathbf{A}_T(\tilde{\mathbf{e}}, t), \mathbf{B}_T(\mathbf{e}, t)$ and **w** are Lebesque continuous functions, and $\forall t \in \mathbb{R}, \mathbf{A}_T(0, t) = 0$

Assumption 5: For any$(\tilde{\mathbf{e}}, t) \in \mathbb{R}^{2(n-m)} \times \mathbb{R}$, the Euclidian norm of w is bounded by some known continuous function, i.e.

$$\|\mathbf{w}\| \leq \rho(\mathbf{e}, t) \tag{28}$$

where $\rho(\tilde{\mathbf{e}}, t)$is a Lebesque continuous function.

Assumption 6: For a given set $E \subset \mathbb{R}^{2(n-m)}$ and for $[a, b] \subset \mathbb{R}$, there exists a Lebesque integral function $m_i(\cdot) : [a, b] \rightarrow \mathbb{R},\ i = 1, 2$, such that for any $(e, t) \in E \times [a, b]$

$$\begin{aligned} &\|\mathbf{A}_T(\mathbf{e}, t)\mathbf{e}\| \leq m_1(t) \\ &\|\mathbf{B}_T(\mathbf{e}, t)\mathbf{e}\|\, \rho(\mathbf{e}, t) \leq m_2(t) \end{aligned} \tag{29}$$

Assumption 7: The origin is an asymptotic stable equilibrium for the nominal system $\dot{\mathbf{e}} = \mathbf{A}_T(\mathbf{e}, t)\mathbf{e}$. In particular, there exist a positive definite function $V(\cdot) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^+$ and continuous decreasing scalar functions $\gamma_i(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+,\ i = 1, 2, 3$ satisfying

$$\gamma_i(0) = 0,\ i = 1, 2, 3 \tag{30}$$

$$\lim_{r\rightarrow\infty} \gamma_i(r) = \infty, i = 1, 2. \tag{31}$$

such that for any $(\tilde{e}, t) \in \mathbb{R}^{2(n-m)} \times \mathbb{R}$

$$\gamma_1(\|\mathbf{e}\|) \leq V(\mathbf{e}, t) \leq \gamma_2(\|\mathbf{e}\|) \tag{32}$$

$$\frac{\partial V(\mathbf{e}, t)}{\partial t} + \nabla_e^\top V(\mathbf{e}, t)A_T(\mathbf{e}, t)\mathbf{e} \leq -\gamma_3(\|\mathbf{e}\|) \tag{33}$$

where $V(\cdot)$ is a Lyapunov candidate function for the nominal system

$$\dot{\mathbf{e}} = \mathbf{A}_T(\mathbf{e}, t)\mathbf{e} + \mathbf{B}_T(\mathbf{e}, t)\mathbf{T}_{11}(-f(\mathbf{e}, t) + \mathbf{u}_2^*) \tag{34}$$

The objective of the tracking design for nominal system involves designing an optimal control whose effect is to minimize the control effort to the input of the system. The nonlinear optimal control which should possess some physical meanings must be concerned in the quadratic performance criterion. One of the noble contributions in the work of Johansson [14] is that a selective applied torque is applied as

$$\tau_2^* = \mathbf{M}_{20}^*(\mathbf{x}_2)\mathbf{T}_1\dot{\mathbf{e}} + \mathbf{C}_{20}^*(\mathbf{x}_2, \dot{\mathbf{x}}_2)\mathbf{T}_1\mathbf{e} \tag{35}$$

which only affects the kinetic energy of the system since, during the process of motion, it may be unnecessary to consider the consumption due to the potential energy and to optimize gravitational-dependant torques or forces. The relation between the control variable u and the applied torque τ in (34)

$$\mathbf{u}_2^* = f(\mathbf{e}, t) + \mathbf{T}_{11}^{-1}\tau_2^* \tag{36}$$

Then the state tracking error equation, driven by the selective applied torque u, for the uncertain system can be written as

$$\dot{\mathbf{e}} = \mathbf{A}_T(\mathbf{e}, t)\mathbf{e} + \mathbf{B}_T(\mathbf{e}, t)\tau_2^* + \mathbf{B}_T(\mathbf{e}, t)w \tag{37}$$

The control problem goal can be stated as: Given the uncertain dynamical equation (27) of an a robotic system satisfying assumptions $A4$-$A7$ design a robust optimal control to:

1) achieve global asymptotic stabilization of the nominal system using an optimal control approach,
2) ensure global asymptotic stability for the uncertain reduced dynamical system.

A. Optimal Control for the Nominal Reduced Dynamics

We consider in this section the nominal system

$$\dot{\mathbf{e}} = \mathbf{A}_T(\mathbf{e},t)\mathbf{e} + \mathbf{B}_T(\mathbf{e},t)\tau_2^* \tag{38}$$

We look here for an optimal control law τ_2^* that minimizes the quadratic index performance

$$J(\mathbf{e},\tau_2^*) = \frac{1}{2}\int_{t_0}^{\infty}\left(\mathbf{e}^\top(t)\mathbf{Q}\mathbf{e}(t) + \tau_2^{*T}(t)\mathbf{R}\tau_2^*(t)\right)dt \tag{39}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are positive definite matrices to be chosen. Johansson in [19] shows that the optimal solution of (39) is given by

$$\tau_2^*(t) = -\mathbf{R}^{-1}\mathbf{B}_T^\top(\mathbf{e},t)\mathbf{P}(\mathbf{e},t)\mathbf{e} \tag{40}$$

where $\mathbf{P}(\tilde{\mathbf{e}},t)$ is the solution of the nonlinear Ricatti equation

$$\dot{\mathbf{P}} + \mathbf{P}\mathbf{A}_T + \mathbf{A}_T^\top\mathbf{P} - \mathbf{P}\mathbf{B}_T\mathbf{R}^{-1}\mathbf{B}_T^\top\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{41}$$

considering Property 2 related to the dynamics of the manipulator, one can show that $P(\tilde{e},t)$ can be explicitly considered as

$$\mathbf{P}(\mathbf{e},t) = \mathbf{T}^\top\begin{bmatrix}\mathbf{M}_{20}^*(\mathbf{x}_2) & \mathbf{0}_{(n-m)\times(n-m)}\\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{K}_{(n-m)\times(n-m)}\end{bmatrix}\mathbf{T} \tag{42}$$

where $\mathbf{K}$ is a positive $(n-m)\times(n-m)$ matrix.

The nonlinear Ricatti equation can be simplified as

$$\mathbf{P}\mathbf{A}_T + \mathbf{A}_T^\top\mathbf{P} = \begin{bmatrix}\mathbf{0}_{(n-m)\times(n-m)} & \mathbf{K}_{(n-m)\times(n-m)}\\ \mathbf{K}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)}\end{bmatrix} + \mathbf{T}^\top\begin{bmatrix}-\dot{\mathbf{M}}_{20}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2) & \mathbf{0}_{(n-m)\times(n-m)}\\ \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)}\end{bmatrix}\mathbf{T}. \tag{43}$$

and

$$\mathbf{B}_T^\top\mathbf{P} = \mathbf{B}^\top\mathbf{T} \tag{44}$$

From equations (40) and (44), $u*$ can be rewritten as

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^\top(\mathbf{e},t)\mathbf{T}\mathbf{e} \tag{45}$$

and the nonlinear Ricatti equation becomes an algebraic equation as

$$\begin{bmatrix}0 & \mathbf{K}\\ \mathbf{K} & 0\end{bmatrix} + \mathbf{Q} - \mathbf{T}^\top\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{T} = \mathbf{0} \tag{46}$$

Theorem 1 Consider the nominal error manipulator dynamics (34), the optimal control law

$$\tau_2^*(t) = -\mathbf{R}^{-1}\mathbf{B}^\top(\mathbf{e},t)\mathbf{T}\mathbf{e}$$

stabilizes asymptotically the system (38), where $\mathbf{K}$>0 and a nonsingular $\mathbf{T}$ solving the algebraic matrix equation (46).

Proof: Consider the Lyapunov candidate function

$$V(\mathbf{e},t) = \frac{1}{2}\mathbf{e}^\top\mathbf{P}\mathbf{e}.$$

The quadratic function $V(\mathbf{e},t)$ is a suitable Lyapunov function candidate because it is positive radially growing with $\|\mathbf{e}\|$. It is continuous and has a unique minimum at the origin of the error space. It remains to show that $\dot{V} < 0$ for all $\|\mathbf{e}\| \neq 0$. It can be shown that for τ_2^* as in (45), the function V satisfies the partial differential equation

$$-\frac{dV(\mathbf{e},t)}{dt} = \left(\frac{\partial V(\mathbf{e},t)}{\partial \mathbf{e}}\right)^\top \dot{\mathbf{e}}\Big|_{\tau_2^*} + \frac{\partial V(\mathbf{e},t)}{\partial t} \tag{47}$$

But

$$\frac{\partial V(\mathbf{e},t)}{\partial t} = -L(\mathbf{e},\tau_2^*) - \left(\frac{\partial V(\mathbf{e},t)}{\partial \mathbf{e}}\right)^\top \dot{\mathbf{e}}\Big|\tau_2^* \tag{48}$$

where $L(\cdot,\cdot)$ is the Lagrangian as

$$L(\mathbf{e},\tau_2^*) = \tfrac{1}{2}\mathbf{e}^\top\mathbf{Q}\mathbf{e} + \tfrac{1}{2}\tau_2^{*T}\mathbf{R}\tau_2^* \tag{49}$$

then

$$\frac{dV(\mathbf{e},t)}{dt} = -\tfrac{1}{2}\mathbf{e}^\top\mathbf{Q}\mathbf{e} - \tfrac{1}{2}\tau_2^{*T}\mathbf{R}\tau_2^* \tag{50}$$

and finally

$$\frac{dV(\mathbf{e},t)}{dt} = -\tfrac{1}{2}\mathbf{e}^\top(\mathbf{T}^\top\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{T} + \mathbf{Q})\mathbf{e} < 0, \quad \forall t > 0,\ \mathbf{e} \neq 0. \tag{51}$$

B. Robust Optimal Stabilization of the Reduced Dynamics

In this section we design a robust control law for the asymptotic stabilization of the uncertain dynamical system

The proposed control law is as

$$\tau_2^* = \tau_{21}^* + \tau_{22}^* \tag{52}$$

where τ_{21}^* is the optimal control developed in the last subsection and which is given by

$$\tau_{21}^* = -\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{T}\mathbf{e} \tag{53}$$

and τ_{22}^* is the robust control component to be designed to overcome the effect of the perturbation vector w satisfying (28).

Substituting (53) in (37), one get

$$\dot{\mathbf{e}} = f(\mathbf{e},t) + \mathbf{B}_T(\mathbf{e},t)(w + \tau_{21}^* + \tau_{22}^*) \tag{54}$$

where

$$f(\mathbf{e},t) = (\mathbf{A}_T(\mathbf{e},t) - \mathbf{B}_T(\mathbf{e},t)\mathbf{K})\,\mathbf{e} \tag{55}$$

and

$$\mathbf{K} = -\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{T}. \tag{56}$$

Consider the Lyapunov function candidate

$$V(\mathbf{e},t) = \tfrac{1}{2}\mathbf{e}^\top\mathbf{T}^\top\begin{bmatrix}\mathbf{M}_{20}^*(\mathbf{x}_2) & \mathbf{0}\\ 0 & \mathbf{K}\end{bmatrix}\mathbf{T}\mathbf{e}$$

The derivative of V with respect to e is given by

$$\begin{aligned}\nabla_e V &= \mathbf{T}^\top \begin{bmatrix} \mathbf{M}_{20}^*(\mathbf{e},t) & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \mathbf{Te} \\ &+\frac{1}{2}\mathbf{e}^\top \mathbf{T}^\top \begin{bmatrix} \nabla_e \mathbf{M}_{20}^*(\mathbf{e},t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Te}. \end{aligned} \quad (57)$$

Define

$$\mu(\mathbf{e},t) := \mathbf{B}^\top(\mathbf{e},t)\nabla_e(\mathbf{e},t)\rho(\mathbf{e},t). \quad (58)$$

Reporting (57) in (58), we get

$$\mu(\mathbf{e},t) = \left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_{20}^*(\mathbf{x}_2) & 0 \end{bmatrix} (\mathbf{T}^{-1})^\top \mathbf{e}^\top \times \mathbf{T} \begin{bmatrix} \nabla_e \mathbf{M}_{20}^*(\mathbf{e},t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \mathbf{Te}\rho(\mathbf{e},t),$$

it is easy to verify that

$$\gamma_1(\|\mathbf{e}\|) \le V(\mathbf{e},t) \le \gamma_2(\|\mathbf{e}\|) \quad (59)$$

and that

$$\nabla_t V(\mathbf{e},t) + \nabla_e^\top V(\mathbf{e},t)\mathbf{A}_T(\mathbf{e},t)\mathbf{e} \le -\gamma_3(\|\mathbf{e}\|) \quad (60)$$

where the scalar functions $\gamma_i(\cdot)$ are given by

$$\begin{aligned}\gamma_1(\|\mathbf{e}\|) &= \lambda_1 \|\mathbf{e}\|^2 \\ \gamma_2(\|\mathbf{e}\|) &= \lambda_2 \|\mathbf{e}\|^2 \\ \gamma_3(\|\mathbf{e}\|) &= \lambda_3 \|\mathbf{e}\|^2\end{aligned}$$

where

$$\begin{aligned}\lambda_1 &:= \lambda_{\min}\left\{\mathbf{T}^\top \begin{bmatrix} \mathbf{M}_{20}^*(\mathbf{x}_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \mathbf{T}\right\}, \\ \lambda_2 &:= \lambda_{\max}\left\{\mathbf{T}^\top \begin{bmatrix} \mathbf{M}_{20}^*(\mathbf{x}_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \mathbf{T}\right\} \\ \lambda_3 &:= \min\{\lambda_{\min}(\mathbf{K}), \lambda_{\min}(\mathbf{T}_{11}^\top \mathbf{M}_{20}^*(\mathbf{x}_2)\mathbf{T}_{11}), \\ &\quad \lambda_{\min}(\mathbf{T}_{12}^\top \mathbf{M}_{20}^*(\mathbf{x}_2)\mathbf{T}_{12}), \lambda_{\min}(\mathbf{T}_{12}\mathbf{M}_{20}^*(\mathbf{x}_2)\mathbf{T}_{11})\},\end{aligned}$$

$\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote respectively the minimum and maximum eigenvalues of $(\cdot)$.

Considering (58), Corless and Leitmann [12] proved that the following discontinuous state feedback function

$$\tau_{22}^*(\mathbf{e},t) = \begin{cases} -\frac{\mu(\mathbf{e},t)}{\|\mu(\mathbf{e},t)\|}\rho(\mathbf{e},t), \text{ if } \|\mu(\mathbf{e},t)\| > \varepsilon, \\ -\frac{\mu(\mathbf{e},t)}{\varepsilon}\rho(\mathbf{e},t), \text{ if } \|\mu(\mathbf{e},t)\| \le \varepsilon, \end{cases} \quad (61)$$

can be used ultimately stabilize the nonlinear uncertain dynamical system

$$\dot{\mathbf{e}} = f(\mathbf{e},t) + \mathbf{B}_T(\mathbf{e},t)(w + \tau_{21}^* + \tau_{22}^*)$$

where $\varepsilon > 0$. One can notice here the discontinuity of the robust control law when $\varepsilon \to 0$.

To overcome the discontinuity problem related to the control (61), define the new class of control as

$$\tau_{22}^*(\mathbf{e},t) = \begin{cases} -\frac{\mu(\mathbf{e},t)}{\|\mu(\mathbf{e},t)\|}\rho(\mathbf{e},t), \text{ if } \|\mu(\mathbf{e},t)\| > \varepsilon\varphi(t), \\ -\frac{\mu(\mathbf{e},t)}{\varepsilon\varphi(t)}\rho(\mathbf{e},t), \text{ if } \|\mu(\mathbf{e},t)\| \le \varepsilon\varphi(t), \end{cases} \quad (62)$$

where $\varphi(t)$ is class of uniformly continuous functions such that $0 < \varphi(t) \le 1$ and $\omega(t) := \int \varphi(t)dt$ satisfying $\omega(t) \le 0$.

Lemma 1

The control law (62) is continuous and stabilizes asymptotically the uncertain dynamical system if there exists a Lyapunov function candidate $V(\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$ such that

1) $\gamma_1(\|\mathbf{e}\|) \le V(\mathbf{e},t) \le \gamma_2(\|\mathbf{e}\|), \forall(\mathbf{e},t) \in \mathbb{R}^n \times \mathbb{R}$,
2) $\dot{V}(\mathbf{e},t) \le -\gamma(\|\mathbf{e}\|) + \gamma(\eta)\varphi(t), \forall(\mathbf{e},t) \in \mathbb{R}^n \times \mathbb{R}$,

where η is a positive constant, $\lim_{r\to\infty}\gamma_i(r) = \infty$, $i = 1, 2$ and γ a positive definite function, such that $\gamma(0) = 0$. If $\gamma, \varphi(t)$, and $\omega(t)$ verify $\gamma(y) - \gamma(\eta) > 0$ for any $y > \eta$, $0 < \varphi(t) \le 1$, $\omega(t) \le 0$ and if η is a positive definite function in $\mathbf{e}_o$, then every solution $\mathbf{e}(t;\mathbf{e}_0,t_0) : [t_0,\infty) \to \mathbb{R}^n$ of the system is globally asymptotically stable equilibrium.

The proof of this Lemma 1 can be found in [15].

Theorem 2 Consider the uncertain dynamical system (34), the control law

$$\tau_2^* = \tau_{21}^* + \tau_{22}^* \quad (63)$$

where

$$\tau_{21}^* = -\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{Te}$$

and

$$\tau_{22}^* = \begin{cases} -\frac{\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \end{bmatrix}\mathbf{e}}{\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \end{bmatrix}\mathbf{e}}\rho(\mathbf{e},t) \\ \text{if } \left\|\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \end{bmatrix}\mathbf{e}\rho(\mathbf{e},t)\right\| > \varepsilon\varphi(t) \\ -\frac{\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \end{bmatrix}\mathbf{e}}{\varepsilon\varphi(t)}\rho(\mathbf{e},t) \\ \text{if } \left\|\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \end{bmatrix}\mathbf{e}\rho(\mathbf{e},t)\right\| \le \varepsilon\varphi(t) \end{cases} \quad (64)$$

stabilizes asymptotically and optimally system trajectories of (64).

Proof .

To prove asymptotic stability of the uncertain system, we should prove that

i every solution $\mathbf{e}(t;\mathbf{e}_0,t_0)$is stable,
ii

$$\lim_{t\to\infty}\mathbf{e}(t;\mathbf{e}_0,t_0) = 0.$$

i. Consider the Lyapunov candidate function

$$V(\mathbf{e},t) = \tfrac{1}{2}\mathbf{e}^\top \mathbf{T}^\top \begin{bmatrix} \mathbf{M}_{20}^*(\mathbf{x}_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \mathbf{Te}$$

the total time derivative of V is given by

$$\begin{aligned}\frac{dV(\mathbf{e},t)}{dt} &= \frac{\partial V(\mathbf{e},t)}{\partial t} + \left(\frac{\partial V(\mathbf{e},t)}{\partial t}\right)^\top \dot{\mathbf{e}} \\ &= -\frac{1}{2}\mathbf{e}^\top(\mathbf{Q} + \mathbf{T}^\top \mathbf{BR}^{-1}\mathbf{B}^\top \mathbf{T})\mathbf{e} \\ &\quad +\mathbf{e}^\top \mathbf{T}^\top \mathbf{B}(w + \tau_{22}^*)\end{aligned}$$

there exists a continuous positive definite function γ such that

$$\frac{dV(\mathbf{e},t)}{dt} \le -\gamma(\|\mathbf{e}\|) + \gamma(\eta)\varphi(t)$$

where η is a positive scalar constant and

$$\begin{aligned}\gamma(\|\mathbf{e}\|) &= \lambda_{\max}(\mathbf{Q}+\mathbf{T}^\top\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{T})\,\|\mathbf{e}\| \\ &= V(\mathbf{e}_0,t)+\int_{t_0}^{t}\dot{V}(\mathbf{e},\tau)d\tau \\ &\le \gamma_2(\|\mathbf{e}_0\|)-\int_{t_0}^{t}\gamma(\|\mathbf{x}(\tau)\|)d\tau+\gamma(\eta)\int_{t_0}^{t}\varphi(\tau)d\tau \\ &\le \gamma_2(\|\mathbf{e}_0\|) \\ &\quad -\int_{t_0}^{t}\gamma(\|\mathbf{x}(\tau)\|)d\tau+\gamma(\eta)\left[\omega(\mathbf{e})-\omega(\mathbf{e}_0)\right] \quad (65) \\ &\le \gamma_2(\|\mathbf{e}_0\|)-\int_{t_0}^{t}\gamma(\|\mathbf{x}(\tau)\|)d\tau-\gamma(\eta)\omega(\mathbf{e}_0) \quad (66)\end{aligned}$$

then for every $\varepsilon > 0$, we have $\|\mathbf{e}\| \le \varepsilon$ if and only if

$$\gamma_2(\|\mathbf{e}_0\|)+\gamma(\eta)\,|\omega(\mathbf{e}_0)| \le \gamma_1(\varepsilon) \quad (67)$$

Since η is a positive definite function of $\|\mathbf{e}_0\|$, there exist $\beta(\varepsilon, t_0)$ so that (65) is verified for any $\|\mathbf{e}_0\| \le \beta$.
ii. from (65), we can write

$$0 \le \gamma_1(\|\tilde{\mathbf{e}}\|) \le \gamma_2(\|\mathbf{e}_0\|)-\int_{t_0}^{t}\gamma(\|\mathbf{x}(\tau)\|)d\tau-\gamma(\eta)\omega(\mathbf{e}_0)$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\gamma(\|\mathbf{x}(\tau)\|)d\tau \le \gamma(\|\tilde{\mathbf{e}}_0\|)+\gamma(\eta)\omega(\mathbf{e}_0) < \infty$$

Considering the continuity of the solution and the property that $\gamma(\cdot)$ is a continuous positive definite function, we then conclude that $\lim_{t\to\infty}\gamma(\|\mathbf{e}\|)=0$ and then $\lim_{t\to\infty}\|\mathbf{e}\|=0$.

V. A SIMULATION EXAMPLE

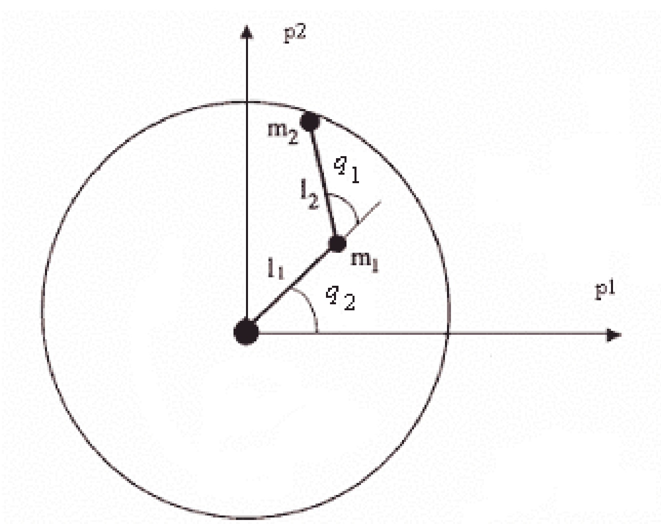

Fig. 1. A 2-DOF mechanical system with circle constraint task.

In this section we present a simulation example to illustrate the validity of the approach used to derive the robust control law presented in this paper. We consider a 2-DOF robot manipulator with a circular path constraint, Fig. 1. The original model of the system has the following components

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} b & b+c\cos(q_1) \\ b+c\cos(q_1) & a+b+2c\cos(q_1) \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} 0 & c\dot{q}_1\sin q_1 \\ -c(\dot{q}_1+\dot{q}_2)\sin q_1 & -c\dot{q}_1\sin q_1 \end{bmatrix}$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} cd\cos(q_1+q_2) \\ ad\cos(q_1)+cd\cos(q_1+q_2) \end{bmatrix}$$

a, b, c and d are functions of uncertain physical parameters of the mechanical system.

In the work space, the constraint in $p_1 - p_2$ plane is supposed to be a circle whose center coincides with the axis of rotation of the first link. This constraint equation is expressed as

$$\phi(\mathbf{p}) = p_1^2 + p_2^2 - r^2$$

where $\mathbf{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}^\top$

The transformation of the constraint equation from the workspace to the joint space yields to the following constraint equation

$$\Phi(\mathbf{q}) = l_1^2 + l_2^2 + 2l_1l_2\cos(q_1) - r^2 = 0$$

which has a unique solution q_{10} such that

$$q_{10} = \Omega(q_2) = \Omega(x_2) = \cos^{-1}\left(\frac{r^2-(l_1^2+l_2^2)}{2l_1l_2}\right)$$

The Jacobian of $\Phi(q)$ is then given by

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -2l_1l_2\sin(q_{10}) \\ 0 \end{bmatrix}$$

By applying the transformation of coordinates we obtain the following components

$$\mathbf{M}^*(\mathbf{x}_2) = \begin{bmatrix} b & b+c\cos(q_{10}) \\ b+c\cos(q_{10}) & a+b+2c\cos(q_{10}) \end{bmatrix}$$

$$\mathbf{C}^*(\mathbf{x}_2,\dot{\mathbf{x}}_2) = \begin{bmatrix} 0 & c\dot{x}_2\sin(q_{10}) \\ -c\dot{x}_2\sin(q_{10}) & 0 \end{bmatrix}$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} cd\cos(q_{10}+x_2) \\ ad\cos(q_{10})+cd\cos(q_{10}+x_2) \end{bmatrix}$$

The nominal values of parameters are

$$a_0 = 0.8, b_0 = 0.2, c_0 = 0.4$$

$$d = g = 9.8ms^{-2}$$

The uncertainty bounds of these parameters have the following values

$$\begin{aligned} 0.3 \le a \le 1.2 \\ 0.1 \le b \le 0.4 \\ 0.2 \le c \le 0.8 \end{aligned}$$

The external torque perturbation vector is supposed to be $\mathbf{p} = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ N·m.

The continuous function $\varphi(t)$ in (62) is taken as $\varphi(t) = e^{-t}$, ε_0 is set at a value of 0.1.

We consider the following continuously derivable desired trajectories $q_{2d}(t) = -\frac{\pi}{2} + 0.9[1 - \cos(1.26t)]$ and a desired contact force of $f_d = 10$N.

Simulation results are shown in Figs. 2 to 6. It is clear from Figs. 3 and 4 that the robust asymptotic stabilization is achieved for the uncertain system with a zero tracking position and force errors by using the control laws (30) and (63)

VI. CONCLUSIONS

A nonlinear mixed optimal/robust control has been proposed for optimal and robust tracking performance design of constrained robotic systems under a class of parametric uncertainties and external disturbances. A PI-like controller was used to ensure the convergence of the force error. The nonlinear time-varying mixed optimal/robust control tracking problem design must first solve a coupled nonlinear Ricatti equation. In order to avoid the difficulty of solving this Ricatti equation, an adequate state transformation and the property of skew symmetry of robotic systems have been employed to obtain an equivalent algebraic equation. A class of nonlinear time varying continuous feedback functions was introduced to solve the problem of robust control.

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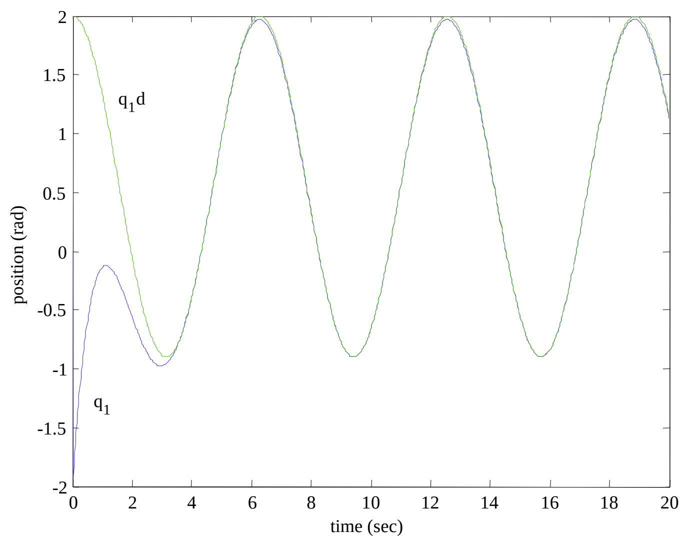

Fig. 2. Trajectory and desired trajectory for the uncertain constrained system with control law (63).

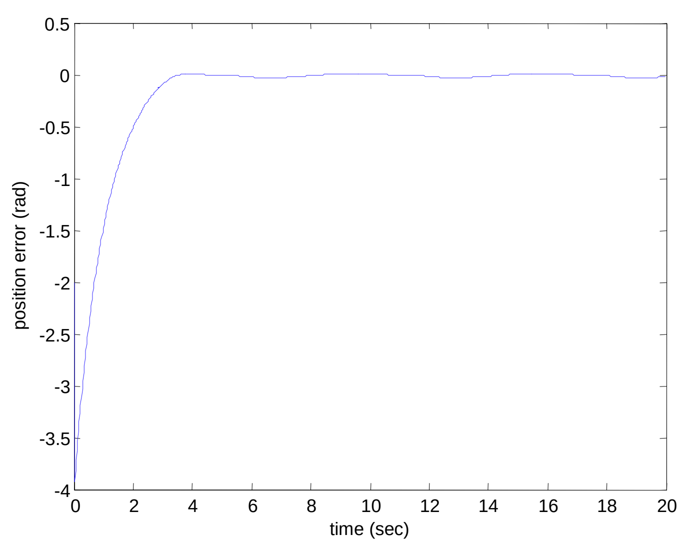

Fig. 3. Position error for the with the system control (63).

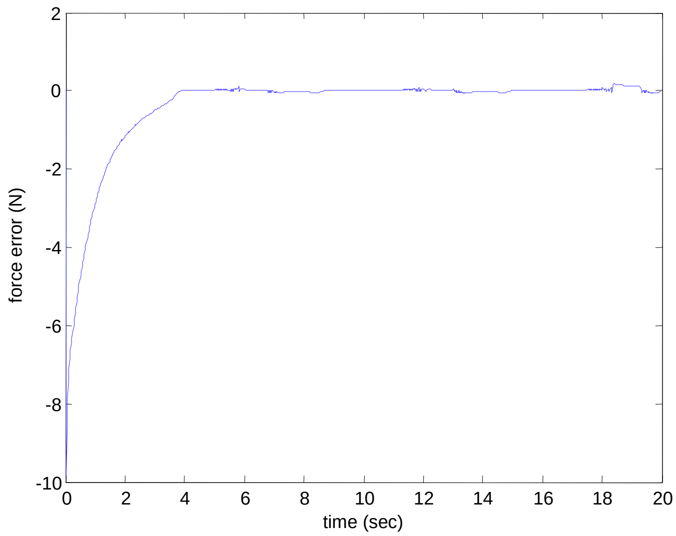

Fig. 4. Force error for the uncertain system with control law (30).

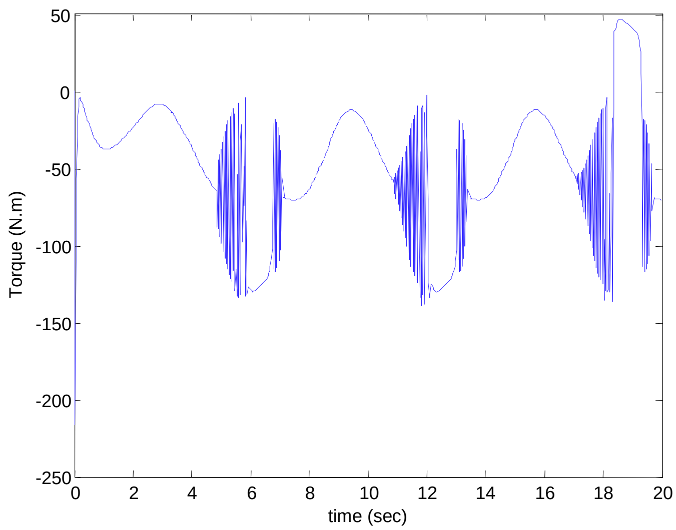

Fig. 5. The applied torque to the first joint.

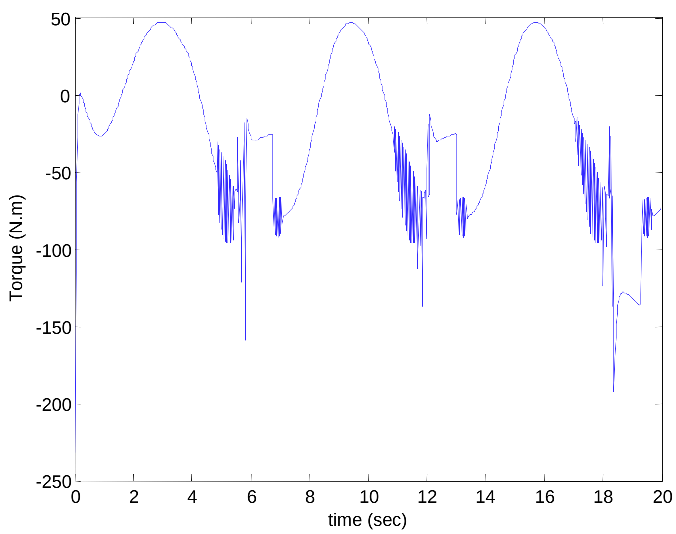

Fig. 6. The applied torque to the second joint.