Unifying approaches and removing unrealistic assumptions in Shape From Shading: Mathematics can help

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Abstract. This article proposes a solution of the Lambertian Shape From Shading (SFS) problem by designing a *new mathematical framework* based on the notion of viscosity solutions. The power of our approach is twofolds: 1) it defines a notion of *weak* solutions (in the viscosity sense) which *does not necessarily require boundary data.* Note that, in the previous SFS work of Rouy et al. [23, 15], Falcone et al. [8], Prados et al. [22, 20], the characterization of a viscosity solution and its computation require the knowledge of its values on the boundary of the image. This was quite unrealistic because in practice such values are not known. 2) it *unifies* the work of Rouy et al. [23, 15], Falcone et al. [8], Prados et al. [22, 20], based on the notion of viscosity solutions and the work of Dupuis and Oliensis [6] dealing with classical (C^1) solutions. Also, we *generalize* their work to the "perspective SFS" problem recently introduced by Prados and Faugeras [20].
Moreover this article introduces a *"generic" formulation of the SFS problem.* This "generic" formulation summarizes various (classical) formulations of the Lambertian SFS problem. In particular it *unifies the orthographic and the perspective SFS problems.* This "generic" formulation significantly simplifies the formalism of the problem. Thanks to this generic formulation, a *single algorithm* can be used to compute numerical solutions of all these previous SFS formulations.
Finally we propose two algorithms which provide numerical approximations of the new weak solutions of the *"generic SFS" problem.* These provably convergent algorithms are quite *robust* and do not necessarily require boundary data.

1 Introduction

The application of the theory of Partial Differential Equations (PDEs) to the Shape from Shading (SFS) problem has been hampered by several types of difficulties. The first type arises from the kind of modelling that is used: orthographic cameras looking at Lambertian objects with a single point light source at infinity is the set of usual assumptions [29, 10]. The second type is mathematical: characterizing the solution(s) of the corresponding PDE has turned out to be a very difficult problem; boundary conditions are assumed to be known, say at image boundary, in contradiction with real practice [23, 22, 8]. The third type is algorithmic: assuming that existence has been proved, coming up with provably convergent numerical schemes has turned out to be quite involved [7].

Our approach is therefore based upon the interaction of the following three areas:

1. Mathematics: We use and "extend" the notion of viscosity solutions to solve such basic problems as the existence and uniqueness of a solution or the characterization of all solutions when uniqueness does not hold.
2. Algorithmic: In [2], Barles and Souganidis propose a large class of approximation schemes (called monotonous) of these solutions. Inspired by their work, we build such schemes for the SFS equations from which we obtain algorithms whose properties we can analyze in detail (stability, convergence, accuracy). This results in provably correct code within a set of well-defined assumptions.
3. Modeling: The classical theory of viscosity solutions (used until now for solving the SFS problem [23, 15, 22, 20, 8]) is not well-adapted to the natural constraints of the SFS problem. In particular it requires that boundary conditions be given, e.g. at the image boundary, and creates undesirable folds (see section 3). In order to be able to get rid of this constraint, we have adapted the notion of viscosity solutions.

Our contributions are first in the area of Mathematics: we adapt the notion of singular viscosity solutions (recently developed by Camilli and Siconolfi [3, 4]) for obtaining a "new" class of viscosity solutions which is really more suitable to the SFS problem than the previous ones. This mathematical framework is very general and allows to improve and unify the work of [23, 15, 6, 22, 20, 8]. Directly connected to the area of modeling, thanks to the introduction of this framework, we are able to relax the very constraining assumption that boundary conditions are known. Concerning the area of modeling, we extend the work of [20]: considering a pinhole camera, we allow the light source to be either at infinity or approximately at the optical center, as in the case of a flash. We also show that the orthographic and pinhole camera SFS equations are special cases of a general equation, thereby simplifying the formalization of the problem. Our contributions are also algorithmic: we propose two provably convergent approximation schemes for our "generic" SFS equation. Moreover, one of the algorithms we propose seems to be the most efficient iterative algorithms of the SFS literature. The article is written in a non mathematical style. The reader interested in the proofs is referred to [19, 21].

2 A unification of the "perspective" and "orthographic SFS"

We deal with Lambertian scenes and suppose that the albedo is constant and equal to 1. The scene is represented by a surface S. Let Ω, the image, be the rectangular domain $]0, X[\times]0, Y[$. S can be explicitly parameterized by using a function defined on the closure $\overline{\Omega}$. The particular type of parametrization is irrelevant here but may vary according to the camera type (orthographic versus pinhole) and the position of light source (finite or infinite distance). We note I the image intensity, a function from $\overline{\Omega}$ into the closed interval $[0,1]$. The Lambertian hypothesis implies:

$$I(x) = \frac{\mathbf{n}(x) \cdot \mathbf{L}}{|\mathbf{n}(x)|}, \tag{1}$$

where $\mathbf{n}(x)$ is a normal vector to the surface S at the point $S(x)$ and $\mathbf{L}$ is the unit vector representing the light direction at this same point (the light source is assumed to be a point). Despite the notation, $\mathbf{L}$ can depend on $S(x)$, if the point source is at a finite distance from the scene.

2.1 "Orthographic SFS" with a point light source at infinity

This is the traditional setup for the SFS problem. We denote by $\mathbf{L} = (\alpha, \beta, \gamma)$ the unit vector representing the direction of the light source ($\gamma > 0$), $\mathbf{l} = (\alpha, \beta)$, and u the distance of the points in the scene to the camera. The SFS problem is then, given I and $\mathbf{L}$, to find a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying the brightness equation:

$$\forall x \in \Omega, \quad I(x) = (-\nabla u(x) \cdot \mathbf{l} + \gamma)/\sqrt{1 + |\nabla u(x)|^2},$$

In the SFS literature, this equation is rewritten in a variety of ways as $H(x,p) = 0$, where $p = \nabla u$:
1) In [23], Rouy and Tourin introduce $H_{R/T}(x,p) = I(x)\sqrt{1 + |p|^2} + p \cdot \mathbf{l} - \gamma$.
2) In [6], Dupuis and Oliensis consider

$$H_{D/O}(x,p) = I(x)\sqrt{1 + |p|^2 - 2p \cdot \mathbf{l}} + p \cdot \mathbf{l} - 1.$$

(use the change of variables: $\Psi(x_1, x_2, z) = (x_1, x_2, x_1\alpha + x_2\beta + z\gamma)$)
3) In the case where $\mathbf{L} = (0, 0, 1)$, Lions et al. [15] deal with:

$$H_{Eiko}(x,p) = |p| - \sqrt{\frac{1}{I(x)^2} - 1}. \qquad \text{(called the Eikonal equation)}$$

The function H is called the *Hamiltonian*.

2.2 "Perspective SFS" with a point light source at infinity

Few SFS approaches deal with the perspective projection problem. To our knowledge, only eight authors [17, 13, 9, 27, 28, 20, 26, 5] consider a pinhole camera model instead of an affine or orthographic model. Among these papers, only the work of Prados and Faugeras [20] proposes a formalism completely based on Partial Differential Equations (PDEs) and provides a rigourous mathematical study. The camera is characterized and represented by the retinal plane R and by the optical center as shown in figure 1. We note f the focal length. We assume that S can be explicitly parameterized by the depth modulation function u defined on $\overline{\Omega}$:

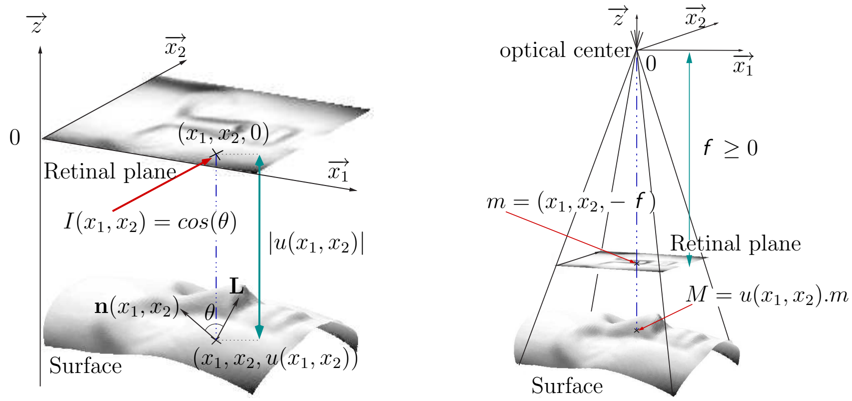
$$S = \left\{ u(x).(x, -f); \quad x \in \overline{\Omega} \right\},$$

and that the surface is visible (in front of the retinal plane) hence $u \geq 1$.
We also note $\mathbf{L} = (\alpha, \beta, \gamma)$ the unit vector representing the direction of the light source ($\gamma > 0$). Combining the expression of $\mathbf{n}(x)$ (easily obtained through

differential calculus) and the change of variables $v = ln(u)$, Prados and Faugeras [18, 20] obtain from the irradiance equation the following Hamiltonian:

$$H_{P/F}(x,p) = I(x)\sqrt{f^2|p|^2 + (x \cdot p + 1)^2} - (f\,\mathbf{l} + \gamma x) \cdot p - \gamma;$$

By using the change of variables $v(x) = \frac{\gamma}{f}[\gamma f - \mathbf{l} \cdot x]u(x)$, we obtain another Hamiltonian $H_{Pers}(x,p)$ which verifies more interesting properties (see [19]).



a) orthogonal projection b) perspective projection

Fig. 1. Images arising from an orthogonal (versus perspective) projection.

2.3 "Perspective SFS" with a single point light source located at the optical center

We present a new formulation of the "perspective SFS". This approximately models the situation encountered when we use a simple camera equiped with a flash and the scene is relatively far from the camera. In this case, we represent the scene by the surface S defined by $S = \left\{ \frac{f\ u(x)}{\sqrt{|x|^2 + f^2}}\,(x,\ f);\quad x \in \overline{\Omega} \right\}$. Using the same trick as in the previous section ($v = ln(u)$), we readily obtain the Hamiltonian:

$$H_F(x,p) = I(x)\sqrt{f^2|p|^2 + (p \cdot x)^2 + Q(x)^2} - Q(x),$$

where $Q(x) = \sqrt{f^2/(|x|^2 + f^2)}$. See [19] for more details.

2.4 A generic Hamiltonian

In [19], we prove that all the previous SFS Hamiltonians are special cases of the following "generic" Hamiltonian:

$$H_g(x,p) = \tilde{H}_g(x, A_x p + \overrightarrow{v_x}) + \overrightarrow{w_x} \cdot p + c_x,$$

with $\quad \tilde{H}_g(x,q) = \kappa_x \sqrt{|q|^2 + K_x^2}$,
$\kappa_x, K_x \geq 0$, $A_x = D_x\, R_x$, $D_x = \begin{pmatrix} \mu_x & 0 \\ 0 & \nu_x \end{pmatrix}$, R_x is the rotation matrix $\frac{1}{|x|}\begin{pmatrix} x_2 & -x_1 \\ x_1 & x_2 \end{pmatrix}$ if $x \neq 0$, $R_x = Id_2$ if $x = 0$, $\mu_x, \nu_x \neq 0$ ($\mu_x, \nu_x \in \mathbb{R}$), $\overrightarrow{v_x}, \overrightarrow{w_x} \in \mathbb{R}^2$ and $c_x \in \mathbb{R}$.
By using the Legendre transform, we rewrite this Hamiltonian as a "generic" Hamilton-Jacobi-Bellman (HJB) Hamiltonian:

$$H_g(x,p) = \sup_{a \in B_2(0,1)} \{-f_g(x,a) \cdot p - l_g(x,a)\}.$$

In [19], we detail the exact expressions of f_g and l_g. The HJB formulations of the Hamiltonians H_{Eiko}, $H_{D/O}$, $H_{R/T}$ and $H_{P/F}$, respectively given in [23, 6, 22, 20], are special cases of the above generic formulation; thereby ours is a generalization and a unification of these works. This generic formulation considerably simplifies the formalism of the problem. All theorems about the characterization and the approximation of the solution are now proved by using this generic SFS Hamiltonian. In particular, this formulation unifies the orthographic and perspective SFS problems. Also, from a practical point of view, a unique code can be used to numerically solve these two problems.

3 Weaknesses of the previous theoretical approaches

The notion of viscosity solutions was first used to solve SFS problems by Lions, Rouy and Tourin [23, 15] in the 90s. Their work was based upon the notion of *continuous* viscosity solution. The viscosity solutions are PDE solutions in a weak sense. In particular, they are not necessarily differentiable and can have edges. This notion allows to define a solution of a PDE which does not have classical solutions. For example, the equation

$$|\nabla u(x)| = 1 \text{ for all } x \text{ in }]0,1[\tag{2}$$

with $u(0) = u(1) = 0$, does not have classical solutions (Rolles theorem) but has a continuous viscosity solution (see figure 2-a)). Let us emphasize that continuous viscosity solutions are continuous (on the closure of the set where it is defined) and that a solution in the classical sense is a viscosity solution. The weakness of this notion is due to the compatibility condition necessary to the existence of the solution (constraint on the variation of the boundary conditions [14]). Also, the same equation (2) with $u(0) = 0$, $u(1) = 1.5$ does not have continuous viscosity solutions. Now let us suppose that we make a large error

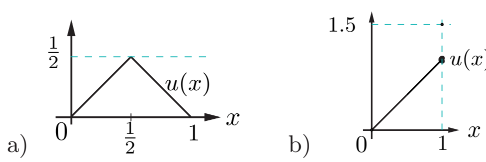


Fig. 2.
a) Continuous viscosity solution of (2) with $u(0) = u(1) = 0$;
b) discontinuous viscosity solution of (2) with $u(0) = 0$ and $u(1) = 1.5$.

on the boundary condition, when we compute a numerical solution of the SFS problems. If this error is too large then there do not exist continuous viscosity solutions. In this case one may wonder what the numerical algorithm of [23, 15] computes. In [22], Prados et al. answer this question by proposing to use the more general idea of *discontinuous* viscosity solutions. For example, equation (2) with $u(0) = 0$, $u(1) = 1.5$ has a discontinuous viscosity solution (see figure 2-b)). Let us emphasize that a "discontinuous viscosity solution" *can* have discontinuities and that a continuous viscosity solution is a discontinuous viscosity solution.
The classical theory of viscosity solutions offers simple and general theorems of existence and uniqueness of solutions for exactly the type of PDEs that arise in the context of SFS. In particular the theory allows to characterize exactly all possible continuous viscosity solutions: given a particular Dirichlet condition on the image boundary (verifying the compatibility condition), if the set of *critical points* (points of maximal intensity, i.e. $I(x) = 1$) is empty, then there exists a unique continuous viscosity solution satisfying the boundary conditions; If the set of critical points is not empty there exists an infinity of continuous solutions which are characterized by their values at the critical points. Note that this result is general and applies equally to all the SFS models described in section 2 (see [19]). As a consequence, the SFS problem is ill-posed and to compute an approximation of a solution, Rouy et al. and Prados et al. [23, 22, 20] must assume that the values of the solutions are given at the image boundary and the critical points. This is quite unsatisfactory, even more so since small errors on these values create undesirable crests, see figure 3-b) or [22] for an example

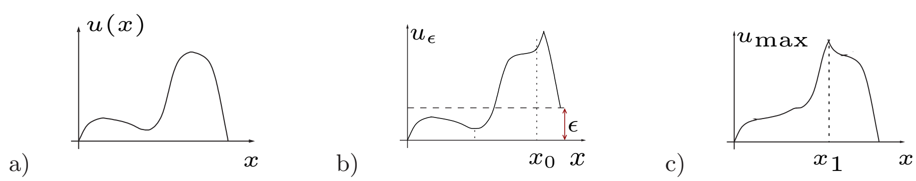


Fig. 3. a) original surface u; b) solution u_ϵ associated to corrupted boundary conditions and to the image obtained from the original surface a) with the Eikonal equation; c) maximal solution $u_{\max}$ (in Falcone's sense [8]) associated to the same image. u_ϵ and $u_{\max}$ present a kink at x_0 and x_1.

with a real image. Falcone [8] proposes not to specify anymore the values of the solution at the critical points (he still requires to specify the values at the image boundary though). In order to achieve this, he uses the notion of maximal viscosity solutions developed by Camilli and Siconolfi [3]. Despite its advantages, this approach is not really adapted to the SFS problem, see for example figure 3-c). In this figure, the maximal solution $u_{\max}$ associated to the image obtained from the original surface u shows a highly visible crest where the surface should be smooth. Even with the correct boundary conditions, Falcone's method does not really provide a suitable solution.

To summarize, the work of Rouy et al. [23], Prados et al. [22, 20] and Falcone et al. [8] suggests theories and numerical methods based on the concept of viscosity solutions and requiring data on the boundary of the image. At the opposite, Dupuis and Oliensis [6] consider C^1 solutions. They characterize a C^1 solution by specifying only its values at the critical points which are local minima. In particular, they do not specify the values of the solution on the boundary of the image. Also, they provide algorithms for approximating these smooth solutions. Nevertheless, in practice, because of noise, of incorrect modelization, errors on parameters or on the depth values enforced at the critical points, there do not exist C^1 solutions to the SFS equations [16]. Therefore, the theory of Dupuis and Oliensis does not apply.
Considering the drawbacks and the advantages of all these methods, it seems important to define a new class of weak solutions such that the characterization of Dupuis and Oliensis holds, and which provides a (theoretical and numerical) solution when there do not exist smooth solutions.

As we show in [21], the classical notion of viscosity solutions, like the notion of singular viscosity solutions (pioneered by Ishii and Ramaswamy [11] and recently upgraded by Camilli and Siconolfi [3]) does not provide a direct extension of the Dupuis and Oliensis work. For such an extension, we must modify these notions and we must consider a "new" type of boundary conditions (called "state constraints" [24]). It turns out that the correct notion of viscosity solution for the SFS problem is the "singular discontinuous viscosity solution with Dirichlet boundary conditions and state constraints". These solutions can be interpreted as maximal solutions and have the great advantage of not necessarily requiring boundary or critical points conditions. Moreover, this notion provides a mathematical framework unifying the work of Rouy et al. [15, 23], Prados et al. [22, 20], Falcone et al. [8] and Dupuis and Oliensis [6].

4 Singular discontinuous viscosity solutions for SFS

In this section we briefly describe the notion of "singular discontinuous viscosity solutions with Dirichlet boundary conditions and state constraints" (SDVS). We refer to [21] for more details. Also we do not recall the classical definition of viscosity solutions: see [1] for a recent overview.
Considering the generic SFS problem, we concentrate on the following HJB equation:

$$\sup_{a \in A}\{f(x,a) \cdot \nabla u(x) - l(x,a)\} = 0, \qquad \forall x \in \Omega \tag{3}$$

To simplify, we assume in this paper[1] that $l \geq 0$ and we denote $\mathcal{S} := \{x \mid l(x,a) = 0 \text{ for some } a \in A\}$. Also assume that $\mathcal{S}$ verifies $\mathcal{S} \cap \partial\Omega = \emptyset$. To equation

[1] In [21], we do not assume that $l \geq 0$. Also, the definition of $\mathcal{S}$ and the developed tools are more sophisticated. Note that, except for $H_{R/T}$ and $H_{P/F}$, all the SFS Hamiltonians verify $l \geq 0$. This justifies our interest for the Hamiltonian $H_{D/O}$ and the original one H_{Pers}.

(3), we add Dirichlet boundary conditions (DBC) on the boundary of the image and on $\mathcal{S}$:

$$u(x) = \varphi(x), \quad \forall x \in \partial\Omega \cup \mathcal{S}; \tag{4}$$

φ being continuous on $\partial\Omega \cup \mathcal{S}$ into $\mathbb{R} \cup \{+\infty\}$ (but $\varphi \neq +\infty$ everywhere). At the points x s.t. $\varphi(x) = +\infty$, we say that we impose a state constraint [24]. In the SFS context, $\mathcal{S}$ is the set of critical points $\{x \mid I(x) = 1\}$.

Definition: *u is a SDVS of (3)-(4), if u is a discontinuous viscosity solution of (3)-(4) on $\overline{\Omega} - \mathcal{S}$ and if $\forall x \in \mathcal{S}$, [$u^*(x) \leq \varphi(x)$] and [$u_*(x) \geq \varphi(x)$ or u_* is a singular viscosity supersolution in Camilli's sense at the point x].*

Definitions of u_* and u^* (not detailed here because of space) can be found in [1]. The notion of singular viscosity supersolution in Camilli's sense is completely described in [3, 4].

In [21], we prove the *existence* and the *uniquess* of the SDVS of all SFS equations as soon as I is Lipschitz continuous and the Hamiltonian is coercive (e.g. $H_{D/O}$ and $H_{R/T}$ are coercive $\Leftrightarrow I(x) > |\mathbf{l}|$) . We also prove the *robustness* of this solution to *pixel noise* and to *errors on the light or focal length parameters*. Finally, note that, when we impose state contraints on the boundary of the images and some critical points, this solution can be interpreted as the *maximal* viscosity solution. See [21] for more details.

5 A general framework for SFS:

The main interest of this "new" class of solutions lies in the possibility to impose the heights of the solution at the critical points when we know them (this is impossible with discontinuous viscosity solutions; it is possible with continuous viscosity solutions but compatibility conditions are required) and in the possibility to "send at infinity" the boundary conditions when we do not know then (this possibility is not considered by Falcone et al. [8]). The relevance of this notion is amplified by its consistency with the work of Dupuis and Oliensis [6]. This is illustrated by the following proposition (see [21]):

Proposition 1 *Let u be a C^1 solution of equation (3). Let $\tilde{\mathcal{S}}$ be the subset of $\mathcal{S}$ corresponding to the local minima of u. If u verifies the assumption 2.1 of [6] [2] then u is the SDVS of (3)-(4) for $\varphi(x) = u(x)$ $\forall x \in \tilde{\mathcal{S}}$ and $\varphi(x) = +\infty$ elsewhere.*

Therefore, when there do not exist C^1 solutions, the SDVS consistently *extend* the work of Dupuis and Oliensis. Moreover, the SDVS *unify* the various theories used for solving the SFS problem. In effect, we can verify that [21]:

- In the case where the DBC are finite on $\partial\Omega \cup \mathcal{S}$ and the compatibility condition (see [14]) holds, then the SDVS of (3)-(4) is the continuous viscosity solution used by [23, 15, 22, 20].

[2] Not stated here because of space.

- When the DBC are finite on the boundary of the image and state constraints are imposed at the critical points, the SDVS of (3)-(4) corresponds to Camilli's singular viscosity solutions [3, 4] used by Falcone [8].
- As seen above, the SDVS corresponds to the C^1 solution of (3), verifying the assumption 2.1 of Dupuis and Oliensis [6].

Consequently, the theoretical results of Falcone et al. [8] Rouy et al. [23, 15], Prados et al. [22, 20] and Dupuis et al. [6] are *automatically extended to the "perspective SFS"* (use H_F and H_{pers}). Finally, one can conjecture that by using the work of [12, 25], the notion of SDVS can be extended to solve SFS problem with discontinuous images. This would be very difficult without the tool of viscosity solutions.

6 Numerical approximation of the SDVS for generic SFS

This section explains how to compute a numerical approximation of the SDVS of the generic SFS equation. This requires three steps. First we "regularize" the equation. Second, we approximate the "regularized" SFS equation by approximation schemes. Finally, from the approximation schemes, we design numerical algorithms.

Regularisation of the generic SFS equation:
For an intensity image I and $\epsilon > 0$, let us consider the truncated image I_ϵ defined by $I_\epsilon(x) = min(I(x), 1-\epsilon)$. By using a stability result, we prove that for the generic SFS Hamiltonian, the SDVS associated with the image I_ϵ converges uniformly toward the SDVS associated with the image I, when $\epsilon \to 0$. Also, $\forall \epsilon > 0$, the generic SFS equation associated with I_ϵ is *no more degenerate*. Thus for approximating this equation, we can use the classical tools developed by Barles and Souganidis [2].

Approximation schemes for the nondegenerate SFS equations:
Let us consider the "regularized" generic SFS equation. The theory of Barles and Souganidis [2] suggests to consider *monotonous* schemes. Therefore, we construct the following monotonous scheme (we call it "implicit") $S(\rho, x, u(x), u) = 0$ with

$$S(\rho, x, t, u) = \max_{s_1, s_2 = \pm 1} S_{s_1,s_2}(\rho, x, t, u),$$

where $\rho = (\Delta x_1, \Delta x_2)$ is the mesh size and where we choose:

$$S^{impl}_{s_1,s_2}(\rho, x, t, u) \; = \; \sup_{a \in A_{s_1,s_2}} \left\{ -f_g(x,a) \cdot \begin{pmatrix} \frac{t-u(x+s_1\Delta x_1 \vec{e_1})}{-s_1 \Delta x_1} \\ \frac{t-u(x+s_2\Delta x_2 \vec{e_2})}{-s_2 \Delta x_2} \end{pmatrix} - l_g(x,a) \right\}.$$

$A_{s_1,s_2} = \{a \in A \mid {f_g}_1(x,a)s_1 \geq 0 \text{ and } {f_g}_2(x,a)s_2 \geq 0\}$.
By introducing a fictitious time $\Delta\tau$, we can transform the implicit scheme in a "semi-implicit" scheme (also monotonous):

$$S^{semi}_{s_1,s_2}(\rho, x, t, u) = t - \; (\; u(x) + \Delta\tau \; S^{impl}_{s_1,s_2}(\rho, x, u(x), u) \;),$$

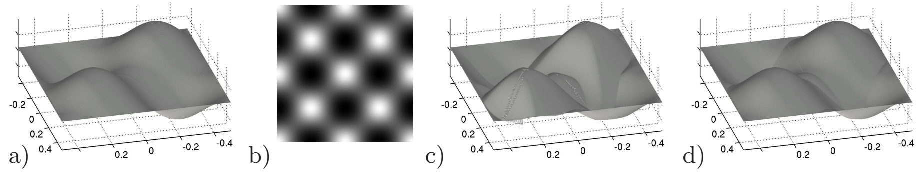


Fig. 4. a) original surface; b) image generated from a) by the Eikonal process [size 400×400]; c) reconstructed surface from b) after 15 iterations of Dupuis and Oliensis' algorithm (based on differential games) enforcing the exact Dirichlet condition on the boundary of the image and at all critical points: $\epsilon_1 = 0.015$, $\epsilon_2 = 5.7e-05$, $\epsilon_\infty = 0.35$; d) reconstructed surface by the implicit algorithm with the same boundary data and after the same number (15) of iterations as c): $\epsilon_1 = 0.002$, $\epsilon_2 = 1.0e-05$, $\epsilon_\infty = 0.014$.

where $\Delta\tau = (f_g(x, a_0) \cdot (1/\Delta x_1, 1/\Delta x_2))^{-1}$; a_0 being the optimal control. Let us emphasize that these two schemes have exactly the same solutions.

Using Barles and Souganidis definitions [2], we prove in [19] that these schemes are always monotonous and *stable*. Also, they are *consistent* with the genereric SFS equation as soon as the intensity image is Lipschitz continuous. Finally, when the Hamiltonian is coercive, we prove that the solutions of these schemes *converge toward the unique SDVS* of the "regularized" generic SFS equation, when $\rho \to 0$.

Remark: These two schemes have also a control interpretation. It is easy to verify that the implicit scheme is an extension of the control-based schemes proposed by [23, 15, 22] and the semi-implicit scheme corresponds to the control-based scheme proposed by [6]. All these schemes have been designed for the "orthographic SFS" problem. Note that for a given Hamiltonian, they all have the same solutions. Therefore we have unified and generalized these various approaches.

Numerical algorithms for the generic SFS problem: In the previous section, we have proposed two schemes whose solutions u^ρ converge toward the unique SDVS of the "regularized" generic SFS equation. For each scheme, we now describe an algorithm that computes an approximation of u^ρ.
For a fixed mesh size $\rho = (\Delta x_1, \Delta x_2)$, let us denote $x_{ij} := (i\Delta x_1, j\Delta x_2)$ and $\mathcal{X} := \{x_{ij} \in \Omega;\ i, j \in \mathbb{Z}\}$. The algorithms consist of the following computation of the sequence of values U_{ij}^n, $n \geq 0$ (U_{ij}^n being an approximation of $u^\rho(x_{ij})$).

Algorithm 1 *1. Initialisation ($n = 0$): $U_{ij}^0 = u_0(x_{ij})$.*

2. *Choice of a pixel $x_{ij} \in \mathcal{X}$ and modification of U_{ij}^n: We choose U^{n+1} such that $\forall (k,l) \neq (i,j)$, $U_{kl}^{n+1} = U_{kl}^n$ and $S(\rho, x_{ij}, U_{ij}^{n+1}, U^n) = 0$.*
3. *Choose the next pixel $x_{ij} \in \mathcal{X}$ in such a way that all pixels of $\mathcal{X}$ are regularly visited and go back to 2.*

We prove in [19] that if u_0 is a subsolution or a supersolution, then the computed numerical approximations converge toward u^ρ. In their work, Rouy, Prados et al. [23, 15, 22, 20] use (some particular cases of) the implicit algorithm starting from a subsolution. When we start from a supersolution, we reduce the number of iterations by 3 orders of magnitude! In [20], Prados and Faugeras need around

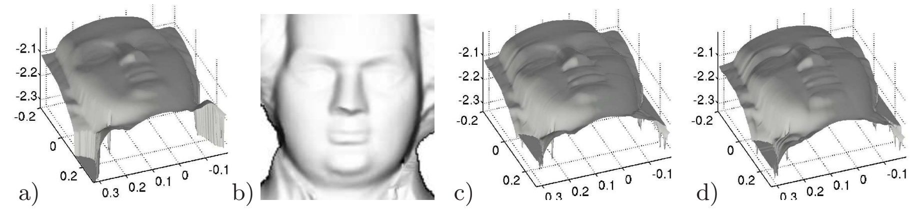


Fig. 5. a) original surface; b) image generated from a) [size $\simeq 200 \times 200$]; c) reconstructed surface from b) with the implicit algorithm (IA) after only 3 iterations, using the exact boundary data at all critical points and with state constraints on the boundary of the image: $\epsilon_1 \simeq 0.58$, $\epsilon_2 \simeq 0.0019$, $\epsilon_\infty \simeq 0.42$; d) reconstructed surface by the IA (after 3 iterations) with state constraints on the boundary of the image and at all the critical points except at that on the nose: $\epsilon_1 \simeq 0.60$, $\epsilon_2 \simeq 0.0020$, $\epsilon_\infty \simeq 0.42$.

4000 iterations for computing the surface of the classical Mozart's face [29]. Starting from a supersolution (in practice, a large constant function u_0 does the trick!), *only three iterations are sufficient* for obtaining a good result; see figure 5. As an example, we show in figure 4 a comparison of our results with those of what we consider to be the most efficient algorithm of the SFS literature [6]. Figures 4-c) and 4-d) show the results returned by our implementation of this algorithm and our algorithm, respectively, after 15 iterations. The results are visually different. This visual difference is confirmed by the computation of the errors with respect to the original surface (ϵ_1, ϵ_2 and ϵ_∞ are the errors of the computed surface measured according to the L_1, L_2 and L_∞ norms, respectively). Nevertheless let us note that the cost of one update is slightly larger for our implicit algorithm than for the (semi-implicit) algorithm of Dupuis and Oliensis. This may also be because we have not optimized our code for this special case. Let us add that in this test, we have constrained the solution by the exact Dirichlet condition on the boundary of the image and at all the critical points. Let us recall that the SDVS method does not necessarily require boundary data. Figure 5 shows some reconstructions of the Mozart face when using the exact boundary data at all the critical points and state constraints on the boundary of the image (Fig.5-c), and with no boundary data, except for the tip of the nose (Fig.5-d). Moreover, let us emphasize that our implicit algorithm (as our semi-implicit one) allows to compute some numerical approximations of the *SDVS* of the *degenerate* (when the intensity reaches 1) and *generic* SFS problem. Thus, we only need to implement a *single* algorithm for all SFS modelizations. Finally, let us remark that, as the theory predicted, our algorithm shows an exceptional robustness to noise and errors on the parameters; This robustness is even bigger when we send the boundary to infinity (apply the state constraints). Figure 6 displays a reconstruction of Mozart's face from an image perturbed by additive uniformly distributed white noise (SNR $\simeq 5$) by using the implicit algorithm with the wrong parameters $\mathbf{l}_\epsilon = (0.2, -0.1)$ and $f_\epsilon = 10.5$ (focal length) and without any boundary data. The original image Fig.6-a) has been synthetized with $\mathbf{l} = (0.1, -0.3)$ and $f = 3.5$. The angle between the initial light vector $\mathbf{L}$ and the corrupted light vector $\mathbf{L}_\epsilon$ is around 13°. More details, experimental comparisons and stability tests can be found in [19, 21]. These reports also contain the proofs of all our statements.

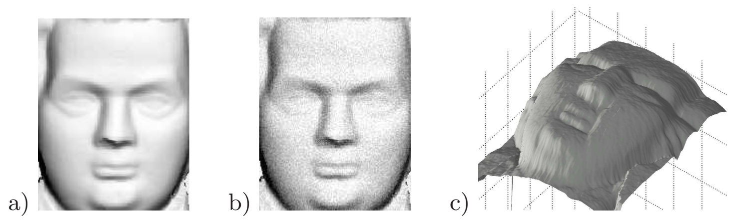


Fig. 6. a) Image generated from Mozart's face represented in Fig.5-a) with $\mathbf{l} = (0.1, -0.3)$ and $f = 3.5$ [size $\simeq 200 \times 200$]; b) noisy image (SNR $\simeq 5$); c) reconstructed surface from b) after 4 iterations of the implicit algorithm, using the incorrect parameters $\mathbf{l}_\epsilon = (0.2, -0.1)$ and $f_\epsilon = 10.5$, and with state constraints on the boundary of the image and at all the critical points except at the critical point on the nose.

7 Conclusion

We have *unified various formulations* of the Lambertian SFS problem; in particular the orthographic and perspective problems. We have developed *a new mathematical framework* which *unifies* some SFS theories and *generalizes* them to all SFS Hamiltonians. Let us emphasize that we do not consider Mathematics as a goal in itself. Mathematics is simply a powerful tool allowing us to

- suggest some numerical methods and algorithms;
- certify algorithms, to guarantee their robustness and to describe their limitations;
- better understand what we compute. In particular, when the problem has several solutions, it allows to characterize all the solutions, a necessary preliminary step for deciding which solution we want to compute.

In effect, our theory ensures the stability and the convergence of our SFS method. Also it suggests a *robust* SFS algorithm which *seems to be the most efficient* iterative algorithm of the SFS literature. Moreover, our new class of weak solutions is really more adapted to the SFS specifications; in particular, it *does not necessarily require boundary data*. We are extending our approach to non Lambertian SFS and to SFS with discontinous images.

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