Constrained Facility Location

Ferran Hurtado[1] Vera Sacristán[1] Godfried Toussaint[2]

1 Introduction

In a classical facility location problem we are given a set of n points C in the plane representing n customers, plants to be serviced, schools, markets, distribution sites or any other locations, depending on the context in which the problem is embedded, and it is desired to determine the location X (find another point in the plane) where a facility (service, transmitter, dispatcher, etc.) should be located so as to minimize the Euclidean distance from X to its furthest customer. Such a *minimax* criterion is particularly useful in locating emergency facilities, such as police stations, fire-fighting stations and hospitals where it is desired to minimize the worst-case response time. This problem has an elegant and succinct geometrical interpretation: find the smallest circle that encloses a given set of n points. The center of this circle is precisely the location of X. This problem is known in the literature under various names such as the *minimum spanning circle* or the *Euclidean 1-center problem*.

In this paper we consider constrained versions of the problem which appear to be very natural. Our main result is an $O(n+m)$ time algorithm for the problem of the minimum enclosing circle with its center constrained to satisfy m linear constraints. There is no need to justify the claim that this problem, in which a constraint polygon is given by a set of linear (possibly redundant) constraints, is natural, for it models a variety of real problems. As a corollary, we obtain a linear time algorithm for the problem when the center is constrained to lie in an m-vertex convex polygon, which improves the best known solution of $O((n+m)\log(n+m))$ time [2].

[1]Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Pau Gargallo 5, 08028 Barcelona, Spain. e-mail: hurtado,vera@ma2.upc.es. Partially supported by Proyecto DGES-SEUID PB96-0005-C02-02.

[2]School of Computer Science, McGill University, 3480 University Street, Montréal, Canada H3A2A7. e-mail: godfried@cs.mcgill.ca.

Similarly, we also show that the smallest circle enclosing n points with the constraint that the circle must pass through a given point or that the circle must be tangent to a given line can be solved in $\Theta(n)$ time.

We also consider a version of the *maximin* problem, that can be considered as a dual of the previous one, namely an obnoxious facility location problem, in which we are given a set of linear constraints, each representing a halfplane where some population may live, and the goal is to locate a point such that the minimum distance to the inhabited region is maximized. Such a maximin criterion es particularly useful in locating obnoxious facilities, such as nuclear plants, chemical factories, and waste disposal centres.

Geometrically, the problem consists of finding the largest circle enclosed in the convex polygon implicitly given by the intersection of the halfplanes defined by a set of linear constraints. The center of this circle is the location of the obnoxious optimal facility. We provide an $O(n)$ time algorithm for this problem on the sphere.

2 Locating Minimax Facilities

In this section we study some two-dimensional constrained versions of the minimax facility location problem. We provide an optimal linear time algorithm that locates the minimum enclosing circle of a set of points, with center constrained to satisfy a set of linear constraints, and some other similar results.

The algorithms we present follow a prune-and-search strategy whose underlying common scheme is inspired by Megiddo's algorithm for finding the minimum spanning center without constraints in [6]. The strategy of Megiddo in [6, 7] was independently studied by Dyer in [3, 4].

In the following, we summarize our results.

Lemma 1 *Given a line s, by solving the problem restricted to s it is possible to find the center of the minimum spanning circle of a set of n points in the plane, constrained to belong to the intersection of a set of m halfplanes, if it lies on s, or to decide on which side of s it lies, in optimal $\Theta(n+m)$ time.*

Theorem 2 *The minimum spanning circle of a set of n points in the plane, with center constrained to satisfy a set of m linear restrictions, can be found in optimal $\Theta(n+m)$ time.*

Corollary 3 *Let P be a convex m-gon given by its vertices, ordered as they appear on the boundary. The minimum spanning circle of a set of n points in the plane, with center constrained to lie in P, can be found in $\Theta(n+m)$ time.*

Theorem 4 *The minimum spanning circle of a set n of points $p_1, \ldots, p_n$ in the plane, constrained to be anchored to a fixed point q, can be found in optimal $\Theta(n)$ time.*

Theorem 5 *The minimum spanning circle of a set of n points $p_1, \ldots, p_n$ in the plane, constrained to be tangent to a given line l, can be found in optimal $\Theta(n)$ time.*

3 Locating Maximin Facilities

In this section, we provide an optimal linear time algorithm that solves the following problem on a sphere: find the maximum spherical cap enclosed in a convex polygon defined on a sphere as the intersection of spherical halfspaces.

The analogous problem in the plane consists of computing the maximum circle such that all its points satisfy a given set of linear constraints, that is the maximum circle enclosed in a given intersection of halfplanes. This problem was first shown [1] to be linear for the particular case in which the circle is constrained to lie in a convex polygon given by the ordered list of its edges, as they appear on the boundary of the polygon, and was then proved [8] to be linear also in the more general case in which the polygon is given as an unordered and possibly redundant intersection of halfplanes, for it can be posed as a linear programming problem in one higher dimension (and hence solved by Megiddo's prune-and-serch strategy).

In the spherical case, the problem is solved by reducing it to a dual minimax problem on the halfsphere. Consider the center of the sphere to be the origin. Each halfspace through the center of the sphere can be associated to the point on the sphere which is the intersection with the external normal ray to the halfspace through the origin.

As for the minimax problem, it is known that the smallest cap enclosing a set of n points on an open halfsphere can be found in $\Theta(n)$ time [5].

Theorem 6 *The maximum spherical cap enclosed in a convex polygon defined on a halfsphere as the intersection of n spherical halfspaces can be found in optimal $\Theta(n)$ time.*

References

[1] A. Aggarwal, L. J. Guibas, J. Saxe, and P. Shor. A linear time algorithm for computing the Voronoi diagram of a convex polygon. In *Proc. 19th Annu. ACM Sympos. Theory Comput.*, pages 39–45, 1987.

[2] P. Bose and G. Toussaint. Computing the constrained euclidean, geodesic and link center of a simple polygon with applications. In *Proc. of Pacific Graphics International*, pages 102–112, 1996.

[3] M. E. Dyer. Linear time algorithms for two- and three-variable linear programs. *SIAM J. Comput.*, 13:31–45, 1984.

[4] M. E. Dyer. On a multidimensional search technique and its application to the Euclidean one-centre problem. *SIAM J. Comput.*, 15:725–738, 1986.

[5] F. Gómez, S. Ramaswami, and G. Toussaint. On removing non-degeneracy assumptions in computational geometry. In *Proc. of the 3rd Italian Conference on Algorithms and Complexity*, 1997.

[6] N. Megiddo. Linear-time algorithms for linear programming in R^3 and related problems. *SIAM J. Comput.*, 12:759–776, 1983.

[7] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31:114–127, 1984.

[8] E. Welzl. Lp with small d - algorithms and applications. manuscript, 1994.