Robust Pole Assignment via Sylvester Equation Based State Feedback Parametrization

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Abstract

By using a Sylvester equation based parametrization, the *minimum norm robust pole assignment* problem for linear time-invariant systems is formulated as an unconstrained minimization problem for a suitably chosen cost function. The derived explicit expression of the gradient of the cost function allows the efficient solution of the minimization problem by using powerful gradient search based minimization techniques. We also discuss how requirements for a particular Jordan structure of the closed-loop state matrix or for partial pole assignment can be accommodated with the proposed approach.

1 Introduction

Pole assignment techniques to modify the dynamic response of linear systems are among the most studied problems in modern control theory. The complete theoretical solution of this problem has been followed by the development of many computational methods (see for example the collection of reprints in [15]). Sensitivity analysis of the pole assignment problem (see [12] and references therein) moves one step forward the understanding of difficulties and practical limitations associated with the usage of solution methods.

Consider the state-space system of the form

$$\lambda x(t) = Ax(t) + Bu(t), \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $\lambda x(t) = \dot{x}(t)$ for a continuous-time system and $\lambda x(t) = x(t+1)$ for a discrete-time system. Let $\Gamma_n = \{\lambda_1, \ldots, \lambda_n\}$ be a given symmetric set of n values in the complex plane. We address the following *eigenvalue assignment problem* (EAP): given the controllable pair (A, B), determine the state feedback matrix $F \in \mathbb{R}^{m\times n}$ such that the eigenvalues of the closed-loop state matrix $A + BF$ are at desired locations Γ_n.

In the multi-input case the EAP has a non-unique solution. Therefore it is reasonable to exploit the non-uniqueness by imposing additional conditions. One aspect which is desirable from a practical point of view is to determine feedback matrices with small gains. Intuitively this must be advantageous since small feedback gains lead to smaller control signals, and thus to less energy consumption. Small gains are also beneficial to reduce noise amplification. A second aspect important in pole assignment is to achieve a small condition number for the eigenvector matrix of the closed-loop system. This is the goal of *robust pole assignment* [9, 4, 17]. In light of recent perturbation results [12], both aspects appear to be decisive for the sensitivity of assigned eigenvalues. It was shown in [12] that high feedback gains or high condition numbers lead to high sensitivity of the closed-loop eigenvalues. Thus the simultaneous minimization of the feedback norm and of the sensitivity of closed-loop eigenvalues is a desirable general goal for solving the EAP.

In this paper we focus on developing a reliable numerical approach to exploit the intrinsic non-uniqueness of the EAP by formulating it as minimum norm robust pole assignment problem. By using a Sylvester equation based parametrization, a solution of the EAP is sought by minimizing a special cost function expressing the weighted requirements for minimum Frobenius-norm of the feedback matrix and the minimum condition number of the closed-loop eigenvector matrix. The derived explicit expression for the gradient of cost functions allows the use of standard gradient search based minimization techniques to compute the optimal state feedback matrix.

The efficient evaluation of the cost function and its gradient is of paramount importance for the usefulness of the Sylvester approach. A transformation technique used in conjunction with the solution of a reduced Sylvester equation is the main ingredient to achieve this goal. It allows to address with practically no extra costs the partial pole assignment problem too. Further we show how requirements for a particular closed-loop

Jordan structure can be also accommodated with the Sylvester equation based approach. We believe that the proposed robust pole assignment approach is a viable way to solve large EAPs in the perspective of the requirements formulated by recent sensitivity analysis results [12].

2 Parametrization of solutions

Our parametrization is based on a straightforward Sylvester equation based formulation [3]. Let assume that F solves the EAP. It follows, that there must exist an invertible transformation matrix X and $\widetilde{A}$ satisfying $\Lambda(\widetilde{A}) = \Gamma_n$ such that

$$X^{-1}(A + BF)X = \widetilde{A}. \qquad (2)$$

If we define $G := FX$ then (2) can be rewritten as a Sylvester matrix equation

$$AX - X\widetilde{A} + BG = 0. \qquad (3)$$

which must be satisfied by X.

Now we can try to solve the EAP assuming that $\widetilde{A}$ is chosen such that $\Lambda(\widetilde{A}) = \Gamma_n$, and G is a given parameter matrix. To solve the EAP, we need to solve (3) for X and, provided X is invertible, we compute the feedback matrix as

$$F = GX^{-1}. \qquad (4)$$

To enforce the invertibility of X, the matrices $\widetilde{A}$ and G and must fulfill some conditions: 1) the pair $(\widetilde{A}, G)$ is observable; 2) $\Lambda(A) \cap \Lambda(\widetilde{A}) = \emptyset$. These conditions together with the controllability of pair (A, B) ensure that X satisfying (3) is generically invertible [6]. Note that if $\widetilde{A}$ is in a Jordan canonical form, then the resulting X plays the role of the eigenvector matrix for the closed-loop state matrix $A + BF$.

3 Solution of the robust EAP

In light of the sensitivity results in [12] it meaningful to exploit the non-uniqueness of the EAP for multi-input systems by minimizing additionally the sensitivity of the closed-loop eigenvalues and the norm of the feedback matrix. This leads to a *minimum norm robust* EAP for which we propose a solution method combining unconstrained optimization techniques with the parametric Sylvester equation based approach. Note that in conjunction with the Sylvester approach the norm minimization problem alone has been considered in [10, 18], while for robust pole assignment the minimization of the trace of $(I - X^T X)^2$ has been proposed in [5]. The simultaneous minimization of feedback norm and eigenvector matrix condition number has been addressed for periodic systems in [20].

As a measure of the sensitivity of closed-loop eigenvalues, we use the condition number $\kappa_F(X)$ of X with respect to the Frobenius norm. For computational convenience, instead of minimizing $\kappa_F(X) := \|X\|_F \|X^{-1}\|_F$, the minimization of the sum $\|X\|_F^2 + \|X^{-1}\|_F^2$ can be alternatively performed, since the two optimization problems are mathematically equivalent [4]. Thus, for the simultaneous minimization of the norm of the state feedback matrix F and of the condition number $\kappa_F(X)$ we can use the following performance index

$$J = \frac{\alpha}{2}\left(\|X\|_F^2 + \|X^{-1}\|_F^2\right) + \frac{1-\alpha}{2}\|F\|_F^2, \qquad (5)$$

where $0 \leq \alpha \leq 1$ is a weighting factor. For $\alpha = 0$ J defines a pure norm minimization problem, while for $\alpha = 1$ we get a pure robust EAP. Intermediary values of α lead to a combination of both aspects.

The main advantage of the Sylvester equation based parametrization is that it allows a straightforward derivation of analytic expressions of gradients of the performance criterion J with respect to the free parameter matrix G. The following result is a specialization to standard systems of a more general result derived for periodic systems in [20]:

Proposition 1 *Let F be the state feedback computed as in (4), assigning the desired eigenvalues Γ_n for given $\widetilde{A}$ and G. Then, the gradient of J with respect to G is given by*

$$\nabla_G J = (1-\alpha)H^T + B^T U^T, \qquad (6)$$

where $H = X^{-1}F^T$, and U satisfies the Sylvester equation

$$\widetilde{A}U - UA + S = 0 \qquad (7)$$

with $S = (1-\alpha)HF + \alpha(-X^T + X^{-1}X^{-T}X^{-1})$.

Each function and gradient evaluation involves the solution of two Sylvester equations (3) and (7) sharing the same coefficient matrices. Efficient algorithms to solve these equations are the Hessenberg-Schur method proposed in [8] and the Schur method (known also as the Bartels-Stewart method) proposed in [2]. In the next section we describe a transformation based approach by which gradient computations can be substantially speeded up.

Having explicit analytical expressions for the function and its gradient it is easy to employ any gradient based technique to minimize J. However, since the dimension of the minimization problem nm could be potentially large, a particularly well suited class of methods to solve our problem is the class of unconstrained descent methods, as for instance, the limited memory BFGS method [11] used in conjunction with a line search procedure with guaranteed decrease as that described in [14]. The guaranteed decrease feature of these methods ensures that for $\alpha > 0$ the condition number $\kappa_F(X)$ progressively decreases and thus the solution X of (3) remains invertible at each iteration once an invertible solution has been determined at the first iteration.

A word of caution is necessary when using gradient techniques to solve the EAP by minimizing J. Since the minimization problems has possibly many local minima, it is likely that the computed solution is a local minimum. By solving the problem repeatedly with different initializations, we can choose that solution which produces the lowest value of the cost function. Note however, that in most of cases the global minimum leads to a condition number of transformation matrix X which has the same order of magnitude as those corresponding to any of local minima. Thus there is practically no difference for solving a robust eigenvalue assignment problem if the global minimum or one of local minima is employed to compute the feedback.

4 Algorithmic features

A satisfactory eigenvalue assignment algorithm must fulfill several functional and numerical requirements to serve as basis for a numerically robust software implementation. In what follows we discuss several algorithmic features of the Sylvester equation approach and we point out how apparent limitations of this method can be overcome.

4.1 Functional features

A satisfactory computational algorithm must be general and flexible, and must be able to exploit all structural aspects of the underlying problem. We will examine these aspects in detail in case of the eigenvalue assignment method based on the Sylvester equation based approach.

Generality

Generality means that an eigenvalue assignment algorithm is able to assign an arbitrary set of eigenvalues and ideally, it can also assign a desired eigenstructure for the closed-loop system. Although the first requirement seems to be trivial, even well-known methods implemented in commercial software are not able to fulfill this requirement. For example, the robust pole assignment method of [9] can not assign poles with multiplicities greater than rank of B and the improved version of this approach has the same limitation [17]. The Sylvester approach has no such limitations, although for a complete generality two aspects must be additionally addressed: the assignment of a given eigenstructure for the closed-loop eigenvalues and the assignment of eigenvalues which possibly coincide with those of the original system.

An arbitrary set of eigenvalues can be assigned with the Sylvester equation based approach by suitably choosing the matrix $\widetilde{A}$ used in EAP. Assume that Γ_n contains p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, and each eigenvalue λ_i has multiplicity k_i. Then, we can always choose $\widetilde{A}$ in a Jordan canonical form

$$\widetilde{A} = \begin{bmatrix} J_{k_1}(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{k_p}(\lambda_p) \end{bmatrix}$$

where $J_k(\lambda)$ denotes a Jordan block for the eigenvalue λ of order k. For a complex eigenvalue λ_i belonging to a multiple pair of complex conjugated eigenvalues $(\lambda_i, \overline{\lambda}_i)$, a $2k_i \times 2k_i$ real Jordan block can be used instead of two $k_i \times k_i$ complex Jordan blocks [7, page 365]. If $\operatorname{rank} B = m > 1$, a better conditioned transformation matrix X can be obtained by employing several Jordan blocks of lower dimensions for each multiple eigenvalue. Thus for each λ_i of multiplicity k_i, up to m Jordan blocks with dimensions at most $\left[\frac{k_i}{m}\right] + 1$ can be used, where $[\cdot]$ denotes the integer part.

Although the Sylvester equation based approach can be used for eigenstructure assignment, there are some limitations with respect to the admissible closed-loop eigenstructure. The eigenstructure assignment problem can be equivalently formulated as the assignment of a set of invariant polynomials $\psi_1, \psi_2, \ldots, \psi_n$ for the closed loop state matrix $A + BF$. Let $q_1 \leq q_2 \leq \ldots \leq q_m$ be the controllability indices of the controllable pair (A, B). Then, according to [16], the eigenstructure assignment problem has a solution if and only if the following set of inequalities is satisfied

$$\sum_{i=1}^{j} q_i \geq \sum_{i=1}^{j} \deg \psi_i, \quad j = 1, \ldots, m \qquad (8)$$

where (8) holds with equality when $j = m$. Thus, for the assignment of a desired Jordan structure for the closed-loop state matrix $A + BF$ the choice of corresponding $\widetilde{A}$ must reflect this structure in accordance with the conditions (8).

The second aspect of overlapping open-loop and closed-loop eigenvalues can be easily solved with the help of a preliminary state-feedback. We can always use, as a *practical* solution, a randomly generated feedback matrix F_0 and solve the EAP for the modified pair $(A + BF_0, B)$. If F_1 is the solution of this problem, then $F = F_0 + F_1$ solves the original problem. An alternative, numerically more robust solution to this aspect is discussed in the next paragraph.

Flexibility

One apparent limitation of the Sylvester equation based approach is the need that the closed-loop and open-loop spectra do not overlap. This condition guarantees the existence of a unique solution to the Sylvester equation (3), and is thus convenient from numerical point of view when using Sylvester equation solvers as that proposed in [8, 2]. Although technical, this condition prevents the Sylvester equation based approach to perform a *partial eigenvalue assignment*, i.e., to keep unmodified some of the open-loop eigenvalues. Since the partial eigenvalue assignment is a very useful feature, especially when stabilizing high order systems, we show how this feature can be easily accommodated within the Sylvester equation based approach and thus substantially increasing its flexibility.

It is easy to see that the performance index J is invariant to an orthogonal system similarity transformation, that is, if F is the optimal feedback matrix for the pair (A, B) then $\widehat{F} = FQ$ is the optimal feedback matrix for the transformed pair $(\widehat{A}, \widehat{B}) := (Q^T AQ, Q^T B)$, where Q is an orthogonal matrix. Thus, if we want to keep unmodified the eigenvalues of A lying in a "good" region $\mathbb{C}_g$ of $\mathbb{C}$ and to modify only those lying in its complement $\mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g$ (the "bad" region), then we can first reduce A to an ordered *real Schur form* (RSF) to obtain the pair

$$Q^T AQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad Q^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad (9)$$

where $\Lambda(A_{11}) \subset \mathbb{C}_g$, $\Lambda(A_{22}) \subset \mathbb{C}_b$ and $A_{22} \in \mathbb{R}^{n_2 \times n_2}$. With this separation, we can perform a partial pole assignment by solving for the optimal solution F_2 the EAP for the reduced pair (A_{22}, B_2) and a corresponding reduced set Γ_{n_2}. The overall optimal feedback matrix results as $F = [\ 0 \ \ F_2\ ]Q^T$.

Structure exploitation

The Sylvester equation approach in conjunction with the optimization based search for a minimum norm and well-conditioned feedback exploits the intrinsic freedom of the multi-input EAP to address an important additional requirement, namely, the well conditioning of the EAP. Note that most of pole assignment algorithms do not exploit this structural feature of the problem and even algorithms for robust pole assignment address only partially this aspect by ignoring norm minimization. Moreover, most methods have also restrictions with respect to the allocation of the closed-loop eigenstructure.

4.2 Numerical features

We focus on discussing numerical properties like the numerical stability and computational efficiency of the Sylvester equation based eigenvalue assignment algorithms, and we address shortly the implementation aspects of this approach in robust numerical software.

Numerical stability

To solve the EAP, the computation of the optimal solution F for the computed optimal parameter matrix G involves the solution of two systems of linear equations: the Sylvester equation (3) to compute X, and the linear system $FX = G$ to compute the feedback matrix F. Thus the Sylvester equation based approach can be considered to be practically numerically stable.

Concerning the accuracy of the results, in a robust pole assignment problem it is expected that the optimal X is reasonably well-conditioned, thus the last computational step is usually very accurate. Thus, the main source of errors appears to be the numerical solution of the Sylvester equation, where the separation of spectra of the pairs A and $\widetilde{A}$ is the essential factor for the accuracy of the computed X. However, a good separation can be always achieved by an initial eigenvalue shifting with a preliminary feedback (see also subsection 4.1), and therefore, for most practical problems, we can expect that the computed results corresponding to an optimal solution are very accurate.

Efficiency

The overall efficiency of the eigenvalue assignment algorithm heavily depends on the costs of function and gradient evaluations. Each function and gradient evaluation involves the solution of two Sylvester equations (3) and (7) sharing the same coefficient matrices. For our purposes, the best suited method to solve Sylvester equations is the well-known Schur method [2]. This approach can be efficiently employed in our case provided the matrix A is reduced first to a RSF using an orthogonal similarity transformation and assuming further that the matrix $\widetilde{A}$ is in a Jordan form (a particular RSF with block-diagonal structure). The reduction of A, performed only once, requires about $10n^3$ operations and can be seamlessly combined with the reordering of the RSF to accommodate with the partial pole assignment requirement. The solution of the minimization problem for the EAP can be performed to obtain the optimal solution $\widehat{F}$ for the transformed pair $(\widehat{A}, \widehat{B}) = (Q^T AQ, Q^T B)$ with $\widehat{A}$ in RSF and $\widetilde{A}$ in Jordan form. The solution of the original EAP results as $F = \widehat{F}Q^T$.

For the transformed problem, the function and gradient evaluations can be performed very efficiently since now we have to solve only reduced Sylvester equations with the coefficient matrices in RSF. This involves about n^3 operations for the solution of each reduced Sylvester equation [2]. Thus the overall cost to evaluate the function and gradient is about $5n^3$ operations, from which $3n^3$ operations amounts to form the free term S in (7).

Implementation aspects

The Sylvester equation based approach is simple to implement. For a FORTRAN implementation, all necessary software to perform the linear algebra computations is available in LAPACK 3.0 [1]. Here routines are provided to compute the RSF of a matrix, to solve the Sylvester equation, as well as systems of linear equations. For optimization, efficient unconstrained minimization routines are available in MINPACK-2 (the successor of MINPACK-1 [13]), offering a convenient reverse communication interface which allows an easy implementation of function and gradient computations.

For testing purposes, a prototype MATLAB function `sylvplace` have been implemented by the author to solve the robust EAP. This function relies on an efficient *mex*-function `linmeq` to solve various matrix equations developed within the NICONET project[1]. For optimization, the `fminunc` unconstrained minimization function available in the Optimization Toolbox 2.0 of MATLAB has been employed.

5 Numerical results

The numerical results have been obtained with three MATLAB *m*-functions implementing three robust pole assignment approaches: `sylvplace` based on our approach, `place` from the MATLAB Control Toolbox based on the algorithm of [9], and `robpole` based on the enhanced robust pole assignment algorithm proposed in [17]. We compared these *m*-functions on the suite of 11 EAPs presented in [4].

In Table 1 we present results for solving the *pure* robust EAP. For each of three methods, the condition numbers of the resulting eigenvector matrix X and the norms of corresponding feedback matrices have been computed.

Table 1: Conditioning results

	`place`		`robpole`		`sylvplace`	
No.	$\kappa_2(X)$	$\|F\|_2$	$\kappa_2(X)$	$\|F\|_2$	$\kappa_2(X)$	$\|F\|_2$
1	3.43	1.45	4.27	1.28	3.39	1.45
2	354.6	2973.8	39.85	225.5	37.68	354.8
3	37.47	59.40	39.29	49.1	35.48	77.25
4	10.78	9.84	10.77	9.44	10.77	9.44
5	91.62	4.54	88.56	5.14	89.05	4.22
6	3.90	13.8	3.63	19.41	3.58	23.0
7	10.29	158.41	4.65	235.2	4.38	270.3
8	6.16	38.5	3.61	19.84	3.61	19.25
9	18.47	844.1	18.44	820.5	18.42	829.2
10	1.05	1.33	1.00	1.41	1.00	1.41
11	12526	6692	12443	6580	12443	6580

[1]see `http://www.win.tue.nl/niconet/niconet.html`

It is apparent from this table that the Sylvester method produced generally better results than the algorithm implemented in `place` and practically the same results as those obtained with the enhanced robust EAP method of [17] implemented `robpole`.

The accuracy of computed closed-loop eigenvalues measured in the number of accurate digits is presented in Table 2. The two functions `robpole` and `sylvplace` perform practically identically on the example set and perform better than `place` on several examples.

Table 2: Number of accurate digits of eigenvalues

No.	`place`	`robpole`	`sylvplace`
1	16	15	16
2	12	14	15
3	15	15	14
4	16	15	15
5	14	14	14
6	15	16	16
7	15	15	15
8	16	15	16
9	15	15	15
10	16	16	16
11	11	12	13

The increased flexibility of `sylvplace` with respect to `place` and `robpole` is illustrated in Table 3, where for each example we computed the condition number of X and the norm of the corresponding feedback F for three values of α.

Table 3: Conditioning results for `sylvplace`

	$\alpha = 1$		$\alpha = .5$		$\alpha = 0$	
No.	$\kappa_2(X)$	$\|F\|_2$	$\kappa_2(X)$	$\|F\|_2$	$\kappa_2(X)$	$\|F\|_2$
1	3.39	1.45	3.23	1.28	52.42	0.58
2	37.68	354.8	258.5	94.0	275.1	92.57
3	35.48	77.25	83.42	10.84	913.5	4.33
4	10.77	9.44	12.71	2.77	50.02	0.027
5	89.05	4.22	90.94	3.80	1306.1	1.97
6	3.58	23.0	4.95	11.56	5.14	11.5
7	4.38	270.3	11.82	2.84	15.93	2.39
8	3.61	19.25	24.12	5.92	84.05	4.17
9	18.42	829.2	208.3	263.1	289.6	206.7
10	1.0	1.41	1.06	1.32	1.29	1.27
11	12443	6580	$6 \cdot 10^5$	370.3	$8.3 \cdot 10^5$	295.4

It can be observed, that for some examples the computed results for the intermediary value $\alpha = 0.5$, are better conditioned than either the *pure* robust or *pure* minimum norm EAPs. This is especially true for problems No. 3, 4, 7, where with a small increase of condition number of X, we achieved an order of magnitude smaller feedback norms.

6 Conclusions

We focused on developing a reliable numerical approach to exploit the intrinsic non-uniqueness of the EAP. One possibility to address the non-uniqueness is by formulating the EAP as a minimum norm robust pole assignment problem. By using a convenient parametrization, a solution of the EAP is sought by minimizing a special cost function expressing the weighted requirements for minimum Frobenius-norm of the feedback matrix and the minimum sensitivity of the closed-loop eigenvalues. The derived explicit expression for the gradient of cost function allows the use of standard gradient search based minimization techniques. The efficient evaluation of the cost functions and gradients is of paramount importance for the usefulness of the proposed approach. Transformation techniques used in conjunction with the solution of reduced Sylvester equation are the main ingredients to achieve this goal. Further, they allow to address with practically no extra costs the partial pole assignment problem too.

We believe that the proposed robust pole assignment approach is a viable way to solve large EAPs in the perspective of the requirements formulated by recent sensitivity analysis results [12]. In a broader context, the Sylvester equation based approach provides a unified framework to solve various eigenvalue assignment problems for standard, descriptor [19] and even periodic systems [20]. Therefore, in light of discussions on generality and flexibility, this approach has the potential to become the standard way to solve all classes of eigenvalue assignment problems.

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