The tail of the stationary distribution of a random coefficient AR(q) model *

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Abstract

We investigate a stationary random coefficient autoregressive process. Using renewal type arguments tailor-made for such processes we show that the stationary distribution has a power-law tail. When the model is normal, we show that the model is in distribution equivalent to an autoregressive process with ARCH errors. Hence we obtain the tail behaviour of any such model of arbitrary order.

Key words: ARCH model, autoregressive model, geometric ergodicity, heteroscedastic model, random coefficient autoregressive process, random recurrence equation, regular variation, renewal theorem for Markov chains, strong mixing.

AMS (2000) Subject Classification : primary: 60J10, 60H25; secondary: 62P05, 91B28, 91B84.

*This work was supported by the German Science Foundation (Deutsche Forschungsgemeinschaft), SFB 386, Statistical Analysis of Discrete Structures. The second author is also partially supported by the RFFI-Grant 00-01-880.

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1 Introduction

We consider the following random coefficient autoregressive model.

$$y_n = \alpha_1(n)y_{n-1} + \cdots + \alpha_q(n)y_{n-q} + \xi_n\,, \quad n \in \mathbb{N}\,, \tag{1.1}$$

with random variables $\alpha_i(n) = a_i + \sigma_i\eta_i(n)$, where $a_i \in \mathbb{R}$ and $\sigma_i \geq 0$. Set

$$\alpha(n) = (\alpha_1(n), \ldots, \alpha_q(n))'\,, \quad \eta(n) = (\eta_1(n), \ldots, \eta_q(n))'\,,$$

where throughout the paper all vectors are column vectors and $'$ denotes transposition. We suppose that the sequences of coefficient vectors $(\eta(n))_{n\in\mathbb{N}}$ and noise variables $(\xi_n)_{n\in\mathbb{N}}$ are independent and both sequences are iid with

$$\mathbf{E}\xi_1 = \mathbf{E}\eta_i(1) = 0 \quad \text{and} \quad \mathbf{E}\xi_1^2 = \mathbf{E}\eta_i^2(1) = 1\,, \quad i = 1, \ldots, q\,.$$

We are interested in the existence of a stationary version of the process $(y_n)_{n\in\mathbb{N}}$, represented by a random variable y_∞ and its properties. In this paper we investigate the tail behaviour

$$\mathbf{P}(y_\infty > t) \quad \text{as} \quad t \to \infty\,. \tag{1.2}$$

This is, in particular, the first step for an investigation of the extremal behaviour of the corresponding stationary process, which we will study in a forthcoming paper. Stationarity of (1.1) is guaranteed by condition $\mathbf{D_0}$ below. To obtain the asymptotic behaviour of the tail of y_∞ we embed $(y_n)_{n\in\mathbb{N}}$ into a multivariate set-up.

Set $Y_n = (y_n, \ldots, y_{n-q+1})'$. Then the multivariate process (Y_n) can be considered in the much wider context of random recurrence equations of the type

$$Y_n = A_nY_{n-1} + \zeta_n\,, \quad n \in \mathbb{N}\,, \tag{1.3}$$

where $((A_n, \zeta_n))_{n\in\mathbb{N}}$ is an iid sequence, the A_n are iid random $(q \times q)$-matrices and the ζ_n are iid q-dimensional vectors. Moreover, for every n, the vector Y_{n-1} is independent of (A_n, ζ_n).

Such equations play an important role in many applications as e.g. in queueing; see Brandt, Franken and Lisek [3] and in financial time series; see Engle [8]. See also Diaconis and Freedman [5] for an interesting review article with a wealth of examples.

If the Markov process defined in (1.3) has a stationary distribution and Y has this stationary distribution, then certain results are known on the tail behaviour of Y. In the one-dimensional case ($q = 1$) Goldie [10] has derived the tail behaviour of Y in a very elegant way by a renewal type argument: the tail decreases like a power-law. For the multivariate model, Kesten [14] and Le Page [20] show - under certain conditions on the matrices A_n - that $t^\lambda \mathbf{P}(x'Y > t)$ is asymptotically equivalent to a renewal function, that is

$$t^\lambda \mathbf{P}(x'Y > t) \sim G(x,t) = \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n)\,, \quad t \to \infty\,, \tag{1.4}$$

where $\sim$ means that the quotient of both sides tends to a positive constant. Note that, if we set $x' = (1, 0, \ldots, 0)$, then we obtain again (1.2). Here $g(\cdot,\cdot)$ is some continuous function satisfying condition (4.1) below, $(x_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are stochastic processes, defined in (1.9) and (1.10) below.

In model (1.1), we have $\zeta_n = (\xi_n, 0, \ldots, 0)'$ and

$$A_n = \begin{pmatrix} \alpha_1(n) & \cdots & \alpha_q(n) \\ I_{q-1} & & 0 \end{pmatrix}, \quad n \in \mathbb{N}, \tag{1.5}$$

where I_{q-1} denotes the identity matrix of order $q-1$.

Standard conditions for the existence of a stationary solution to (1.3) are given in Kesten [15] (see also Goldie and Maller [11]) and require that

$$\mathbf{E} \log^+ |A_1| < \infty \quad \text{and} \quad \mathbf{E} \log^+ |\zeta_1| < \infty \tag{1.6}$$

and that the top Lyapunov exponent

$$\widetilde{\gamma} = \inf\{n^{-1}\mathbf{E} \log |A_1 \cdots A_n| : n \in \mathbb{N}\} < 0\,. \tag{1.7}$$

In our case, conditions (1.6) are satisfied. Moreover, we can replace (1.7) by the following simpler condition; see e.g. Nicholls and Quinns [19].

$\mathbf{D_0}$) The eigenvalues of the matrix

$$\mathbf{E}\, A_1 \otimes A_1 \tag{1.8}$$

have moduli less than one, where $\otimes$ denotes the Kronecker product of matrices.

In the context of model (1.1) the processes $(x_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are defined as

$$x_0 = x \in S\,, \qquad x'_n = \frac{x'_{n-1} A_n}{|x'_{n-1} A_n|} = \frac{x' A_1 \cdots A_n}{|x' A_1 \cdots A_n|}\,, \quad n \in \mathbb{N}, \tag{1.9}$$

and

$$v_0 = 0\,, \quad v_n = \sum_{i=1}^{n} u_i = \log |x' A_1 \cdots A_n|\,, \qquad \text{with } u_n = \log |x'_{n-1} A_n|\,, \quad n \in \mathbb{N}. \tag{1.10}$$

Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^q$ and $|A|^2 = \operatorname{tr} A\, A'$ is the corresponding matrix norm; we denote furthermore $S = \{z \in \mathbb{R}^q \,:\, |z| = 1\}$ and $\overline{x} = x/|x|$ for $x \neq 0$.

Since GARCH models are commonly used as volatility models, modelling the (positive) standard deviation of a financial time series, Kesten's work can be applied to such models; see e.g. Diebolt and Guegan [6]. Kesten [14, 15] proved and applied a Key Renewal Theorem to the right-hand side of (1.4) under certain conditions on the function g, the Markov chain $(x_n)_{n\geq 0}$ and the stochastic process $(v_n)_{n\geq 0}$; a special case is the random recurrence model (1.3) with $P(A_n > 0) = 1$. By completely different, namely point process methods, Davis, Mikosch and Basrak [4] show that for a stationary model (1.3) – again with positive matrices A_n – the stationary distribution has a (multivariate) regularly varying tail. Some special examples have been worked out as ARCH(1) and GARCH(1,1); see Goldie [10], de Haan et al. [12] and Mikosch and Starica [18].

The random coefficient model (1.1), however, does not necessarily satisfy the positivity condition on the matrices A_n; see Section 2 for examples. On the other hand, it is a special case within Kesten's very general framework. Consequently, we derived a new Key Renewal Theorem in the spirit of Kesten's results, but tailor-made for Markov chains with compact state space, general matrices A_n and functions g satisfying condition (4.1); see Klüppelberg and Pergamenchtchikov [16], Theorem 2.1. We apply this theorem to the random coefficient model (1.1).

The paper is organised as follows. Our main results are stated in Section 2. We give weak conditions implying a power-law tail for the stationary distribution of the random coefficient model (1.1). For the Gaussian model (all random coefficients and noise variables are Gaussian) we show that model (1.1) is in distribution equivalent to an autoregressive model with ARCH errors of the same order as the random coefficient model. Since the limit variable of the random recurrence model (1.5) is obtained by iteration, products of random matrices have to be investigated. This is done in Section 3. In Section 4 we check the sufficient conditions and apply the Key Renewal Theorem from [16] to model (1.1). Some auxiliary results are summarized in the Appendix.

2 Main results

Our first result concerns stationarity of the multivariate process $(Y_n)_{n\in\mathbb{N}}$ given by (1.3). We need some notions from Markov process theory, which can be found e.g. in Meyn and Tweedie [17]; see also Appendix A0.

The following result is an immediate consequence of Theorem 3 of Feigin and Tweedie [9].

Theorem 2.1. *Consider model (1.1) with A_n given by (1.5) and $\zeta_n = (\xi_n, 0, \ldots, 0)'$, where ξ_1 has a positive density on $\mathbb{R}$. If $\mathbf{D_0}$ holds, then $Y_n = (y_n, \ldots, y_{n-q+1})'$ converges* *in distribution to*

$$Y = \zeta_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} \zeta_k \,. \tag{2.1}$$

Moreover, $(Y_n)_{n\in\mathbb{N}}$ *is uniform geometric ergodic.*

Remark 2.2. (a) From equation (2.1) we obtain

$$Y \stackrel{d}{=} A_1 Y_1 + \zeta_1 \,, \tag{2.2}$$

where

$$Y_1 = \zeta_2 + \sum_{k=3}^{\infty} A_2 \cdots A_{k-1} \zeta_k \stackrel{d}{=} Y$$

and Y_1 is independent of (A_1, ζ_1).

(b) Since $\mathbf{E}\left((A_1 \cdots A_n) \otimes (A_1 \cdots A_n)\right) = \left(\mathbf{E}\left(A_1 \otimes A_1\right)\right)^n$ condition $\mathbf{D_0}$ guarantees that

$$\mathbf{E}|A_1 \cdots A_n|^2 \le c e^{-\gamma n} \tag{2.3}$$

for some constants $c, \gamma > 0$. From this follows that the series in (2.1) converges a.s. and the second moment of Y is finite; see Theorem 4 of [9]. □

We require the following additional conditions for the distributions of the coefficient vectors $(\eta(n))_{n\in\mathbb{N}}$ and the noise variables $(\xi_n)_{n\in\mathbb{N}}$ in model (1.1).

$\mathbf{D_1}$) *The random variables* $\{\eta_i(n)\,, 1 \le i \le q\,, n \in \mathbb{N}\}$ *are iid with symmetric continuous positive density* $\phi(\cdot)$ *which is non-increasing on* $\mathbb{R}_+$ *and moments of all order exist.*

$\mathbf{D_2}$) *For some* $m \in \mathbb{N}$ *we assume that* $\mathbf{E}(\alpha_1(1) - a_1)^{2m} = \sigma_1^{2m} \mathbf{E}\eta_1(1)^{2m} \in (1, \infty)$. *In particular,* $\sigma_1 > 0$.

$\mathbf{D_3}$) $\mathbf{E}|\xi_1|^m < \infty$ *for all* $m \in \mathbb{N}$.

$\mathbf{D_4}$) *For every real sequence* $(c_k)_{k\in\mathbb{N}}$ *with* $0 < \sum_{k=1}^{\infty} |c_k| < \infty$, *the random variable*

$$\tau = \sum_{k=1}^{\infty} c_k \, \xi_k$$

has a symmetric density, which is non-increasing on $\mathbb{R}_+$.

Condition $\mathbf{D_4}$ looks rather awkward and complicated to verify. Therefore, we give a simple sufficient condition, which is satisfied by many distributions. The proof is given in Appendix A1.

Proposition 2.3. *If the random variable* ξ_1 *has bounded, symmetric density* f, *which is continuously differentiable on* $[0, \infty)$ *with bounded derivative* $f' \le 0$, *then condition* $\mathbf{D_4}$ *holds.*

The following is our main result.

Theorem 2.4. *Assume model (1.1) such that conditions* $\mathbf{D_0} - \mathbf{D_4}$ *hold and* $a_q^2 + \sigma_q^2 > 0$. *Then the distribution of the vector (2.1) satisfies*

$$\lim_{t\to\infty} t^{\lambda}\mathbf{P}(x'Y > t) = h(x)\,, \quad x \in S\,.$$

The function $h(\cdot)$ *is strictly positive and continuous on* S *and the parameter* λ *is given as the unique positive solution of*

$$\kappa(\lambda) = 1\,, \tag{2.4}$$

where

$$\kappa(\lambda) = \lim_{n\to\infty} \left(\mathbf{E}|A_1\cdots A_n|^{\lambda}\right)^{1/n} \tag{2.5}$$

and the solution of (2.4) satisfies $\lambda > 2$.

The following model describes an important special case.

Definition 2.5. *If in model (1.1) all coefficients and the noise are Gaussian; i.e* $\eta_i(1) \sim \mathcal{N}(0,1)$ *for* $i = 1,\ldots,q$ *and* $\xi_1 \sim \mathcal{N}(0,1)$, *we call the model (1.1) a* Gaussian linear random coefficient model.

The proof of the following result is given in Appendix A2.

Proposition 2.6. *We assume the Gaussian model (1.1) with* $\sigma_1 > 0$. *This process satisfies conditions* $\mathbf{D_1} - \mathbf{D_4}$. *Furthermore, under condition* $\mathbf{D_0}$ *the conditional correlation matrix of* Y *is given by*

$$R = \mathbf{E}(YY'|A_i,\ i\geq 1) = B + \sum_{k=2}^{\infty} A_1\cdots A_{k-1}BA'_{k-1}\cdots A'_1\,, \tag{2.6}$$

where

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdot & 0 \\ . & . & \cdot & . \\ 0 & 0 & \cdot & 0 \end{pmatrix}.$$

Moreoever, R *is positive definite a.s., i.e. the vector* Y *is conditionally non-degenerate Gaussian with finite second moment.*

We show that the Gaussian model is in distribution equivalent to an autoregressive model with uncorrelated Gaussian errors, which we specify as an autoregressive process with ARCH errors, an often used class of models for financial time series.

Lemma 2.7. *Define for the same $q \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\sigma_i \geq 0$ as in model (1.1),*

$$x_n = a_1 x_{n-1} + \cdots + a_q x_{n-q} + \sqrt{1 + \sigma_1^2 x_{n-1}^2 + \cdots + \sigma_q^2 x_{n-q}^2}\, \varepsilon_n\,, \quad n \in \mathbb{N}, \tag{2.7}$$

with the same initial values $(x_0, \ldots, x_{-q+1}) = (y_0, \ldots, y_{-q+1})$ as for the process (1.1). Furthermore, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be iid $\mathcal{N}(0,1)$ random variables with initial values $(x_0, \ldots, x_{-q+1})$ independent of the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Then the stochastic processes $(x_n)_{n\geq 0}$ and the Gaussian linear random coefficient model (1.1) have the same distribution.

Proof. We can rewrite model (1.1) in the form

$$y_n = a_1 y_{n-1} + \ldots + a_q y_{n-q} + \sqrt{1 + \sigma_1^2 y_{n-1}^2 + \ldots + \sigma_q^2 y_{n-q}^2}\, \tilde{\varepsilon}_n\,, \quad n \in \mathbb{N}, \tag{2.8}$$

where

$$\tilde{\varepsilon}_n = \frac{\xi_n + \sigma_1 y_{n-1}\eta_1(n) + \ldots + \sigma_q y_{n-q}\eta_q(n)}{\sqrt{1 + \sigma_1^2 y_{n-1}^2 + \ldots + \sigma_q^2 y_{n-q}^2}}\,, \quad n \in \mathbb{N},$$

are iid $\mathcal{N}(0,1)$. This can be seen by calculating characteristic functions. □

Remark 2.8. For $q = 1$ this model was investigated in Borkovec and Klüppelberg [2] by different, purely analytic methods. Stationarity of the model was shown for $a_1^2 + \sigma_1^2 < 1$. Under quite general conditions on the noise variables, defining

$$\kappa(\lambda) = \mathbf{E}|a_1 + \sigma_1 \varepsilon|^\lambda\,, \tag{2.9}$$

the equation $\kappa(\cdot) = 1$ has a unique positive solution λ and the tail of the stationary random variable x_∞ satisfies

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(x_\infty > t) = c\,.$$

Moreover, this also covers infinite variance cases, i.e. λ can be any positive value. □

Example 2.9. Consider the autoregressive process (2.1) of order 2 with $\sigma_2 = 0$; i.e.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \sqrt{1 + \sigma_1^2 x_{n-1}^2}\, \varepsilon_n\,, \quad n \in \mathbb{N}. \tag{2.10}$$

In this case the corresponding random matrices (1.5) have the following form

$$A_n = \begin{pmatrix} \alpha_1(n) & a_2 \\ 1 & 0 \end{pmatrix}, \quad n \in \mathbb{N}, \tag{2.11}$$

where $\alpha_1(n) = a_1 + \sigma_1 \eta_1(n)$ and

$$\mathbf{E}A_1 \otimes A_1 = \begin{pmatrix} a_1^2 + \sigma_1^2 & a_1 a_2 & a_1 a_2 & a_2^2 \\ a_1 & 0 & a_2 & 0 \\ a_1 & a_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The stationary distribution for the process (2.10) is the distribution of the first component of the two-dimensional random vector (2.1) with matrices (2.11) and vectors $\zeta_n = (\xi_n, 0)'$. Theorem 2.4 applies if $\sigma_1^2 > 0$ and $a_2 \neq 0$.

Kesten proved a similar theorem for the process (1.3) (see Theorem 6 in [14]) under the following condition : there exists $m > 0$ such that $\mathbf{E}\,(\lambda_{\min}(A_1 A_1'))^m \geq 1$, where $\lambda_{\min}(A_1 A_1')$ is the minimal eigenvalue of $A_1 A_1'$. However, we calculate for example (2.10)

$$\lambda_{\min}(A_1 A_1') = \frac{2a_2^2}{\alpha_1^2(1) + a_2^2 + 1 + \sqrt{(\alpha_1^2(1) + a_2^2 + 1)^2 - 4a_2^2}} \leq 2a_2^2 \quad \text{a.s.}$$

which is less than 1 only for $|a_2| < 1/4$.

Notice that for $\sigma_1 = 0$ Theorem 2.4 cannot be applied. In this case the vector Y is Gaussian.

3 Products of random matrices

In this section we investigate the function $\kappa(\lambda)$ as defined in (2.5) for matrices $(A_j)_{j\in\mathbb{N}}$ presented in (1.5) derived from model (1.1). Notice that $A_1 \cdots A_n \stackrel{d}{=} A_n \cdots A_1$ for all $n \in \mathbb{N}$, since the A_j are iid. Furthermore, for any function $f : \mathbb{R}^q \to \mathbb{R}$ we write $f(x') = f(x)$ for all $x \in \mathbb{R}^q$.

Lemma 3.1. *Assume that conditions* $\mathbf{D_1} - \mathbf{D_2}$ *are satisfied and* $a_q^2 + \sigma_q^2 > 0$*. Then the limit* $\kappa(\lambda)$ *in (2.5) exists for every* $\lambda > 0$*.*

Proof. Denote by $\mathbf{B}(S)$ the set of bounded measurable functions and by $\mathbf{C}(S)$ the set of continuous functions on S. Define for $\lambda > 0$,

$$Q_\lambda \;:\; \mathbf{B}(S) \to \mathbf{B}(S) \quad \text{by} \quad Q_\lambda(f)(x) = \mathbf{E}\,|x'A_1|^\lambda f(\overline{x'A_1}) \tag{3.1}$$

for $x \in S$ and $f \in \mathbf{B}(S)$, where $\overline{v} = v/|v|$ for $v \neq 0$. Notice that, if f is continuous, then also $Q_\lambda(f)$ is continuous, i.e. $Q_\lambda \;:\; \mathbf{C}(S) \to \mathbf{C}(S)$.

Denote by $\mathcal{P}(S)$ the set of probability measures on S. Since S is a compact in $\mathbb{R}^q$, $\mathcal{P}(S)$ is a compact convex set with respect to the weak topology. Furthermore, for every probability measure $\sigma \in \mathcal{P}(S)$ we define the measure $T_\sigma \in \mathcal{P}(S)$ by

$$T_\sigma(f) = \int_S f(x) T_\sigma(\mathrm{d}x) = \frac{\int_S Q_\lambda(f)(x)\sigma(\mathrm{d}x)}{\int_S Q_\lambda(e)(x)\sigma(\mathrm{d}x)}\,, \tag{3.2}$$

where $e(x) \equiv 1$, $f \in \mathbf{B}(S)$.

The operator $T_\sigma : \mathcal{P}(S) \to \mathcal{P}(S)$ is continuous with respect to the weak topology and, by the Schauder-Tykhonov theorem (see Dunford and Schwartz [7], p. 450) there exists a fixpoint $\nu \in \mathcal{P}(S)$ such that $T_\nu = \nu$, i.e. $T_\nu(f) = \nu(f)$ for all $f \in \mathbf{B}(S)$. This implies that

$$\int_S Q_\lambda(f)(x)\nu(\mathrm{d}x) = \kappa(\lambda)\int_S f(x)\nu(\mathrm{d}x)\,,$$

where

$$\kappa(\lambda) = \int_S Q_\lambda(e)(x)\nu(\mathrm{d}x).$$

Notice that for all $n \in \mathbb{N}$,

$$\int_S Q_\lambda^{(n)}(f)(x)\nu(\mathrm{d}x) = \kappa^n(\lambda)\int_S f(x)\nu(\mathrm{d}x)\,. \tag{3.3}$$

Here $Q^{(n)}$ is the nth power of the operator Q. From (3.1) follows for every $f \in \mathbf{B}(S)$

$$Q_\lambda^{(n)}(f)(x) = \mathbf{E}\,|x'A_1\cdots A_n|^\lambda f(\overline{x'A_1\cdots A_n})\,, \quad x \in S\,. \tag{3.4}$$

Therefore, by (3.3) we have

$$\kappa^n(\lambda) = \int_S Q_\lambda^{(n)}(e)(x)\nu(\mathrm{d}x) = \int_S \mathbf{E}|x'A_1\cdots A_n|^\lambda \nu(\mathrm{d}x)\,.$$

This implies that

$$\kappa^n(\lambda) \le \mathbf{E}|A_1\cdots A_n|^\lambda\,.$$

On the other hand we have

$$\kappa^n(\lambda) = \mathbf{E}|A_1\cdots A_n|^\lambda \int_S |x'B_n|^\lambda \nu(\mathrm{d}x)\,, \tag{3.5}$$

where $B_n = A_1\cdots A_n/|A_1\cdots A_n|$. We show that

$$c_* = \inf_{|B|=1}\int_S |x'B|^\lambda \nu(\mathrm{d}x) > 0\,. \tag{3.6}$$

Indeed (taking into account that $\int_S |x'B|^\lambda \nu(\mathrm{d}x)$ is a continuous function of B), if $c_* = 0$ there exists B with $|B| = 1$ such that

$$\int_S |x'B|^\lambda \nu(\mathrm{d}x) = 0\,,$$

which means that $\nu\{x \in S : \ x'B \ne 0\} = 0$. Set $\mathcal{N} = \{x \in S : \ x'B = 0\}$ and $g(x) = \chi_{\mathcal{N}^c}$, where $\mathcal{N}^c = S \setminus \mathcal{N}$ and χ_A denotes the indicator function of a set A. If $\mathcal{N} \ne \emptyset$ there exist vectors $b_1 \ne 0, \dots, b_l \ne 0$ with $1 \le l \le q$ such that

$$\mathcal{N} \subset \{x \in \mathbb{R}^q \ : \ x'B = 0\} = \{x \in \mathbb{R}^q \ : \ x'b_1 = 0, \dots, x'b_l = 0\}\,.$$

Furthermore, by (3.3) we obtain for all $n \in \mathbb{N}$

$$\int_S Q_\lambda^{(n)}(g)(x)\nu(\mathrm{d}x) = \kappa^n(\lambda)\int_S g(x)\nu(\mathrm{d}x) = 0\,.$$

By (3.4) this implies for $n = 2q+1$

$$\begin{aligned} &\mathbf{E}\int_S |x'A_1\cdots A_{2q+1}|^\lambda g(\overline{x'A_1\cdots A_{2q+1}})\nu(\mathrm{d}x) \\ &= \int_{\mathcal{N}} \mathbf{E}\,|x'A_1\cdots A_{2q+1}|^\lambda g(\overline{x'A_1\cdots A_{2q+1}})\nu(\mathrm{d}x) = 0\,. \end{aligned}$$

Since $\nu(\mathcal{N}) = 1$ there exists some $x \in \mathcal{N}$ such that $\overline{x'A_1\cdots A_{2q+1}} \in \mathcal{N}$ a.s., i.e. for all $1 \le j \le l$

$$\mathbf{P}(x'A_1\cdots A_{2q+1}b_j = 0) = 1\,.$$

By Lemma A.7 this is only possibly if $b_j = 0$ for all $1 \le j \le l$; i.e if $B = 0$. But this contradicts $|B| = 1$. Thus we obtained (3.6).

Consequently,

$$\mathbf{E}|A_n\cdots A_1|^\lambda \ge \kappa^n(\lambda) = \mathbf{E}|A_n\cdots A_1|^\lambda \int_S |x'B_n|^\lambda\nu(\mathrm{d}x) \ge c_*\mathbf{E}|A_n\cdots A_1|^\lambda\,,$$

i.e.

$$\kappa(\lambda) \le (\mathbf{E}|A_n\cdots A_1|^\lambda)^{1/n} \le \frac{\kappa(\lambda)}{(c_*)^{1/n}}$$

and from this inequality Lemma 3.1 follows by taking the limit as $n \to \infty$. □

Lemma 3.2. *Assume that conditions* $\mathbf{D_0} - \mathbf{D_2}$ *are satisfied and* $a_q^2 + \sigma_q^2 > 0$. *Then equation (2.4) has a unique positive solution.*

Proof. Denote

$$\Psi(n) = A_n\cdots A_1 = (\Psi_{ij}(n))_{i,j=1,\ldots,q}\,.$$

Then

$$\begin{aligned} \Psi_{11}(n) &= \alpha_1(n)\Psi_{11}(n-1) + \ldots + \alpha_q(n)\Psi_{q1}(n-1) \\ &= (\alpha_1(n) - a_1)\Psi_{11}(n-1) + \mu_n\,, \end{aligned}$$

where

$$\mu_n = a_1\Psi_{11}(n-1) + \alpha_2(n)\Psi_{21}(n-1) + \ldots + \alpha_q(n)\Psi_{q1}(n-1)\,,$$

independent of $\eta_1(n)$. By the binomial formula and condition $\mathbf{D_1}$ (which implies that all odd moments of η are equal to zero) we have for arbitrary $m \in \mathbb{N}$ with $C_{2m}^j = \binom{2m}{j}$,

$$\begin{aligned}
\mathbf{E}(\Psi_{11}(n))^{2m} &= \sum_{j=0}^{2m} C_{2m}^j \sigma_1^j \mathbf{E}[\eta_1^j(n)] \mathbf{E}[(\Psi_{11}(n-1))^j \mu_n^{2m-j}] \\
&= \sum_{j=0}^{m} C_{2m}^{2j} \mathbf{E}[(\alpha_1(n) - a_1)^{2j}] \mathbf{E}[(\Psi_{11}(n-1))^{2j} \mu_n^{2(m-j)}] \\
&\geq s(m) \mathbf{E}(\Psi_{11}(n-1))^{2m} ,
\end{aligned}$$

where by $\mathbf{D_2}$

$$s(m) = \mathbf{E}\left(\alpha_1(n) - a_1\right)^{2m} > 1$$

for some $m > 1$. Thus $\mathbf{E}(\Psi_{11}(n))^{2m} \geq s(m)^n$, i.e.

$$\mathbf{E}|\Psi(n)|^{2m} \geq \mathbf{E}(\Psi_{11}(n))^{2m} \geq s(m)^n$$

which implies that $\kappa(2m) = \lim_{n\to\infty} (\mathbf{E}\,|\Psi(n)|^{2m})^{1/n} \geq s(m) > 1$.

We show now that $\log \kappa(\lambda)$ is convex for all $\lambda > 0$ and hence continuous on $\mathbb{R}_+$. To see the convexity, set

$$\varsigma_n(\lambda) = \frac{1}{n} \log \mathbf{E}|\Psi(n)|^{\lambda} , \quad \lambda > 0 ,$$

and recall that $\log \kappa(\lambda) = \lim_{n\to\infty} \varsigma_n(\lambda)$. Then for $\alpha \in (0,1)$ and $\beta = 1 - \alpha$ we obtain by Hölder's inequality for $\lambda, \mu > 0$,

$$\varsigma_n(\alpha\lambda + \beta\mu) \leq \alpha\varsigma_n(\lambda) + \beta\varsigma_n(\mu) .$$

By Remark 2.2(b) condition $\mathbf{D_0}$ implies inequality (2.3) which ensures that $\kappa(\mu) < 1$ for all $0 < \mu \leq 2$. Therefore equation (2.4) has a unique positive root. □

The proof of the following Lemma is a simplification of Step 2 of Theorem 3 of Kesten [15] adapted to model (1.1); see also Le Page [20], Step 2 of Proposition 1.2.

Lemma 3.3. *Assume that conditions $\mathbf{D_1} - \mathbf{D_2}$ are satisfied and $a_q^2 + \sigma_q^2 > 0$. For every $\lambda > 0$ there exists a continuous function $h(\cdot) > 0$ such that for Q_λ as defined in (3.1),*

$$Q_\lambda(h)(x) = \kappa(\lambda) h(x), \qquad x \in S. \tag{3.7}$$

The function h is unique up to a positive constant. Moreover, for $q = 1$ it is independent of x.

Proof. For $q = 1$ we have $S = \{1, -1\}$ and then equation (3.7) is equivalent to the following two equations:

$$\begin{aligned}\kappa(\lambda)\, h(1) &= p\, h(1) + q\, h(-1)\\ \kappa(\lambda)\, h(-1) &= q\, h(1) + p\, h(-1)\,,\end{aligned}$$

where $p = \mathbf{E}\,|\alpha_1|^\lambda\,\chi_{(\alpha_1>0)}$ and $q = \mathbf{E}\,|\alpha_1|^\lambda\,\chi_{(\alpha_1\le 0)}$. By (2.9) we have $\kappa(\lambda) = \mathbf{E}\,|\alpha_1|^\lambda = p+q$. Since by $\mathbf{D}_1$ the random variable α_1 has a positive density, p and q are strictly positive. Hence the solution to this system satisfies $h(1) = h(-1)$, i.e. any solution of (3.7) is constant on S.

For $q \ge 2$ we first recall the notation of the proof of Lemma 3.1, in particular (3.4) and (3.5). Set for $\lambda > 0$

$$s_n(x) = \frac{Q_\lambda^{(n)}(e)(x)}{\kappa^n(\lambda)} = \frac{\mathbf{E}|x'A_1\cdots A_n|^\lambda}{\kappa^n(\lambda)}\,, \quad x\in S\,.$$

Using (3.5)-(3.6) we obtain

$$\sup_{x\in S} s_n(x) \le 1/c_*\,.$$

Notice that for any $(q\times q)$-matrix A and $\lambda > 0$, choosing $\lambda_* = \min(\lambda, 1)$,

$$||x'A|^\lambda - |y'A|^\lambda| \le \max(1,\lambda)|x-y|^{\lambda_*}|A|^\lambda\,, \quad x,y\in S\,,$$

which implies

$$|s_n(x) - s_n(y)| \le (\max(1,\lambda)/c_*)|x-y|^{\lambda_*}\,, \quad x,y\in S\,.$$

By the principle of Arzéla-Ascoli there exists a sequence $(n_k)_{k\in\mathbb{N}}$ with $n_k\to\infty$ as $k\to\infty$ and a continuous function $h(\cdot)$, such that uniformly for $x\in S$,

$$h_k(x) = \frac{1}{n_k}\sum_{j=1}^{n_k} s_j(x) \to h(x)\,,$$

and

$$\begin{aligned}Q_\lambda(h)(x) &= \lim_{k\to\infty} Q_\lambda(h_k)(x) = \lim_{k\to\infty}\frac{1}{n_k}\sum_{j=1}^{n_k} Q_\lambda(s_j)(x)\\ &= \lim_{k\to\infty}\frac{\kappa(\lambda)}{n_k}\sum_{j=1}^{n_k} s_{j+1}(x) = \kappa(\lambda)h(x)\,.\end{aligned}$$

Furthermore, if $h(x) = 0$ for some $x\in S$, then $Q_\lambda^{(n)}(h)(x) = 0$ for all $n\in\mathbb{N}$, i.e.

$$\mathbf{E}|x'A_1\cdots A_n|^\lambda h(x_n) = 0\,,$$

where $x'_n = \overline{x' A_1 \cdots A_n}$, which means that $h(x_n) = 0$ $\mathbf{P}$-a.s. for all $n \in \mathbb{N}$. From Lemma A.11, where $\pi(\cdot)$ denotes the invariant measure of the Markov chain $(x_n)_{n\geq 0}$ we conclude

$$\mathbf{E}_x\, h(x_n) = 0 \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} \mathbf{E}_x\, h(x_n) = \int_S h(z)\pi(\mathrm{d}z) = 0$$
$$\Rightarrow \quad \lim_{k\to\infty} \int_S h_k(z)\pi(\mathrm{d}z) = \int_S h(z)\pi(\mathrm{d}z) = 0\,.$$

But on the other hand

$$\begin{aligned} \int_S h_k(z)\pi(\mathrm{d}z) &= \frac{1}{n_k}\sum_{j=1}^{n_k}\int_S s_j(z)\pi(\mathrm{d}z) \\ &= \frac{1}{n_k}\sum_{j=1}^{n_k}\frac{1}{\kappa^j(\lambda)}\int_S Q_\lambda^{(j)}(e)(z)\pi(\mathrm{d}z) \\ &= \frac{1}{n_k}\sum_{j=1}^{n_k}\frac{1}{\kappa^j(\lambda)}\mathbf{E}|A_1\cdots A_j|^\lambda \int_S \frac{|z'A_1\cdots A_j|^\lambda}{|A_1\cdots A_j|^\lambda}\pi(\mathrm{d}z) \\ &\geq c_1\frac{1}{n_k}\sum_{j=1}^{n_k}\frac{\mathbf{E}|A_1\cdots A_j|^\lambda}{\kappa^j(\lambda)} \geq c_1\,, \end{aligned}$$

where

$$c_1 = \inf_{|B|=1}\int_S |z'B|^\lambda \pi(\mathrm{d}z)\,.$$

Assume that $c_1 = 0$. Then there exists a matrix B with $|B| = 1$ such that $\pi(\mathcal{N}^c \cap S) = 0$ for $\mathcal{N} = \{x \in \mathbb{R}^q \;:\; x'B = 0\}$. Denote by $\Lambda(\cdot)$ the Lebesgue measure on S, then $\Lambda(\mathcal{N}\cap S) = 0$ because $\mathcal{N}$ is a linear subspace of $\mathbb{R}^q$. By Lemma A.11 π is equivalent to Λ; i.e. $\pi(\mathcal{N}\cap S) = 0$. This implies that $\pi(S) = \pi(\mathcal{N}^c\cap S) + \pi(\mathcal{N}\cap S) = 0$, which contradicts $\pi(S) = 1$. Hence $c_1 > 0$ and $h(x) > 0$ for all $x \in S$.

Now assume that there exists some positive function $g \neq h$ satisfying equation (3.7). Define

$$\Pi_n = A_1\cdots A_n\,, \quad n \in \mathbb{N}\,.$$

Then for every $n \in \mathbb{N}$ we have

$$g(x) = \frac{Q_\lambda^{(n)}(g)(x)}{\kappa^n(\lambda)} = \frac{\mathbf{E}\,|x'\Pi_n|^\lambda\, g(\overline{x'\Pi_n})}{\kappa^n(\lambda)} = \frac{h(x)}{\kappa^n(\lambda)}\,\tilde{\mathbf{E}}_x\, f(x'\Pi_n)\,, \quad x \in S\,,$$

where $f(z) = g(\overline{z})/h(\overline{z})$, and for every $n \in \mathbb{N}$,

$$\tilde{\mathbf{E}}_x\, f(x'\Pi_n) = \frac{1}{h(x)}\mathbf{E}\,|x'\Pi_n|^\lambda\, h(\overline{x'\Pi_n})\, f(x'\Pi_n)\,, \quad x \in S\,,$$

i.e. $\tilde{\mathbf{E}}_x$ denotes expectation with respect to the measure defined in (4.3) below. Since the representation for g holds for all $n = 2q+1$, the function g is continuous by Lemma A.9. Define

$$\rho = \sup_{x\in S} \frac{g(x)}{h(x)} = \frac{g(x_0)}{h(x_0)} \quad \text{and} \quad l(x) = \rho h(x) - g(x)\,, \quad x \in S\,.$$

Notice that $l(x) \geq 0$ and $l(x_0) = 0$. Next set

$$L(y) = \frac{l(y)}{h(y)} = \frac{Q_\lambda(l)(y)}{\kappa(\lambda)h(y)} = \ldots = \frac{Q_\lambda^{(n)}(l)(y)}{\kappa^n(\lambda)h(y)} = \frac{Q_\lambda^{(n)}(hL)(y)}{\kappa^n(\lambda)h(y)}\,, \quad y \in S\,.$$

We write

$$\sup_{y\in S} L(y) = L(y_0) = \frac{Q_\lambda^{(n)}(hL)(y_0)}{\kappa^n(\lambda)h(y_0)}\,,$$

equivalently, for $x_n' = \overline{y_0'\Pi_n}$,

$$\mathbf{E}\,|y_0'\Pi_n|^\lambda h(x_n)L(x_n) = L(y_0)h(y_0)\kappa^n(\lambda)\,.$$

Moreover, equation (3.7) implies that $\mathbf{E}\,|y_0'\Pi_n|^\lambda h(x_n) = \kappa^n(\lambda)h(y_0)$ for this sequence $(x_n)_{n\geq 0}$ and therefore

$$\mathbf{E}\,|y_0'\Pi_n|^\lambda h(x_n)(L(y_0) - L(x_n)) = 0\,.$$

Thus, for all $n \in \mathbb{N}$, $L(x_n) = L(y_0)$ $\mathbf{P}$-a.s. and therefore

$$\mathbf{E}_{y_0} L(x_n) = \mathbf{E} L(\overline{y_0'\Pi_n}) = L(y_0)\,.$$

By Lemma A.11, with $\pi(\cdot)$ the invariant measure of $(x_n)_{n\geq 0}$ we get

$$\int_S L(z)\pi(\mathrm{d}z) = \lim_{n\to\infty} \mathbf{E}_{y_0} L(x_n) = L(y_0)\,.$$

Since $L(\cdot)$ is continuous and the measure $\pi(\cdot)$ is equivalent to Lebesgue measure, we have

$$L(y_0) = L(z) = L(x_0) = \frac{l(x_0)}{h(x_0)} = 0\,, \quad z \in S\,.$$

Thus $l(z) = 0$ for all $z \in S$ and Lemma 3.3 follows. $\square$

4 Renewal theorem for the associated Markov chain

The next result is based on the renewal theorem in Klüppelberg and Pergamenchtchikov [16] for the stationary Markov chain $(x_n)_{n\geq 0}$ and the processes $(v_n)_{n\geq 0}$ and $(u_n)_{n\geq 1}$ as defined in (1.9) and (1.10), respectively. Some general properties of $(x_n)_{n\geq 0}$ are summarized in Appendix A4.

Moreover, let $g : S \times \mathbb{R} \to \mathbb{R}$ be a continuous bounded function satisfying

$$\sum_{l=-\infty}^{\infty} \sup_{x \in S} \sup_{l \le t \le l+1} |g(x,t)| < \infty \,. \tag{4.1}$$

The renewal theorem in [16] gives the asymptotic behaviour of the renewal function

$$G(x,t) \;=\; \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k)$$

under the following conditions:

$\mathbf{C_1}$) *For the processes* $(x_n)_{n \ge 0}$ *and* $(u_n)_{n \ge 1}$ *define the* σ*-algebras*

$$\mathcal{F}_0 = \sigma\{x_0\}\,, \quad \mathcal{F}_n = \sigma\{x_0, x_1, u_1, \ldots, x_n, u_n\}\,, \quad n \in \mathbb{N}\,,$$

with some initial value x_0*, which is independent of* $(A_n)_{n \in \mathbb{N}}$*.*
For every bounded measurable function $f \,:\, \Pi_{i=0}^{\infty}(S \times \mathbb{R}) \to \mathbb{R}$ *and for every* $\mathcal{F}_n$*-measurable random variable* η*,*

$$\begin{aligned} & \mathbf{E}(f(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+l}, u_{n+l}, \ldots)|\mathcal{F}_n) \\ & \quad = \quad \mathbf{E}_{x_n} f(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+l}, u_{n+l}, \ldots) \\ & \quad =: \quad \Phi(x_n, \eta)\,, \end{aligned} \tag{4.2}$$

i.e. $\Phi(x,a) = \mathbf{E}_x f(a, x_1, u_1, \ldots, x_l, u_l, \ldots)$ *for all* $x \in S$ *and* $a \in \mathbb{R}$*. Moreover, if for* $m \in \mathbb{N}$ *the function* $f \,:\, (S \times \mathbb{R})^m \to \mathbb{R}$ *is continuous then* $\Phi(x) = \mathbf{E}_x\, f(x_1, u_1, \ldots, x_m, u_m)$ *is continuous on* S*.*

$\mathbf{C_2}$) *There exists a probability measure* $\pi(\cdot)$ *on* S*, which is equivalent to Lebesgue measure such that*

$$\|\mathbf{P}_x^{(n)}(\cdot) - \pi(\cdot)\| \to 0\,, \quad n \to \infty\,,$$

for all $x \in S$*, where* $\|\mu\| = \sup_{|f| \le 1} \int_S f(y)\mu(dy)$ *denotes total variation of any measures* μ *on* S*. Moreover, there exists a constant* $\beta > 0$ *such that for all* $x \in S$

$$\lim_{n \to \infty} \frac{v_n}{n} = \beta \qquad \mathbf{P}_x - a.s.$$

$\mathbf{C_3}$) *There exists some number* $m \in \mathbb{N}$ *such that for all* $\nu \in \mathbb{R}$ *and for all* $\delta > 0$ *there exist* $y_{\nu,\delta} \in S$ *and* $\varepsilon_0 = \varepsilon_0(\nu, \delta) > 0$ *such that* $\forall\ 0 < \varepsilon < \varepsilon_0$

$$\inf_{x \in B_{\delta,\nu}} \mathbf{P}_x(|x_m - y_{\nu,\delta}| < \varepsilon\,,\, |v_m - \nu| < \delta) > 0\,,$$

where $B_{\delta,\nu} = \{x \in S :\ |x - y_{\nu,\delta}| < \delta\}$*.*

$\mathbf{C_4}$) *There exists some* $l \in \mathbb{N}$ *such that the function* $\Phi_1(x,t) = \mathbf{E}_x\, \Phi(x_l, v_l, t)$ *satisfies*

$$\sup_{|x-y|<\varepsilon} \sup_{t\in\mathbb{R}} |\Phi_1(x,t) - \Phi_1(y,t)| \to 0\,, \quad \varepsilon \to 0\,,$$

for every bounded measurable function $\Phi \;:\; S \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 4.1. (Klüppelberg and Pergamenchtchikov [16])
Assume that conditions $\mathbf{C_1} - \mathbf{C_4}$ *are satisfied. Then for any function* g *satisfying (4.1)*

$$\lim_{t\to\infty} G(x,t) = \lim_{t\to\infty} \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k) = \frac{1}{\beta} \int_S \pi(dx) \int_{-\infty}^{\infty} g(x,t)\, dt.$$

We apply this renewal theorem to

$$G(x,t) = \frac{1}{e^t} \int_0^{e^t} u^\lambda \mathbf{P}(x'Y > u) \mathrm{d}u\,, \quad x \in S\,, t \in \mathbb{R}\,,$$

where the vector Y is given by (2.1) and λ is the unique positive solution of (2.4).

This definition corresponds to an exponential change of measure, equivalently, to an exponential tilting of the bivariate Markov process $(x_n, v_n)_{n\geq 0}$ as follows. Denote by $\tilde{\mathbf{E}}_x$ the expectation with respect to the probability measure $\tilde{\mathbf{P}}_x$, which is defined by

$$\tilde{\mathbf{E}}_x\, F(x_1, u_1, \ldots, x_n, u_n) = \frac{1}{h(x)} \mathbf{E} |x' A_1 \cdots A_n|^\lambda h(x_n) F(x_1, u_1, \ldots, x_n, u_n) \tag{4.3}$$

for each measurable function F. Then $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ are the corresponding quantities (as $\mathbf{P}$ and $\mathbf{E}$ are for $(x_n, v_n)_{n\geq 0}$) of the Markov chain $(\tilde{x}_n, \tilde{v}_n)_{n\geq 0}$ defined by the n-step transition densities

$$\tilde{p}^{(n)}_{x,v}(\mathrm{d}y, \mathrm{d}w) = \frac{e^{\lambda w} h(y)}{e^{\lambda v} h(x)} p^{(n)}_{x,v}(\mathrm{d}y, \mathrm{d}w)\,,$$

where $p^{(n)}_{x,v}(\mathrm{d}y, \mathrm{d}w)$ is the n-step transition density of the original Markov chain $(x_n, v_n)_{n\geq 0}$.

In order to apply Theorem 4.1 we need to check conditions $\mathbf{C_1} - \mathbf{C_4}$.

However, before we treat the general case for arbitrary dimension q, we consider the case $q = 1$ in the next example.

Example 4.2. Consider model (1.1) for $q = 1$ and $0 < a_1^2 + \sigma_1^2 < 1$, den $\mathbf{D_0}$ holds. Define $(x_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ and $(u_n)_{n\in\mathbb{N}}$ as in (1.9) and (1.10), respectively. Assume that conditions $\mathbf{D_1} - \mathbf{D_2}$ are satisfied.

First note that in this case the function $\kappa(\cdot)$ is defined by (2.9), and Lemma 3.2 implies that equation $\kappa(\lambda) = 1$ has a unique positive solution. From Lemma 3.3 we conclude that only constant functions satisfy equation (3.7), and we simply set $h(x) = 1$ in (4.3).

This case is special in the sense that $S = \{1, -1\}$, i.e. the sphere degenerates to two points, and we define the "Lebesgue measure" on S as any point measure with $\Lambda(1) > 0$ and $\Lambda(-1) > 0$. We show now directly that the Markov chain $(x_n)_{n\geq 1}$ (defined in (1.9)) is uniform geometric ergodic with unique invariant distribution $\pi = \tilde{\pi} = (1/2, 1/2)$ with respect to both measures $\mathbf{P}$ and $\tilde{\mathbf{P}}$. To this end we calculate the limit of the nth power transition matrix. Setting $p = \mathbf{P}(\alpha_1 > 0)$ (for $\mathbf{P}$) and $p = \mathbf{E}\,|\alpha_1|^\lambda \chi_{(\alpha_1>0)}$ (for $\tilde{\mathbf{P}}$), the transition matrix has the form

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2p-1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Since for any $p \in (0,1)$ we have $(2p-1)^n \to 0$, we conclude that P^n converges to the matrix with all entries equal to $1/2$. Obviously, for any $0 < p < 1$ the matrix P is irreducible. Therefore, by the ergodic theorem for finite Markov chains we can directly (without Lemma A.10) conclude that condition $\mathbf{C}_2$ holds with $\beta = \mathbf{E}\,|\alpha_1|^\lambda \log|\alpha_1|$, which is positive (cf. Goldie [10], Lemma 2.2).

To prove condition $\mathbf{C}_3$ for the measure $\tilde{\mathbf{P}}$ set $m = 1$ and $y_{\nu,\delta} = 1$ for $\nu > 0$ and $\delta > 0$. Then $B_{\nu,\delta} \supseteq \{1\}$. Therefore, for every $0 < \varepsilon < 1$

$$\begin{aligned} \inf_{x\in B_{\nu,\delta}} \tilde{\mathbf{P}}_x(|x_1 - y_{\nu,\delta}| < \varepsilon\,, |v_1 - \nu| < \delta) &\geq \tilde{\mathbf{P}}_1(x_1 = 1\,, |v_1 - \nu| < \delta) \\ = \mathbf{E}\,|\alpha_1|^\lambda \chi_{(\alpha_1>0\,,|\log|\alpha_1|-\nu|<\delta)} &= \mathbf{E}\,|\alpha_1|^\lambda \chi_{(e^{\nu-\delta}<\alpha_1<e^{\nu+\delta})} > 0, \end{aligned}$$

since by $\mathbf{D}_1$ the random variable α_1 has a positive density.

Proposition 4.3. *Consider model (1.1) with $(x_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ and $(u_n)_{n\in\mathbb{N}}$ defined in (1.9) and (1.10), respectively. Assume that conditions $\mathbf{D_0} - \mathbf{D_2}$ are satisfied and $a_q^2 + \sigma_q^2 > 0$. Then conditions $\mathbf{C_1} - \mathbf{C_4}$ hold with respect to the measure $\tilde{\mathbf{P}}_x$ generated by the finite dimensional distributions (4.3).*

Proof. First recall $\Pi_n = A_1 \cdots A_n$ and $x'_n = \overline{x'\Pi_n} = x'\Pi_n/|x'\Pi_n|$ and $v'_n = \log|x'\Pi_n|$. For every bounded measurable function $\Phi(x_n, v_n, t) = f(x'\Pi_n, t)$, with $f(z,t) = \Phi(\overline{z}, \log|z|, t)$ we have by Lemma A.9 immediately that condition $\mathbf{C_4}$ holds.

Next we check $\mathbf{C_1}$. For $n, l \in \mathbb{N}$ we have

$$x'_{n+l} = \frac{x'_n A_{n+1}\cdots A_{n+l}}{|x'_n A_{n+1}\cdots A_{n+l}|} = h_l(x_n, A_{n+1}, \ldots, A_{n+l})$$

and

$$u_{n+l} = \log|x'_{n+l-1}A_{n+l}| = \log|\frac{x'_n A_{n+l-1}\cdots A_{n+1}}{|x'_n A_{n+l-1}\cdots A_{n+1}|}A_{n+l}| = g_l(x_n, A_{n+1}, \ldots, A_{n+l})\,.$$

Now for every function $f\ :\ \Pi_{i=0}^{\infty}(S\times\mathbb{R})\ \to\ \mathbb{R}$ and some η - $\mathcal{F}_n$ measurable random variable we calculate

$$
\begin{aligned}
& f(\eta, x_{n+1}, u_{n+1}, ..., x_{n+l}, u_{n+l}, ...)\\
&\quad = f(\eta, h_1(x_n, A_{n+1}), g_1(x_n, A_{n+1}), \ldots,\\
&\qquad h_l(x_n, A_{n+1}, \ldots, A_{n+l}), g_l(x_n, A_{n+1}, \ldots, A_{n+l}), \ldots)\\
&\quad = f_1(\eta, x_n, A_{n+1}, \ldots, A_{n+l}, \ldots)\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathbf{E}(f(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+l}, u_{n+l}, \ldots)|\mathcal{F}_n)\\
&\quad = \mathbf{E}(f_1(\eta, x_n, A_{n+1}, \ldots, A_{n+l}, \ldots)|\mathcal{F}_n)\\
&\quad = \Phi(x_n, \eta_n)\,,
\end{aligned}
$$

where (notice that (η_n, x_n) is independent of $(A_{n+1}, \ldots, A_{n+l}, \ldots)$)

$$
\begin{aligned}
\Phi(x,a) &= \mathbf{E}f_1(a, x, A_{n+1}, \ldots, A_{n+l}, \ldots)\\
&= \mathbf{E}f_1(a, x, A_1, \ldots, A_l, \ldots)\\
&= \mathbf{E}f(a, h_1(x, A_1), g_1(x, A_1), \ldots, h_l(x, A_1, \ldots, A_l), g_l(x, A_1, \ldots, A_l), \ldots)\\
&= \mathbf{E}_x f(a, x_1, u_1, \ldots, x_l, u_l, \ldots)\,.
\end{aligned}
$$

From this and (4.3) we get for every $m \in \mathbb{N}$ and every bounded function $f_m\ :\ \mathbb{R}\times(S\times\mathbb{R})^m\ \to\ \mathbb{R}$

$$
\tilde{\mathbf{E}}_x(f_m(\eta, x_{n+1}, u_{n+1}, \ldots, x_{n+m}, u_{n+m})|\mathcal{F}_n) = \Phi_m(\eta, x_n)\,, \tag{4.4}
$$

where $\Phi_m(a,x) = \tilde{\mathbf{E}}_x(f_m(a, x_1, u_1, \ldots, x_m, u_m)$.

Denote by μ_x the measure on the cylindric σ-algebra $\mathcal{B}$ in $\Pi_{i=0}^{\infty}(S\times\mathbb{R})$ generated by the finite dimensional distributions of $(x_1, u_1, \ldots, x_k, u_k)$ (defined by (4.3) with initial value x) on $\mathcal{B}_k$, where $\mathcal{B}_k$ is the Borel σ-algebra on $(S\times\mathbb{R})^k$ and $\mathcal{B} = \sigma\{\cup_{k=1}^{\infty}\mathcal{B}_k\}$. Let furthermore $\mu_{x|\mathcal{F}_n}$ be the conditional (on $\mathcal{F}_n$) infinite dimensional distribution of $(x_{n+1}, u_{n+1}, \ldots, x_{n+k}, u_{n+k}, \ldots)$. Equality (4.4) implies that the finite dimensional distributions of the measure $\mu_{x|\mathcal{F}_n}$ coincide with the finite dimensional distributions of the measure μ_{x_n}; i.e. $\mu_{x|\mathcal{F}_n} \equiv \mu_{x_n}$ on $\mathcal{B}$. This implies (4.2) for the measure defined in (4.3). Furthermore, the definitions of $(x_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ imply that for every continuous f also $\Phi(x) = \tilde{\mathbf{E}}_x\, f(x_1, v_1, \ldots, x_m, v_m)$ is continuous in $x \in \mathbb{R}$. Hence condition $\mathbf{C_1}$ holds.

Next we check condition $\mathbf{C_2}$ for $q \geq 2$. The case $q = 1$ has been treated in Example 4.2. We first show

$$
\sup_{x\in S}\, \tilde{\mathbf{E}}_x\,(\log\,|x' A_1|)^2 < \infty\,. \tag{4.5}
$$

To see this notice that for every $\lambda > 0$

$$\sup_{x\in\mathbb{R}} \frac{|x|^\lambda (\log|x|)^2}{1+|x|^{\lambda+1}} =: c^* < \infty\,.$$

Hence for every $x \in S$

$$\begin{aligned}\tilde{\mathbf{E}}_x (\log|x'A_1|)^2 &= \frac{1}{h(x)}\mathbf{E}\,|x'A_1|^\lambda\, h(\overline{x'A_1})\,(\log|x'A_1|)^2 \\ &\le c^*\frac{h^*}{h_*}(1+\mathbf{E}\,|x'A_1|^{\lambda+1}) \\ &\le c^*\frac{h^*}{h_*}(1+\mathbf{E}\,|A_1|^{\lambda+1}) < \infty\,,\end{aligned}$$

where $h_* = \inf_{x\in S} h(x)$ and $h^* = \sup_{x\in S} h(x)$. This implies (4.5).

Define

$$f(x) = \frac{1}{h(x)}\mathbf{E}\,|x'A_1|^\lambda \log|x'A_1| h(\overline{x'A_1}) = \tilde{\mathbf{E}}_x \log|x'A_1|\,,$$

and

$$m_k = \log|x'_{k-1}A_k| - \tilde{\mathbf{E}}_x\left(\log|x'_{k-1}A_k||\mathcal{F}_{k-1}\right) = \log|x'_{k-1}A_k| - f(x_{k-1})\,,$$

then

$$\frac{v_n}{n} = \frac{1}{n}\sum_{k=1}^n f(x_{k-1}) + \frac{1}{n}\sum_{k=1}^n m_k\,, \quad n\in\mathbb{N}. \tag{4.6}$$

By the strong law of large numbers for square integrable martingales and (4.5) the last term in (4.6) converges to zero $\tilde{\mathbf{P}}$-a.s.

By Lemma A.11 $(x_n)_{n\in\mathbb{N}}$ is positive Harris recurrent with respect to the measure $\tilde{\mathbf{P}}$ as defined in (4.3). Hence we can apply the ergodic theorem to the first term of the right-hand side of (4.6) (see Theorem 17.0.1, p. 411 in [17]). This term then converges to the expectation of f with respect to the invariant measure $\tilde{\pi}$:

$$\lim_{n\to\infty}\frac{v_n}{n} = \beta = \int_S \tilde{\pi}(\mathrm{d}z)\frac{1}{h(z)}\mathbf{E}|z'A_1|^\lambda \log|z'A_1| h(\overline{z'A_1})\,, \quad \tilde{\pi}\ -\ \text{a.s.}. \tag{4.7}$$

This implies

$$\int_S \tilde{\mathbf{P}}_x(\lim_{n\to\infty}\frac{v_n}{n} = \beta)\tilde{\pi}(\mathrm{d}x) = 1\,.$$

By Lemma A.11 the measure $\tilde{\pi}$ is equivalent to Lebesgue measure, hence

$$\tilde{\mathbf{P}}_x(\lim_{n\to\infty}\frac{v_n}{n} = \beta) = 1 \tag{4.8}$$

for Λ-almost all $x \in S$. From condition $\mathbf{C_1}$ we conclude

$$\tilde{\mathbf{P}}_x(\lim_{n\to\infty}\frac{v_n}{n} = \beta) = \tilde{\mathbf{E}}_x f(x_l, v_l)\,,$$

where $l = 2q + 1$ and
$$f(x, v) = \tilde{\mathbf{P}}_x(\lim_{n\to\infty} \frac{v_n + v}{n} = \beta).$$
By condition $\mathbf{C_4}$ the function $\tilde{\mathbf{P}}_x(\lim_{n\to\infty} \frac{v_n}{n} = \beta)$ is continuous on S and therefore (4.8) holds for all $x \in S$.

It remains to show that the constant β in (4.7) is positive. By inequality (2.3) there exist $c > 0$ and $\gamma > 0$ such that
$$\mathbf{E}\,|\Pi_n|^2 \le c\,e^{-\gamma n}\,.$$
Choose $\delta > 0$ such that $d = \gamma - 2\delta > 0$. Then by Chebyshev's inequality,
$$\mathbf{P}(|x'\Pi_n| \ge e^{-\delta n}) \le e^{2\delta n}\,\mathbf{E}\,|x'\Pi_n|^2 \le e^{2\delta n}\,\mathbf{E}\,|\Pi_n|^2 \le c\,e^{-dn}\,.$$
Moreover, for every $0 < \rho < d/\lambda$ and $x'_n = \overline{x'\Pi_n}$ we have
$$\begin{aligned}\tilde{\mathbf{P}}_x(|x'\Pi_n| < e^{\rho n}) &= \frac{1}{h(x)}\,\mathbf{E}\,|x'\Pi_n|^{\lambda} h(x_n)\chi_{\{|x'\Pi_n|<e^{\rho n}\}} \\ &\le \frac{h^*}{h_*}\left(e^{-\lambda\delta n} + \mathbf{E}|x'\Pi_n|^{\lambda}\chi_{\{e^{-\delta n}\le|x'\Pi_n|<e^{\rho n}\}}\right) \\ &\le \frac{h^*}{h_*}\left(e^{-\lambda\delta n} + e^{\lambda\rho n}\mathbf{P}(|x'\Pi_n| \ge e^{-\delta n})\right) \\ &\le \frac{h^*}{h_*}(e^{-\lambda\delta n} + c\,e^{-(d-\lambda\rho)n})\,.\end{aligned}$$
By the Lemma of Borel-Cantelli we conclude that for all $x \in S$
$$\lim_{n\to\infty} \frac{v_n}{n} \ge \rho > 0 \quad \tilde{\mathbf{P}}_x - \text{a.s.}$$
This verifies condition $\mathbf{C_2}$.

Finally, we check condition $\mathbf{C_3}$ for $q \ge 2$. The case $q = 1$ has already been treated in Example 4.2. We shall show that for $m = 2q + 1$ and $\forall \nu \in \mathbb{R}$, $\forall \delta > 0$, $\forall y \in S$, $\forall \varepsilon > 0$,
$$\inf_{x\in S} \tilde{\mathbf{P}}_x(|x_m - y| < \varepsilon\,,\ |v_m - \nu| < \delta) > 0. \tag{4.9}$$
Indeed, with $L(z) = z/|z|$, consider
$$\begin{aligned}\tilde{\mathbf{P}}_x(|x_m - y| < \varepsilon\,,\ |v_m - \nu| < \delta) &= \tilde{\mathbf{P}}_x(|L(x'\Pi_m) - y| < \varepsilon\,,\ |\log|x'\Pi_m| - \nu| < \delta) \\ &= \tilde{\mathbf{P}}_x(x'\Pi_m \in \Gamma_{y,\varepsilon,\delta})\,,\end{aligned}$$
where
$$\Gamma_{y,\varepsilon,\delta} = \{z \in \mathbb{R}^q \setminus \{0\}\ :\ |L(z) - y| < \varepsilon\ ,\ \ |\log|z| - \nu| < \delta)\}\,.$$

For every $y \in S$ and every $\nu \in \mathbb{R}$ this set is a non-empty open set in $\mathbb{R}^q$, because the vector $z_0 = e^\nu y \in \Gamma_{y,\varepsilon,\delta}$ ($\forall \nu \in \mathbb{R}$, $\forall \delta > 0$, $\forall y \in S$, $\forall \varepsilon > 0$). This implies that the Lebesgue measure of $\Gamma_{y,\varepsilon,\delta}$ is positive. By Lemma A.8 we conclude that

$$\inf_{x \in S} \tilde{\mathbf{P}}_x(x'\Pi_m \in \Gamma_{y,\varepsilon,\delta}) > 0\,.$$

This ensures (4.9), which implies $\mathbf{C_3}$. □

Define

$$\tilde{G}(x,t) = \frac{G(x,t)}{h(x)}\,,$$

where $h(\cdot) > 0$ satisfies equation (3.7) with positive λ for which $\kappa(\lambda) = 1$. Further, recall that by Remark 2.2

$$Y \stackrel{d}{=} A_1 Y_1 + \zeta_1\,,$$

where $Y_1 = \zeta_2 + \sum_{k=3}^{\infty} A_2 \cdots A_{k-1}\zeta_k$ is independent of (A_1, ζ_1) and $Y_1 \stackrel{d}{=} Y$. Therefore,

$$\begin{aligned} \tilde{G}(x,t) &= \frac{1}{h(x)e^t}\int_0^{e^t} u^\lambda \mathbf{P}(x'A_1Y_1 + x'\zeta_1 > u)\mathrm{d}u \\ &=: \tilde{g}(x,t) + \psi(x,t)\,, \end{aligned} \tag{4.10}$$

where, setting $\tau_1 = x'A_1Y_1$ and $\tau_2 = x'\zeta_1$,

$$\tilde{g}(x,t) = \frac{1}{h(x)e^t}\int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u)\mathrm{d}u\,, \tag{4.11}$$

$$\psi(x,t) = \frac{1}{h(x)e^t}\int_0^{e^t} u^\lambda \psi_0(x,u)\mathrm{d}u\,, \tag{4.12}$$

$$\psi_0(x,u) = \mathbf{P}(\tau_1 + \tau_2 > u) - \mathbf{P}(\tau_1 > u)\,. \tag{4.13}$$

Proposition 4.4. *Assume that conditions $\mathbf{D_0} - \mathbf{D_2}$ are satisfied and $a_q^2 + \sigma_q^2 > 0$. Then*

$$\tilde{G}(x,t) = \sum_{n=0}^{\infty} \tilde{\mathbf{E}}_x \psi(x_n, t - v_n)\,. \tag{4.14}$$

Proof. Lemmata 3.1–3.3 ensure the existence of positive solutions of equations (2.4) and (3.7) which are used in the definition of the measure $\tilde{\mathbf{P}}$ in (4.3). Now consider first $\tilde{g}(x,t)$ as defined in (4.11). Mapping $u \mapsto u/|x'A_1|$ and using $x_1' = x'A_1/|x'A_1|$, we obtain

$$\tilde{g}(x,t) = \mathbf{E}\frac{|x'A_1|^\lambda}{h(x)e^{t-\log|x'A_1|}}\int_0^{e^t/|x'A_1|} u^\lambda \mathbf{P}(x_1'Y > u)\mathrm{d}u = \tilde{\mathbf{E}}_x \tilde{G}(x_1, t - \log|x'A_1|)\,.$$

Let $\mathbf{B}(S \times \mathbb{R})$ be a linear space of bounded measurable functions $S \times \mathbb{R} \to \mathbb{R}$. Define the linear operator $\Theta : \mathbf{B}(S \times \mathbb{R}) \to \mathbf{B}(S \times \mathbb{R})$ by

$$\Theta(f)(x,t) = \tilde{\mathbf{E}}_x f(x_1, t - v_1)\,, \tag{4.15}$$

where we have used that $v_1 = u_1 = \log|x'A_1|$. Next recall that by Proposition 4.3 condition $\mathbf{C}_1$ holds for the measure (4.3). This implies that the nth power of the operator Θ is defined by

$$\Theta^{(n)}(f)(x,t) = \tilde{\mathbf{E}}_x f(x_n, t - v_n)\,.$$

Then equation (4.10) translates into

$$\tilde{G}(x,t) = \Theta(\tilde{G})(x,t) + \psi(x,t)\,,$$

and we obtain for all $n \in \mathbb{N}$ from (4.5) iteratively,

$$\tilde{G}(x,t) = \Theta^{(n)}(\tilde{G})(x,t) + \psi(x,t) + \Theta(\psi)(x,t) + \ldots + \Theta^{(n-1)}(\psi)(x,t).$$

Moreover, condition $\mathbf{D_0}$ implies $\lim_{n\to\infty} \mathbf{E}|\Pi_n| = 0$ giving

$$\begin{aligned}\Theta^{(n)}(\tilde{G})(x,t) &= \tilde{\mathbf{E}}_x \tilde{G}(x_n, t - v_n)\\ &= \frac{1}{h(x)e^t}\int_0^{e^t} u^\lambda\, \mathbf{P}\left(x'\Pi_n Y > u\right) \mathrm{d}u \to 0\,, \quad n \to \infty\,.\end{aligned}$$

This implies (4.14). □

Lemma 4.5. *Assume the conditions of Theorem 2.4. Then for every $x \in S$ there exists*

$$\lim_{t\to\infty} G(x,t) = h(x)\frac{1}{\beta}\int_S \tilde{\pi}(dz)\frac{1}{h(z)}\int_0^\infty u^{\lambda-1}\psi_0(z,u)\,du = h(x)\,\gamma^* > 0\,. \tag{4.16}$$

Here $h(\cdot) > 0$ satisfies equation (3.7) with positive λ for which $\kappa(\lambda) = 1$, $\beta > 0$ is defined in (4.7) and $\tilde{\pi}(\cdot)$ is the stationary measure of the Markov process $(x_n)_{n\geq 0}$ under the distribution $\tilde{\mathbf{P}}$ as defined in (4.3).

Proof. By Proposition 4.4 it suffices to find the limit for the sum in (4.14). We apply Theorem 4.1 to (4.14). Conditions $\mathbf{C_1} - \mathbf{C_4}$ hold for $q \geq 1$ by Example 4.2 and Proposition 4.3.

It remains to show that the function ψ given by (4.12) satisfies condition (4.1). By Lemma A.12 follows that $\psi(x,t) \geq 0$ and therefore

$$\psi(x,t) \leq \frac{1}{h_*}(\psi_1^*(x,t) + \psi_2^*(x,t)),$$

where $h_* = \min_{x\in S} h(x)$ and, with $n(t) = e^{\mu t}$ for some $\mu > 0$,

$$\begin{aligned}\psi_1^*(x,t) &= \frac{1}{e^t}\int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u - n(t))\mathrm{d}u - \frac{1}{e^t}\int_0^{e^t} u^\lambda \mathbf{P}(\tau_1 > u)\mathrm{d}u\,,\\ \psi_2^*(x,t) &= \frac{e^{\lambda t}}{\lambda+1}\mathbf{P}(\tau_2 > n(t))\,.\end{aligned}$$

We show that the functions $\psi_i^*(x,t)$ satisfy for sufficiently large $t>0$ the inequality

$$\psi_i^*(x,t)\le ce^{-c_1 t} \tag{4.17}$$

for constants $c, c_1>0$. First notice that immediately by Lemma 3.2 we have $\kappa(\theta)<1$ for every $1<\theta<\lambda$. Hence by the defintion of $\kappa(\theta)$ in (2.5), for every $\nu\in(\kappa(\theta)\,,1)$, there exists some $C=C_\nu>0$ such that for all $n\in\mathbb{N}$

$$\mathbf{E}\,|A_1\cdots A_n|^\theta\le C\nu^n\,.$$

From this and Hölder's inequality we obtain for arbitrary $\rho>0$

$$\begin{aligned}
\mathbf{E}\,|\tau_1|^\theta &\le \mathbf{E}\,|A_1|^\theta\mathbf{E}|Y_1|^\theta\\
&\le 2^{\theta-1}\mathbf{E}\,|A_1|^\theta\left(\mathbf{E}|\xi_1|^\theta+\mathbf{E}\left(\sum_{k=3}^\infty|A_2\cdots A_{k-1}||\xi_k|\right)^\theta\right)\\
&\le 2^{\theta-1}\mathbf{E}\,|A_1|^\theta\left(\mathbf{E}\,|\xi_1|^\theta+\mathbf{E}\,|\xi_1|^\theta\sum_{k=3}^\infty\rho^{-\theta(k-2)}\mathbf{E}|A_2\cdots A_{k-1}|^\theta\left(\sum_{k=3}^\infty\rho^{\theta(k-2)/(\theta-1)}\right)^{\theta-1}\right)\\
&\le 2^{\theta-1}\mathbf{E}|A_1|^\theta\left(\mathbf{E}\,|\xi_1|^\theta+C\mathbf{E}\,|\xi_1|^\theta\sum_{k=3}^\infty\rho^{-\theta(k-2)}\nu^{k-2}\left(\sum_{k=3}^\infty\rho^{\theta(k-2)/(\theta-1)}\right)^{\theta-1}\right).
\end{aligned}$$

Now choose in the last term $\rho=\nu^{1/(2\theta)}$. Then for every $1<\theta<\lambda$ there exists some $m(\theta)>0$ such that

$$\sup_{x\in S}\mathbf{E}\,|\tau_1|^\theta=\sup_{x\in S}\mathbf{E}|x'A_1Y_1|^\theta<m(\theta)<\infty\,. \tag{4.18}$$

This means that we can find some $\theta<\lambda$ such that (4.18) holds and $\delta=\lambda-\theta$ is arbitrarily small. Keeping this in mind we study now the function $\psi_1^*(x,t)$. Indeed, for sufficiently large $t>0$ we have

$$\begin{aligned}
\psi_1^*(x,t) &\le \frac{1}{e^t}\int_0^{e^t-n(t)}(n(t)+u)^\lambda\mathbf{P}(\tau_1>u)\mathrm{d}u-\frac{1}{e^t}\int_0^{e^t}u^\lambda\mathbf{P}(\tau_1>u)\mathrm{d}u+\frac{(n(t))^{\lambda+1}}{e^t}\\
&\le (2^\lambda+1)\frac{(n(t))^{\lambda+1}}{e^t}+\frac{1}{e^t}\int_{n(t)}^{e^t-n(t)}u^\lambda\left(\left(1+\frac{n(t)}{u}\right)^\lambda-1\right)\mathbf{P}(\tau_1>u)\mathrm{d}u\\
&\le (2^\lambda+1)\frac{(n(t))^{\lambda+1}}{e^t}+M^*\frac{n(t)}{e^t}\int_{n(t)}^{e^t-n(t)}u^{\lambda-\theta-1}\mathrm{d}u\,\mathbf{E}\,|\tau_1|^\theta\\
&\le (2^\lambda+1)\frac{(n(t))^{\lambda+1}}{e^t}+M^*\frac{m(\theta)n(t)}{e^t}\int_{n(t)}^{e^t-n(t)}u^{\delta-1}\mathrm{d}u\\
&\le (2^\lambda+1)\frac{(n(t))^{\lambda+1}}{e^t}+M^*\frac{m(\theta)n(t)}{\delta\,e^{(1-\delta)t}}\\
&\le (2^\lambda+1)e^{-(1-\mu(\lambda+1))t}+\frac{M^*m(\theta)}{\delta}\,e^{-(1-\delta-\mu)t}\,,
\end{aligned}$$

where $M^* = \sup_{0<x\le 1}((1+x)^\lambda - 1)/x$ and $c > 0$ is some constant. To obtain (4.17) for the function $\psi_1^*(x,t)$ choose the parameters δ and μ such that $\delta + \mu < 1$ and $0 < \mu < (1+\lambda)^{-1}$. The function $\psi_2^*(x,t)$ satisfies inequality (4.17), because for every $m > 0$ by condition $\mathbf{D_3}$,

$$\sup_{x\in S} \mathbf{E}|\tau_2|^m = \sup_{x\in S} \mathbf{E}| < x >_1 \xi_1|^m \le \mathbf{E}|\xi_1|^m < \infty\,.$$

On the other hand, if $t \to -\infty$, we have immediately from definition (4.13),

$$\psi(x,t) \le \frac{1}{h_* e^t} \int_0^{e^t} u^\lambda \mathrm{d}u \le \frac{1}{h_*} e^{\lambda t}$$

and hence condition (4.1) holds.

Furthermore, taking into account that $\tilde{\pi}$ is equivalent to Lebesgue measure Λ on S (see Lemma A.11), by Theorem 4.1 and Lemma A.12 we conclude

$$\begin{aligned}
\lim_{t\to\infty} \frac{G(x,t)}{h(x)} = \lim_{t\to\infty} \tilde{G}(x,t) &= \frac{1}{\beta}\int_S \tilde{\pi}(\mathrm{d}z) \int_{-\infty}^{+\infty} \psi(z,s)\,\mathrm{d}s \\
&= \frac{1}{\beta}\int_S \tilde{\pi}(\mathrm{d}z) \frac{1}{h(z)} \int_{-\infty}^{+\infty} \frac{1}{e^s} \int_0^{e^s} u^\lambda \psi_0(z,u)\,\mathrm{d}u\,\mathrm{d}s \\
&= \frac{1}{\beta}\int_S \tilde{\pi}(\mathrm{d}z) \frac{1}{h(z)} \int_0^{+\infty} u^{\lambda-1}\psi_0(z,u)\,\mathrm{d}u \\
&= \gamma^* > 0\,.
\end{aligned}$$

□

Lemma 4.6. *Assume the conditions of Theorem 2.4. Then for every $x \in S$ there exists*

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(x'Y > t) = \gamma^*\, h(x) > 0\,,$$

with $h(\cdot)$ and γ^* as in Lemma 4.5.

Proof. We use a similar argument as for the proof of the monotone density theorem in regular variation (see e.g. Bingham, Goldie and Teugels [1], Theorem 1.7.2). For $x \in S$ set

$$F_x(t) = \int_0^t u^\lambda\, l_x(u)\mathrm{d}u\,, \quad l_x(t) = \mathbf{P}(x'Y > t)\,, \quad t > 0\,.$$

By Lemma 4.5 we have

$$\lim_{t\to\infty} \frac{F_x(t)}{t} = \gamma^*\, h(x) > 0\,.$$

Monotonocity of the function $l_x(\cdot)$ yields for any $0 < a < b < \infty$

$$t^\lambda\, l_x(bt) \frac{b^{\lambda+1} - a^{\lambda+1}}{\lambda+1} \le \frac{F_x(bt) - F_x(at)}{t} \le t^\lambda\, l_x(at) \frac{b^{\lambda+1} - a^{\lambda+1}}{\lambda+1}\,.$$

This implies

$$\gamma^*\, h(x) \frac{(b-a)(\lambda+1)}{b^{\lambda+1}-a^{\lambda+1}} \le \liminf_{t\to\infty} t^\lambda\, l_x(at) \le \limsup_{t\to\infty} t^\lambda\, l_x(bt) \le \gamma^*\, h(x) \frac{(b-a)(\lambda+1)}{b^{\lambda+1}-a^{\lambda+1}}\,.$$

Taking $a = 1$ in the left inequality and letting $b \downarrow 1$ gives

$$\liminf_{t\to\infty} t^\lambda\, l_x(t) \ge \gamma^*\, h(x)\,.$$

By a similar treatment of the right inequality with $b = 1$ and $a \uparrow 1$ we find that $\limsup_{t\to\infty} t^\lambda\, l_x(t) \le \gamma^*\, h(x)$ and the conclusion follows. □

Example 4.7. *(Continuation of Example 4.2)*
Lemmata 4.5 and 4.6 imply Theorem 2.4 with the limiting constant

$$\gamma^* = \frac{1}{\beta} \int_0^\infty u^{\lambda-1} \frac{(\psi_0(1,u) + \psi_0(-1,u))}{2}\, \mathrm{d}u\,.$$

Symmetry of the distribution of ξ implies that $\psi_0(1,u) = \psi_0(-1,u)$, hence

$$\lim_{t\to\infty} t^\lambda \mathbf{P}(xY > t) = \frac{1}{\beta} \int_0^{+\infty} u^{\lambda-1} \left(\mathbf{P}(Y > u) - \mathbf{P}(\alpha_1 Y_1 > u)\right) \mathrm{d}u$$

for any $x \in S = \{1, -1\}$.

Note that this special case is already covered by Theorem 2.3 of Goldie [10].

Appendix

A0) Criteria for uniform geometric ergodicity

We recall some definitions from Markov chain theory (see e.g. Meyn and Tweedie [17]).

Let $(x_n)_{n\in\mathbb{N}}$ be a homogeneous Markov chain with state space $S \subseteq \mathbb{R}^q$ and $\mathcal{B}(S)$ the Borel σ-algebra in S. We denote by

$$\begin{aligned} \mathbf{P}(x,A) &= \mathbf{P}_x(x_1 \in A) = \mathbf{P}(x_1 \in A | x_0 = x) \\ \mathbf{P}^n(x,A) &= \mathbf{P}_x(x_n \in A) = \mathbf{P}(x_n \in A | x_0 = x)\,, \quad n \in \mathbb{N}\,. \end{aligned}$$

For every set $A \in \mathcal{B}(S)$ we introduce the *return time* and the *return number* to A by

$$\tau_A = \inf\{n \ge 1 \,:\, x_n \in A\}\,, \qquad N_A = \sum_{n=1}^\infty \chi_{\{x_n \in A\}}\,,$$

(as usual we set $\inf\{\emptyset\} = \infty$). We define also $L(x,A) = \mathbf{P}_x(\tau_A < \infty)$.

The Markov chain $(x_n)_{n\in\mathbb{N}}$ is called *φ-irreducible* if there exists a non-negative measure φ on $\mathcal{B}(S)$ such that

$$\forall A \in \mathcal{B}(S) \quad \text{with} \quad \varphi(A) > 0 \quad \Rightarrow \quad L(x, A) > 0 \quad \forall x \in S\,.$$

In this case φ is called an *irreducibility measure*. If the Markov chain $(x_n)_{n\in\mathbb{N}}$ is irreducible, then there exists a unique *maximal irreducibility measure* $\psi(\cdot)$ on $\mathcal{B}(S)$, i.e. for any irreducibility measure φ the relation $\varphi \prec \psi$ holds; i.e. the measure φ is absolutely continuous with respect to ψ. We denote by

$$\mathcal{B}_+(S) = \{A \in \mathcal{B}(S) \,:\, \psi(A) > 0\}\,.$$

The set A is called *Harris recurrent* if $\mathbf{P}_x(N_A = \infty) = 1$ for all $x \in A$. A Markov chain is called *Harris reccurent chain* if it is irreducible and every set in $\mathcal{B}_+(S)$ is Harris recurrent.

A Markov chain is called *uniform geometric ergodic* if there exists an invariant probability measure $\pi(\cdot)$ on $\mathcal{B}(S)$ such that for some $M > 0$ and $0 < \rho < 1$,

$$\sup_{x\in S} \|\mathbf{P}^n(x,\cdot) - \pi(\cdot)\| \le M\rho^n\,.$$

Here $\|\cdot\|$ means total variation, i.e. $\|\mu\| = \sup_{|f|\le 1} \int_S f(y)\mu(\mathrm{d}y)$.

Criteria for uniform ergodicity are often based on "small" sets. A set $\Gamma \in \mathcal{B}(S)$ is called a *small set* if there exists an $m \in \mathbb{N}$ and a non-trivial measure ν_m on $\mathcal{B}(S)$ (i.e. $\nu_m(S) > 0$) such that

$$\mathbf{P}^m(x, A) \ge \nu_m(A)\,, \quad x \in \Gamma\,, \quad A \in \mathcal{B}(S)\,.$$

Moreover, for any φ-irreducible Markov chain $(x_n)_{n\in\mathbb{N}}$ there exist disjoint sets $D_1, \ldots, D_d \in \mathcal{B}(S)$ (a so-called "d-cycle") with $d \ge 1$, such that

- $\varphi(\cap_{i=1}^d D_i^c) = 0$ and
- $\mathbf{P}(x, D_{i+1}) = 1$ for $x \in D_i$,

where $D_i = D_{j(i)}$ for $j(i) \in \{1, \ldots, d\}$ and $j(i) = i \pmod d$. The largest d for which a d-cycle occurs for $(x_n)_{n\in\mathbb{N}}$ is called *period* of $(x_n)_{n\in\mathbb{N}}$. When $d = 1$, the chain is called *aperiodic*.

Lemma A.1. (Meyn and Tweedie [17], p. 355)
Suppose that $(x_n)_{n\in\mathbb{N}}$ is irreducible and aperiodic. Let Γ be a small set and assume that the measurable bounded function $V : S \to [1, \infty)$ satisfies

$$\sup_{x\in\Gamma} \int_S V(y)p(x, dy) < \infty$$

and that for some $\varepsilon > 0$

$$\int_S V(y)p(x, dy) < (1-\varepsilon)\,V(x), \qquad \text{for all } \; x \in \Gamma^c.$$

Then $(x_n)_{n\in\mathbb{N}}$ is positive Harris recurrent and uniform geometric ergodic.

A1) A simple sufficient condition for D_4

Proof of Proposition 2.3. Let $l = \inf\{k \geq 1 : |c_k| > 0\}$. For $n \geq l$ set $\tau_n = \sum_{k=l}^n c_k \xi_k$. If $|c_k| > 0$ then by the condition of this proposition $c_k \xi_k$ has a symmetric density $p_k(\cdot)$, continuously differentiable with derivative $p'_k(\cdot) \leq 0$ on $[0, \infty)$. Therefore τ_l has a symmetric density, which is non-increasing on $[0, \infty)$. We proceed by induction. Suppose that τ_{n-1} has a symmetric density $\varphi_{\tau_{n-1}}(\cdot)$, non-increasing on $[0, \infty)$. We show that τ_n has a density with these properties. Indeed, if $c_n = 0$ then $\tau_n = \tau_{n-1}$ and we have the same distribution for τ_n. Consider now the case $|c_n| > 0$. By the properties of $p_n(\cdot)$ and of $\varphi_{\tau_{n-1}}(\cdot)$, we can write the density $\varphi_{\tau_n}(\cdot)$ of τ_n in the following form

$$
\begin{aligned}
\varphi_{\tau_n}(z) &= \int_{-\infty}^{\infty} p_n(z-u)\varphi_{\tau_{n-1}}(u)\mathrm{d}u \\
&= \int_0^{\infty} p_n(z+u)\varphi_{\tau_{n-1}}(u)\mathrm{d}u + \int_0^z p_n(z-u)\varphi_{\tau_{n-1}}(u)\mathrm{d}u \\
&\quad + \int_z^{\infty} p_n(u-z)\varphi_{\tau_{n-1}}(u)\mathrm{d}u\,, \quad z > 0\,.
\end{aligned}
$$

Therefore the derivative of this function equals

$$
\begin{aligned}
\varphi'_{\tau_n}(z) &= \int_0^{\infty} p'_n(z+u)\varphi_{\tau_{n-1}}(u)\mathrm{d}u + \int_0^z p'_n(z-u)\varphi_{\tau_{n-1}}(u)\mathrm{d}u - \int_z^{\infty} p'_n(u-z)\varphi_{\tau_{n-1}}(u)\mathrm{d}u \\
&= \int_z^{\infty} p'_n(u)\Big(\varphi_{\tau_{n-1}}(u-z) - \varphi_{\tau_{n-1}}(u+z)\Big)\mathrm{d}u \\
&\quad + \int_0^z p'_n(u)\Big(\varphi_{\tau_{n-1}}(z-u) - \varphi_{\tau_{n-1}}(u+z)\Big)\mathrm{d}u \leq 0\,, \quad z > 0\,,
\end{aligned}
$$

since $p'_n(\cdot) \leq 0$ and $\varphi_{\tau_{n-1}}(\cdot)$ is non-increasing on $[0, \infty)$. Therefore we obtained that for all $n \geq l$ the random variable τ_n has a symmetric continuously differentiable density, which is non-increasing on $[0, \infty)$. Moreover, since $\tau = \lim_{n\to\infty} \tau_n$ a.s. and the sequence $(\varphi_{\tau_n}(\cdot))_{n\geq l}$ is uniformly bounded, i.e.

$$
\sup_{z\in\mathbb{R}\,,\, n\geq l} \varphi_{\tau_n}(z) \leq \varphi_{\tau_l}(0) < \infty\,,
$$

we have that for every bounded measurable function g with finite support in $\mathbb{R}$

$$
\lim_{n\to\infty} \int_{-\infty}^{\infty} g(z)\varphi_{\tau_n}(z)\mathrm{d}z = \int_{-\infty}^{\infty} g(z)\varphi_{\tau}(z)\mathrm{d}z\,,
$$

where $\varphi_\tau(\cdot)$ is the density of τ. Since ξ_1 has a continuous density, also φ_τ is continuous. Therefore, for $0 < a < b$ we have for all $0 < \delta < a$,

$$
\int_{b-\delta}^{b+\delta} \varphi_\tau(z)\mathrm{d}z - \int_{a-\delta}^{a+\delta} \varphi_\tau(z)\mathrm{d}z = \lim_{n\to\infty} \left(\int_{b-\delta}^{b+\delta} \varphi_{\tau_n}(z)\mathrm{d}z - \int_{a-\delta}^{a+\delta} \varphi_{\tau_n}(z)\mathrm{d}z \right) \leq 0\,.
$$

Since $\varphi_\tau(\cdot)$ is continuous we conclude

$$\varphi_\tau(b) - \varphi_\tau(a) = \lim_{\delta\to 0} \frac{1}{2\delta}\left(\int_{b-\delta}^{b+\delta} \varphi_\tau(z)\mathrm{d}z - \int_{a-\delta}^{a+\delta} \varphi_\tau(z)\mathrm{d}z\right) \le 0\,. \qquad \square$$

A2) Gaussian linear random coefficient models

Proof of Proposition 2.6. It is evident that conditions $\mathbf{D_1} - \mathbf{D_4}$ hold for this model.

To show that the conditional correlation matrix (2.6) is positive definite a.s. take some $x \in \mathbb{R}^q$ such that $x'Rx = 0$. Then for $\Pi_k = A_1 \cdots A_k$, $k \in \mathbb{N}$, and B as defined in (2.6)

$$x'Bx + \sum_{k=1}^{\infty} x'\Pi_k B \Pi_k' x = 0\,.$$

If we denote by $< x >_i$ the ith coordinate of $x \in \mathbb{R}^q$, the equality above means that $< \Pi_k' x >_1 = 0$ for all $k \in \mathbb{N}$. Set $\theta_k(x) =< \Pi_k' x >_1$ for $k \in \mathbb{N}$ and $\theta_0(x) =< x >_1$. Taking the special form of the matrices (1.5) into account one can show by induction that

$$\theta_k(x) = \begin{cases} \alpha_1(k)\theta_{k-1}(x) + \ldots + \alpha_k(1) < x >_1 + < x >_{k+1} & \text{if} \quad 1 \le k < q\,; \\ \alpha_1(k)\theta_{k-1}(x) + \ldots + \alpha_q(k-q+1)\theta_{k-q}(x) & \text{if} \quad k \ge q\,. \end{cases} \tag{A.1}$$

Consequenly, if $\theta_k(x) = 0$ for all $0 \le k \le q$ then $< x >_1 = \ldots =< x >_q = 0$. From this we conclude that $x'Rx = 0$ implies $x = 0$, which means that R is positive definite a.s. $\square$

A3) Auxiliary properties of $\Pi_n = A_1 \cdots A_n$.

We study the asymptotic properties of $\theta_k(x)$ as defined in (A.1). First recall the classical Anderson inequality; see Ibragimov and Hasminskii [13], p. 214.

Lemma A.2. (Anderson's inequality)
Let η be a random variable with symmetric continuous density, which is non-increasing on $[0, \infty)$. Then for every $c \in \mathbb{R}$ and $a > 0$

$$\mathbf{P}(|\eta + c| \le a) \le \mathbf{P}(|\eta| \le a)\,.$$

Lemma A.3. *Assume model (1.1) such that $\mathbf{D_1} - \mathbf{D_2}$ hold and $a_q^2 + \sigma_q^2 > 0$. Then for every $\mu > 0$ and $k \in \mathbb{N}$*

$$\lim_{\delta\to 0} \sup_{|<x>_1|>\mu} \mathbf{P}(|\theta_k(x)| < \delta) = 0\,. \tag{A.2}$$

Furthermore, for $k = q$ we have

$$\lim_{\delta\to 0} \sup_{|x|>\mu} \mathbf{P}(|\theta_q(x)| < \delta) = 0\,. \tag{A.3}$$

Proof. We show first that for $1 \le j \le q$ and for every $\epsilon > 0$ such that $\delta/\epsilon \to 0$ as $\delta \to 0$,

$$\lim_{\delta\to 0}\sup_{x\in\mathbb{R}^q}\mathbf{P}(|\theta_j(x)| < \delta\,,\ |\theta_{j-1}(x)| \ge \epsilon) = 0\,. \tag{A.4}$$

Recall that $\theta_0(x) = < x >_1$.

To prove (A.4) notice first that by (A.1)

$$\begin{aligned}\theta_j(x) &= \eta_1(j)\sigma_1\theta_{j-1}(x) + m_j(x)\,,\\ m_j(x) &= a_1\theta_{j-1}(x) + \alpha_2(j-1)\theta_{j-2}(x) + \ldots + \alpha_j(1) < x >_1 + < x >_{j+1}\chi_{\{j<q\}}\,.\end{aligned}$$

Moreover, condition $\mathbf{D}_2$ implies that $\sigma_1 > 0$ and therefore by Anderson's inequality (taking into account that $\eta_1(j)$ is independent of $\theta_{j-1}(x)$ and $m_j(x)$) we obtain

$$\begin{aligned}\mathbf{P}(|\theta_j(x)| < \delta\,, |\theta_{j-1}(x)| \ge \epsilon) &= \mathbf{P}(|\eta_1(j)\sigma_1\theta_{j-1}(x) + m_j(x)| < \delta\,, |\theta_{j-1}(x)| \ge \epsilon)\\ &\le \mathbf{P}(|\eta_1(j)\sigma_1\theta_{j-1}(x)| < \delta\,, |\theta_{j-1}(x)| \ge \epsilon)\\ &\le \mathbf{P}(|\eta_1(j)| < \delta/(\epsilon\sigma_1))\,.\end{aligned}$$

From this and condition $\mathbf{D}_1$ we obtain (A.4). Then (A.2) follows by induction.

Next we show (A.3). Introduce for $\delta > 0$ and $1 \le j \le q$ the sets

$$\Gamma_\delta = \bigcap_{j=1}^{q}\Gamma_{j,\delta}\,,\quad \text{where}\quad \Gamma_{j,\delta} = \{|\theta_j(x)| < \epsilon_j\}\,,$$

for $\epsilon_j = \epsilon_j(\delta) = \delta^{j/q}$. Notice that (A.4) implies

$$\lim_{\delta\to 0}\sup_{x\in\mathbb{R}^q}\mathbf{P}(\Gamma_{j,\delta}\cap\Gamma^c_{j-1,\delta}) = 0\,.$$

Set $\alpha^* = \max_{i+j\le q}|\alpha_i(j)|$ and define

$$F_\nu = \{|\alpha_q(1)| \ge \nu\}\,,\qquad B_N = \{\alpha^* \le N\}\,.$$

Take for any fixed $\nu > 0$, $N > 0$ the set $\Gamma_\delta \cap F_\nu \cap B_N$. The definition of $\theta_j(x)$ in (A.1) implies that on this set $|x| \to 0$ as $\delta \to 0$. Hence, if - as in (A.3) - $|x| \ge \mu$, there exists $\delta_0 = \delta_0(\mu, \nu, N) > 0$ such that $\Gamma_\delta \cap F_\nu \cap B_N = \emptyset$ for all $\delta \le \delta_0$. Therefore for this $\delta > 0$ and for $x \in \mathbb{R}^q$ with $|x| > \mu$ we obtain

$$\begin{aligned}\mathbf{P}(|\theta_q(x)| < \delta) &\le \mathbf{P}(\Gamma_\delta) + \sum_{j=2}^{q}\mathbf{P}(\Gamma_{j,\delta}\cap\Gamma^c_{j-1,\delta})\\ &\le \mathbf{P}(|\alpha_q(1)| < \nu) + \mathbf{P}(\alpha^* > N) + \sum_{j=2}^{q}\mathbf{P}(\Gamma_{j,\delta}\cap\Gamma_{j-1,\delta})\\ &\le \mathbf{P}(|a_q + \sigma_q\eta_q(1)| < \nu) + \frac{\mathbf{E}\alpha^*}{N} + \sum_{j=2}^{q}\mathbf{P}(\Gamma_{j,\delta}\cap\Gamma_{j-1,\delta})\,.\end{aligned}$$

Notice that the conditions $a_q^2+\sigma_q^2>0$ and $\mathbf{D}_1$ guarantee that the first term in the last line tends to zero as $\nu\to 0$. Hence we obtain (A.3). $\square$

Corollary A.4. *Under the conditions of Lemma A.3 relation (A.3) holds with respect to the distribution* $\tilde{\mathbf{P}}$ *as defined in (4.3), i.e.*

$$\lim_{\delta\to 0}\sup_{x\in S}\tilde{\mathbf{P}}_x(|\theta_q(x)|<\delta)=0\,.$$

Proof. By definition (4.14) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\tilde{\mathbf{P}}_x(|\theta_q(x)|<\delta) &= \frac{1}{h(x)}\mathbf{E}|x'\Pi_q|^\lambda h(\overline{x'\Pi_q})\chi_{\{|\theta_q(x)|<\delta\}}\\ &\le \frac{h^*}{h_*}\sqrt{\mathbf{E}(|\Pi_q|^{2\lambda})}\sqrt{\sup_{x\in S}\mathbf{P}(|\theta_q(x)|<\delta)}\,,\end{aligned}$$

where $h_*=\inf_{x\in S}h(x)$, $h^*=\sup_{x\in S}h(x)$. The result follows from (A.3). $\square$

In the following lemma we compute the conditional density of $\Pi'_{2q+1}x$ in $\mathbb{R}^q$ with respect to the random vector $\rho=\rho(x)=\Pi'_q x$.

Lemma A.5. *Assume that* $\mathbf{D_1}-\mathbf{D_2}$ *hold,* $a_q^2+\sigma_q^2>0$ *and* $x\ne 0$*. Then the random vector* $\Pi'_{2q+1}x$ *has conditional* $\mathbf{P}$*-density* $p_1(z|\rho(x))=f(z,\rho(x))$ *with respect to* $\rho(x)$*. The function* $f(\cdot,\cdot):\mathbb{R}^q\times\mathbb{R}^q\to[0,\infty)$ *is given by*

$$f(z,y)=\mathbf{E}\,\frac{1}{|\det T|}\,p_0(z'T^{-1},y)\,, \tag{A.5}$$

where

$$T=\begin{pmatrix}\alpha_1(q+1) & \alpha_2(q+1) & \cdots & \alpha_q(q+1)\\ \vdots & \vdots & \vdots & 0\\ \alpha_{q-1}(3) & \alpha_q(3) & \cdots & 0\\ \alpha_q(2) & 0 & \cdots & 0\end{pmatrix} \tag{A.6}$$

and for $z=(z_1,\dots,z_q)\in\mathbb{R}^q$, $y=(y_1,\dots,y_q)\in\mathbb{R}^q$

$$\begin{aligned} p_0(z,y) &= \prod_{j=1}^q\varphi_j(z_j|z_{j-1},\dots,z_1,y)\,,\\ \varphi_j(z_j|z_{j-1},\dots,z_1,y) &= \chi_{\{|z_{j-1}|>0\}}\mathbf{E}\,\frac{1}{\sigma_1|z_{j-1}|}\,\phi\left(\frac{z_j-m_j(z,y)}{\sigma_1 z_{j-1}}\right)\,,\\ m_1(z,y) &= a_1y_1+y_2\,,\quad \text{and for}\quad j>1\\ m_j(z,y) &= a_1z_{j-1}+\alpha_2(j-1)z_{j-2}\dots+\alpha_j(1)y_1+y_{j+1}\chi_{\{j<q\}}\,,\end{aligned} \tag{A.7}$$

where $z_0=y_1$ *and the density* ϕ *is defined in condition* $\mathbf{D}_1$.

Proof. Let $x=(x_1,\ldots,x_q)'\in\mathbb{R}^q$ such that $x_q\neq 0$. We show that the vector $\Pi_{q+1}'x$ has density $f(\cdot,x)$ as defined in (A.5). To this end we show first that $x'\Pi_{q+1}=\theta(x)'T$, where the matrix T is defined in (A.6) and $\theta(x)=(\theta_q(x),\ldots,\theta_1(x))'\in\mathbb{R}^q$. By the definition of A_j in (1.5) we have

$$\begin{aligned}
< x'\Pi_{q+1} >_q &= < x'\Pi_q A_{q+1} >_q = \ \alpha_q(q+1) < x'\Pi_q >_1 \\
&= \alpha_q(q+1) < \Pi_q' x >_1 = \alpha_q(q+1)\theta_q(x)\,, \\
< x'\Pi_{q+1} >_{q-1} &= < x'\Pi_q A_{q+1} >_{q-1} = \ \alpha_{q-1}(q+1) < x'\Pi_q >_1 + < x'\Pi_q >_q \\
&= \alpha_{q-1}(q+1)\theta_q(x) + \alpha_q(q) < x'\Pi_{q-1} >_1 = \ \alpha_{q-1}(q+1)\theta_q(x) + \alpha_q(q)\theta_{q-1}(x)\,,
\end{aligned}$$

and for $1\le j<q-1$

$$\begin{aligned}
< x'\Pi_{q+1} >_j &= < x'\Pi_q A_{q+1} >_j = \alpha_j(q+1) < x'\Pi_q >_1 + < x'\Pi_q >_{j+1} \\
&= \alpha_j(q+1) < x'\Pi_q >_1 + \alpha_{j+1}(q) < x'\Pi_{q-1} >_1 + < x'\Pi_{q-1} >_{j+2} \\
&= \ldots = \ \alpha_j(q+1) < x'\Pi_q >_1 + \cdots + \alpha_{q-1}(j+2) < x'\Pi_{j+1} >_1 + < x'\Pi_{j+1} >_q \\
&= \alpha_j(q+1)\theta_q(x) + \ldots + \alpha_{q-1}(j+2)\theta_{j+1}(x) + \alpha_q(j+1)\theta_j(x)\,.
\end{aligned}$$

This gives $x'\Pi_{q+1}=\theta(x)'T$.

Next note that $a_q^2+\sigma_q^2>0$ implies

$$|\det T| = \prod_{j=1}^{q} |\alpha_q(j+1)| = \prod_{j=1}^{q} |a_q+\sigma_q\,\eta_q(j+1)| > 0 \qquad \mathbf{P}-\text{a.s.}\,.$$

Immediately by (A.1) the vector $\theta(x)$ is measurable with respect to $\sigma\{\alpha_i(k)\,,\ 1\le i\le q\,,1\le k\le q\,,i+k\le q+1\}$. Hence, T is independent of $\theta(x)$. Therefore to prove that the vector $\Pi_{q+1}'x$ has density $f(\cdot,x)$ it suffices to prove that $\theta(x)$ has density $p_0(\cdot,x)$ as in (A.7). Indeed, if $x_1\neq 0$, then condition $\mathbf{D}_2$ guarantees $\sigma_1^2>0$ and $\theta_1(x)=\alpha_1(1)x_1+x_2$ has positive density $\varphi_1(\cdot|x)$ as defined in (A.7). This implies that $\theta_1(x)\neq 0$ a.s., and therefore

$$\theta_2(x)=\alpha_1(2)\theta_1(x)+\alpha_2(1)x_1+x_3$$

has conditional density with respect to $\theta_1(x)$

$$p_{\theta_2}(z_2|\theta_1(x))=\varphi_2(z_2|\theta_1(x),x)\,,$$

where the function φ_2 is also defined in (A.7). Similarly we can show that

$$p_{\theta_j}(z_j|\theta_{j-1}(x),\ldots,\theta_1(x))=\varphi_j(z_j|\theta_{j-1}(x),\ldots,\theta_1(x),x)$$

for every $2\le j\le q$. Therefore $\theta(x)=(\theta_q(x),\ldots,\theta_1(x))'$ has density (A.7) in $\mathbb{R}^q$ provided $x_1\neq 0$.

To complete the proof we show that the conditional density of the vector $\Pi'_{2q+1}x$ with respect to $\rho(x)$ equals $f(\cdot,\rho(x))$ a.s. for $x\neq 0$. To this end recall that (A.3) implies $<\rho(x)>_1=\theta_q(x)\neq 0$ a.s. for every vector $x\neq 0$. Now taking into account that the A_n are iid we obtain for every bounded measurable function $F:\mathbb{R}^q\to\mathbb{R}$

$$\mathbf{E}(F(x'\Pi_{2q+1})|\rho(x))=\mathbf{E}(F(\rho(x)'A_{q+1}\cdots A_{2q+1})|\rho(x))=\Psi(\rho(x))\,,$$

where

$$\Psi(y)=\mathbf{E}\,F(y'\Pi_{q+1})=\int_{\mathbb{R}^q}F(z)\,f(z,y)\,\mathrm{d}z\,,\quad y\in\mathbb{R}^q\,,y_1\neq 0\,.$$

This concludes the proof. □

The followig result is an immediate consequence of the definition of $\tilde{\mathbf{P}}$ in (4.3) and Lemma A.7.

Corollary A.6. *Under the conditions of Lemma A.5 the random vector $\Pi'_{2q+1}x$ has a conditional $\tilde{\mathbf{P}}$-density with respect to $\rho(x)$ given by*

$$\tilde{p}_1(z|\rho)=\frac{|z|^\lambda\,h(\overline{z})}{|\rho|^\lambda\,h(\overline{\rho})}\,p_1(z|\rho)\,,\qquad z,\rho\in\mathbb{R}^q\,,\qquad z\neq 0\,,\rho\neq 0\,,$$

for $p_1(z|x)$ as defined in (A.5).

Lemma A.7. *Assume that conditions $\mathbf{D_1}-\mathbf{D_2}$ hold and $a_q^2+\sigma_q^2>0$. Then for $b\,,x\in\mathbb{R}^q$ and $x\neq 0$*

$$\mathbf{P}(x'\Pi_{2q+1}b=0)>0\ \Rightarrow\ b=0\,.$$

Proof. Lemma A.5 implies that

$$\mathbf{P}(x'\Pi_{2q+1}b=0)=\mathbf{E}\,\mathbf{P}(x'\Pi_{2q+1}b=0|\rho(x))=\mathbf{E}\int_{\{z\in\mathbb{R}^q\,:\,z'b=0\}}p_1(z|\rho(x))\,\mathrm{d}z\,.$$

If this probability is positive, then there exists a vector $\rho\in\mathbb{R}^q$ with $<\rho>_1\neq 0$ such that

$$\int_{\{z\in\mathbb{R}^q\,:\,z'b=0\}}p_1(z|\rho)\,\mathrm{d}z>0\,.$$

This is possible if and only if $b=0$ since the Lebesgue measure of the set $\{z\in\mathbb{R}^q\,:\,b'z=0\}$ equals to zero for all $b\neq 0$. □

Denote by $\mathrm{mes}(\cdot)$ the Lebesgue measure in $\mathbb{R}^q$.

Lemma A.8. *Assume that conditions* $\mathbf{D_1} - \mathbf{D_2}$ *hold,* $q \geq 2$ *and* $a_q^2 + \sigma_q^2 > 0$*. Then there exists some* $\delta_0 > 0$ *such that for all* $0 < \delta < \delta_0$

$$\inf_{x\in S} \mathbf{P}(x'\Pi_{2q+1} \in B) \;\geq\; p_*(\delta)\mu_\delta(B)\,, \tag{A.8}$$
$$\inf_{x\in S} \tilde{\mathbf{P}}_x(x'\Pi_{2q+1} \in B) \;\geq\; \tilde{p}_*(\delta)\tilde{\mu}_\delta(B)\,, \tag{A.9}$$

for every measurable set $B \subseteq \mathbb{R}^q$*. Here* $p_*(\delta), \tilde{p}_*(\delta) > 0$ *and*

$$\begin{aligned} \mu_\delta(B) &= \mathbf{E}\int_{\Omega_\delta} \chi_B(z'T)\,dz\,, \qquad \tilde{\mu}_\delta(B) = \mathbf{E}\int_{\Omega_\delta} |z'T|^\lambda\,\chi_B(z'T)\,dz\,, \\ \Omega_\delta &= \{y = (y_1,\ldots,y_q)' \in \mathbb{R}^q \,:\, \delta \leq |y_j| \leq \delta^{-1}\,, j = 1,\ldots,q\}\,, \end{aligned} \tag{A.10}$$

and the matrix T *is defined in (A.6). Furthermore, if mes*$(B) > 0$ *then there exists some* $\delta_0 > 0$ *such that* $\mu_\delta(B) > 0$ *and* $\tilde{\mu}_\delta(B) > 0$ *for all* $0 < \delta < \delta_0$.

Proof. From Lemma A.5 we know that for a some $0 < \delta < 1$

$$\begin{aligned} \mathbf{P}(x'\Pi_{2q+1} \in B) &= \mathbf{E}\mathbf{P}(x'\Pi_{2q+1} \in B)|\rho(x)) \\ &\geq \mathbf{E}\chi_{\{\rho(x)\in K_\delta\}}\mathbf{P}(x'\Pi_{2q+1} \in B)|\rho(x)) \\ &= \mathbf{E}\chi_{\{\rho(x)\in K_\delta\}}\, I_B(\rho(x))\,, \end{aligned}$$

where $K_\delta = \{y = (y_1,\ldots,y_q)' \in \mathbb{R}^q \,:\, \delta \leq |y_1| \quad \text{and} \quad |y| \leq \delta^{-1}\}$ and

$$I_B(\rho) = \int_{\mathbb{R}^q} \chi_B(z)\,p_1(z|\rho)\mathrm{d}z = \mathbf{E}\int_{\mathbb{R}^q} \chi_B(z'T)\,p_0(z,\rho)\mathrm{d}z \geq \mathbf{E}\int_{\Omega_\delta} \chi_B(z'T)\,p_0(z,\rho)\mathrm{d}z\,.$$

Next we show for $K_\delta^c = \mathbb{R}^q \setminus K_\delta$

$$\lim_{\delta\to 0}\sup_{x\in S}\mathbf{P}(\rho(x) \in K_\delta^c) = 0\,, \qquad \lim_{\delta\to 0}\sup_{x\in S}\tilde{\mathbf{P}}_x(\rho(x) \in K_\delta^c) = 0\,. \tag{A.11}$$

Indeed, we have

$$\begin{aligned} \mathbf{P}(\rho(x) \in K_\delta^c) &\leq \mathbf{P}(|<\rho(x)>_1| < \delta) + \mathbf{P}(|\rho(x)| > \delta^{-1}) \\ &\leq \sup_{x\in S}\mathbf{P}(|\theta_q(x)| < \delta) + \delta(\mathbf{E}|A_1|)^q\,. \end{aligned}$$

(A.3) gives the left limit in (A.11); from Corollary A.4 we obtain the right limit.

Notice that (A.7) implies that for every $\delta > 0$

$$M_*(\delta) = \inf_{z\in\Omega_\delta\,,\,x\in K_\delta} p_0(z,x) > 0\,,$$

which yields

$$\mathbf{P}(x'\Pi_{2q+1} \in B) \geq M_*(\delta)\mathbf{P}(\rho(x) \in K_\delta)\mu_\delta(B)\,.$$

From this and (A.11) we obtain (A.8). Similarly

$$\tilde{\mathbf{P}}_x(x'\Pi_{2q+1} \in B) \geq \tilde{\mathbf{E}}_x \chi_{\{\rho(x)\in K_\delta\}}\, \tilde{I}_B(\rho(x))\,,$$

where

$$\begin{aligned}
\tilde{I}_B(\rho) &= \int_{\mathbb{R}^q} \chi_B(z)\,\tilde{p}_1(z|\rho)\mathrm{d}z \;=\; \frac{1}{|\rho|^\lambda}\int_{\mathbb{R}^q} \frac{|z|^\lambda\, h(\overline{z})}{h(\overline{\rho})}\,\chi_B(z)\,p_1(z|\rho)\mathrm{d}z\\
&= \frac{1}{|\rho|^\lambda}\,\mathbf{E}\int_{\mathbb{R}^q} \frac{|z'T\,|^\lambda\, h(\overline{z'T})}{h(\overline{\rho})}\,\chi_B(z)\,p_0(z,\rho)\mathrm{d}z\\
&\geq \frac{h_*}{h^*|\rho|^\lambda}\,\mathbf{E}\int_{\Omega_\delta} |z'T\,|^\lambda\;\chi_B(z)\,p_0(z,\rho)\mathrm{d}z\,,
\end{aligned}$$

with $h^* = \sup_{x\in S} h(x)$ and $h_* = \inf_{x\in S} h(x)$. Therefore, on K_δ we have

$$\tilde{I}_B(\rho) \geq \delta^\lambda M_*(\delta)\frac{h_*}{h^*}\tilde{\mu}_\delta(B)\,,$$

which together with (A.11) implies (A.9).

Let now B be a measurable set in $\mathbb{R}^q$. By the monotone convergence theorem we have

$$\begin{aligned}
\lim_{\delta\to 0}\mu_\delta(B) &= \mathbf{E}\,\lim_{\delta\to 0}\int_{\Omega_\delta} \chi_B(z'T\,)\mathrm{d}z = \mathbf{E}\int_{\mathbb{R}^q} \chi_B(z'T\,)\mathrm{d}z\\
&= \mathrm{mes}(B)\,\mathbf{E}\,\frac{1}{|\det T|}\,,\\
\lim_{\delta\to 0}\tilde{\mu}_\delta(B) &= \mathbf{E}\,\lim_{\delta\to 0}\int_{\Omega_\delta} |z'T\,|^\lambda\chi_B(z'T\,)\mathrm{d}z = \mathbf{E}\,\int_{\mathbb{R}^q} |z'T\,|^\lambda\chi_B(z'T\,)\mathrm{d}z\\
&= \int_{\mathbb{R}^q} |z|^\lambda\chi_B(z)\mathrm{d}z\,\mathbf{E}\,\frac{1}{|\det T|}\,.
\end{aligned}$$

This implies the second part of the lemma. □

The following Lemma is needed to verify condition $\mathbf{C_4}$.

Lemma A.9. *Assume that* $\mathbf{D_1} - \mathbf{D_2}$ *hold and* $a_q^2 + \sigma_q^2 > 0$. *Then*

$$\Phi(x,t) = \tilde{\mathbf{E}}_x\, f(x'\Pi_{2q+1},t)\,,\quad x\in S\,,\, t\in\mathbb{R},$$

is uniformly continuous on S *for every measurable bounded function* $f\,:\,S\times\mathbb{R}\to\mathbb{R}$*; i.e.*

$$\lim_{\varepsilon\to 0}\;\sup_{|x-y|\leq\varepsilon}\;\sup_{t\in\mathbb{R}}\,|\Phi(x,t)-\Phi(y,t)| = 0\,.$$

Proof. Let $V\,:\,\mathbb{R}^q\to[0,\infty)$ be a continuous function such that $V(z)=0$ for $|z|\geq 1$ and $\int_{\mathbb{R}^q} V(z)\mathrm{d}z = 1$. For example take

$$V(z) = v_*^{-1}\,\exp(-\frac{1}{1-|z|^2})\,\chi_{\{|z|\leq 1\}}\,,\quad \text{where}\quad v_* = \int_{|z|\leq 1}\exp(-\frac{1}{1-|z|^2})\,\mathrm{d}z\,.$$

For some $\epsilon \in (0,1)$ define $K_\epsilon = \{y \in \mathbb{R}^q \,:\, |<y>_1| \ge \epsilon\,,\, |y| \le 1/\epsilon\}$ and $\nu_\epsilon = \epsilon/4$ and

$$g_\epsilon(x) = \frac{1}{(\nu_\epsilon)^q} \int_{\mathbb{R}^q} \chi_{K_\epsilon}(y) V\left(\frac{y-x}{\nu_\epsilon}\right) \mathrm{d}y = \int_{|y|\le 1} \chi_{K_\epsilon}(x+\nu_\epsilon y) V(y)\,\mathrm{d}y\,.$$

Then $g_\epsilon \,:\, \mathbb{R}^q \to [0,1]$ is continuous and for every $x \in \mathbb{R}^q$

$$g_\epsilon(x) \le \chi_{K_{\epsilon/4}}(x) \quad \text{and} \quad \overline{g}_\epsilon(x) = 1 - g_\epsilon(x) \le \chi_{K^c_{4\epsilon}}(x)\,. \tag{A.12}$$

We can represent the function Φ in the following form

$$\Phi(x,t) = \tilde{\mathbf{E}}_x f(x'\Pi_{2q+1},t) = \tilde{\mathbf{E}}_x\, g_\epsilon(\rho(x))\, f(x'\Pi_{2q+1},t) + \Delta_\epsilon(x)\,,$$

where $\Delta_\epsilon(x) = \tilde{\mathbf{E}}_x\, \overline{g}_\epsilon(\rho(x))\, f(x'\Pi_{2q+1},t)$. By (A.12) and (A.11), setting $f^* = \sup |f|$ we obtain

$$\Delta^*_\epsilon = \sup_{x\in S} |\Delta_\epsilon(x)| \le f^* \sup_{x\in S} \tilde{\mathbf{P}}_x(\rho(x) \in K^c_{4\epsilon}) \to 0 \qquad \text{as} \quad \epsilon \to 0\,.$$

From the definition of $\tilde{\mathbf{E}}$ in (4.3) we obtain

$$\tilde{\mathbf{E}}_x\, g_\epsilon(\rho(x))\, f(x'\Pi_{2q+1},t) = \frac{1}{h(x)}\, \mathbf{E}\, g_\epsilon(\rho(x))\, f_1(x'\Pi_{2q+1},t)\,,$$

where $f_1(z,t) = |z|^\lambda h(\overline{z}) f(z,t)$. By Lemma A.5 we can represent this term as

$$\mathbf{E}\, g_\epsilon(\rho(x))\, f_1(x'\Pi_{2q+1},t) = \mathbf{E} \int_{\mathbb{R}^q} \overline{f}_1(z,t)\, \psi_\epsilon(z,\rho(x))\mathrm{d}z = \mathbf{E}\, \Psi_\epsilon(\rho(x),t)$$

with $\overline{f}_1(z,t) = \mathbf{E}\, f_1(z'T\,,t)$ and $\psi_\epsilon(z,\rho) = p_0(z,\rho) g_\epsilon(\rho)$. Here Ψ_ϵ allows the representation

$$\begin{aligned} \Psi_\epsilon(\rho,t) &= \int_{\Omega_\delta} \overline{f}_1(z,t)\, \psi_\epsilon(z,\rho)\mathrm{d}z + \int_{\Omega^c_\delta} \overline{f}_1(z,t)\, \psi_\epsilon(z,\rho)\mathrm{d}z \\ &= \Psi_{\epsilon,\delta}(\rho,t) + \Delta_{\epsilon,\delta}(\rho,t)\,, \end{aligned} \tag{A.13}$$

where $\Omega_\delta = \{y \in \mathbb{R}^q \,:\, \delta \le |<y>_j| \le \delta^{-1}\,, j = 1,\ldots,q\}$. Next we show that for every $\epsilon > 0$

$$\lim_{\delta\to 0} \sup_{\rho \in K_{\epsilon/4}} \mathbf{P}(\theta(\rho) \in \Omega^c_\delta) = 0\,. \tag{A.14}$$

To this end note

$$\begin{aligned} \sup_{\rho\in K_{\epsilon/4}} \mathbf{P}(\theta(\rho) \in \Omega^c_\delta) &\le \sum_{j=1}^q \sup_{|<\rho>_1|\ge\epsilon/4} \mathbf{P}(|\theta_j(\rho)| < \delta) + \sup_{|\rho|\le 4/\epsilon} \mathbf{P}(|\theta(\rho)| > 1/\delta) \\ &\le \sum_{j=1}^q \sup_{|<\rho>_1|\ge\epsilon/4} \mathbf{P}(|\theta_j(\rho)| < \delta) + \delta \sup_{|\rho|\le 4/\epsilon} \mathbf{E}\,|\theta(\rho)| \end{aligned}$$

By the definition of $\theta(\rho)$ in (A.1) we find for every $m > 0$ some constant $c_m > 0$ such that

$$\sup_{|\rho|\le 4/\epsilon} \mathbf{E}\,|\theta(\rho)|^m \le c_m/\epsilon^m < \infty\,.$$

Therefore the limit relation (A.2) implies (A.14). Moreover, notice that the last inequality yields

$$\lim_{N\to\infty} \sup_{|\rho|\le 4/\epsilon} \mathbf{E}\,\chi_{\{|\theta(\rho)|>N\}}\,|\theta(\rho)|^\lambda = 0\,.$$

Next we estimate $\Delta_{\epsilon,\delta}(\rho,t)$ as defined in (A.13). Taking into account that

$$|\overline{f}_1(z,t)| \le f^* h^*\,\mathbf{E}|T|\,|z|^\lambda = f_1^*\,|z|^\lambda$$

we obtain for $\rho\in\mathbb{R}^q$ and $N>0$

$$\begin{aligned} |\Delta_{\epsilon,\delta}(\rho,t)| &\le f_1^* g_\epsilon(\rho)\int_{\Omega_\delta^c} |z|^\lambda\, p_0(z,\rho)\mathrm{d}z \\ &= f_1^* g_\epsilon(\rho)\,\mathbf{E}\,|\theta(\rho)|^\lambda\,\chi_{\{\theta(\rho)\in\Omega_\delta^c\}} \\ &\le f_1^*\chi_{\{\rho\in K_{\epsilon/4}\}}\Big(N^\lambda \mathbf{P}(\theta(\rho)\in\Omega_\delta^c) + \mathbf{E}\,\chi_{\{|\theta(\rho)|>N\}}\,|\theta(\rho)|^\lambda\Big)\,. \end{aligned}$$

This together with (A.14) ensures for every $\epsilon>0$

$$\Delta^*_{\epsilon,\delta} = \sup_{\rho\in\mathbb{R}^q\,,\,t\in\mathbb{R}} |\Delta_{\epsilon,\delta}(\rho,t)| \to 0 \quad \text{as} \quad \delta\to 0\,.$$

From this we conclude for $x,y\in S$ such that $|x-y|\le\eta$ and for $\mu>0$

$$\begin{aligned} |\Phi(x,t)-\Phi(y,t)| \;&\le\; \mathbf{E}\left|\frac{1}{h(x)}\Psi_{\epsilon,\delta}(\rho(x),t) - \frac{1}{h(y)}\Psi_{\epsilon,\delta}(\rho(y),t)\right| + 2\Delta^*_\epsilon + 2\Delta^*_{\epsilon,\delta} \\ &\le\; \Psi^*_{\epsilon,\delta}\left|\frac{1}{h(x)} - \frac{1}{h(y)}\right| + 2\Delta^*_\epsilon + 2\Delta^*_{\epsilon,\delta} \\ &\quad + \frac{q}{\delta^\lambda}\, f_1^*\,\mathrm{mes}(\Omega_\delta)\mathbf{E}\,\sup_{z\in\Omega_\delta}|\psi_\epsilon(z,\rho(x)) - \psi_\epsilon(z,\rho(y))| \\ &\le\; \Psi^*_{\epsilon,\delta}\sup_{|x-y|\le\eta}\left|\frac{1}{h(x)} - \frac{1}{h(y)}\right| + 2\Delta^*_\epsilon + 2\Delta^*_{\epsilon,\delta} \\ &\quad + \frac{q}{\delta^\lambda}\, f_1^*\,\mathrm{mes}(\Omega_\delta)\sup_{z\in\Omega_\delta\,,|\rho_1-\rho_2|\le\mu}|\psi_\epsilon(z,\rho_1) - \psi_\epsilon(z,\rho_2)| \\ &\quad + \frac{q}{\delta^\lambda}\, f_1^*\,\mathrm{mes}(\Omega_\delta)\,\mathbf{P}(|\rho(x)-\rho(y)|>\mu)\,, \end{aligned}$$

where $\Psi^*_{\epsilon,\delta} = \sup|\Psi_{\epsilon,\delta}|$. We take into account that the function $\psi_\epsilon(z,\rho)$ is uniformly continuous on $\Omega_\delta\times\mathbb{R}^q$. Moreover, the last probability is bounded by Chebyshev's inequality:

$$\mathbf{P}(|\rho(x)-\rho(y)|>\mu) \le \frac{1}{\mu}\mathbf{E}\,|\rho(x)-\rho(y)| \le \frac{1}{\mu}\mathbf{E}\,|\Pi_q|\,|x-y| \le \frac{\eta}{\mu}\mathbf{E}\,|\Pi_q|\,.$$

Finally, taking the limits $\lim_{\epsilon\to 0}\lim_{\delta\to 0}\lim_{\mu\to 0}\lim_{\eta\to 0}$ implies Lemma A.9. □

A4) General Markov properties of $(x_n)_{n\in\mathbb{N}}$

We consider now the Markov chain $(x_n)_{n\in\mathbb{N}}$ as defined in (1.9).

Lemma A.10. *Assume that conditions $\mathbf{D_1}-\mathbf{D_2}$ hold, $q\ge 2$ and $a_q^2+\sigma_q^2>0$. Then the following hold.*

(a) The distribution of the random vector x_{2q+1} has the following properties: let A be a measurable set in S and denote by $\Lambda(\cdot)$ the Lebesgue measure on $\mathcal{B}(S)$, then

(i) if $\Lambda(A)>0$ then $\inf_{y\in S}\mathbf{P}_y(x_{2q+1}\in A)>0$ and $\inf_{y\in S}\tilde{\mathbf{P}}_y(x_{2q+1}\in A)>0$;

(ii) if $\Lambda(A)=0$ then $\mathbf{P}_y(x_{2q+1}\in A)=0$ and $\tilde{\mathbf{P}}_y(x_{2q+1}\in A)=0$ for all $y\in S$.

(b) The Markov chain $(x_n)_{n\in\mathbb{N}}$ (with respect to both measures $\mathbf{P}$ and $\tilde{\mathbf{P}}$) is Λ-irreducible and aperiodic. Moreover, every measurable subset of S is small.

Proof. (a) Recall that $x'_n=x'\Pi_n/|x'\Pi_n|$. Note that for every $x\in S$ and every measurable set $A\in S$,

$$\mathbf{P}_x(x_{2q+1}\in A)=\mathbf{P}(x'\Pi_{2q+1}\in B_A)\,,\qquad \tilde{\mathbf{P}}_x(x_{2q+1}\in A)=\tilde{\mathbf{P}}_x(x'\Pi_{2q+1}\in B_A)\,,$$

where $B_A=L^{-1}(A)=\{y\in\mathbb{R}^q\setminus\{0\}\,:\,L(y)\in A\}$ and $L(y)=y/|y|$. From (A.8)–(A.9) we obtain for some $0<\delta<1$

$$\mathbf{P}^{2q+1}(x,A)\ge p_*(\delta)\,\mu_\delta(B_A)=\nu_\delta(A)\,,\qquad \tilde{\mathbf{P}}^{2q+1}(x,A)\ge \tilde{p}_*(\delta)\,\tilde{\mu}_\delta(B_A)=\tilde{\nu}_\delta(A)\qquad\text{(A.15)}$$

for positive constants $p_*(\delta)$ and $\tilde{p}_*(\delta)$.

Next we show

$$\Lambda(A)>0\quad\Rightarrow\quad \mathrm{mes}(B_A)>0\,.\qquad\text{(A.16)}$$

Recall that $q\ge 2$, hence if $\Lambda(A)>0$ there exists a open set $V\subseteq A\subseteq S$ with $\Lambda(V)>0$. Then $L^{-1}(V)\subseteq B_A$, but this set is open and nonempty in $\mathbb{R}^q$ ($L(\cdot)$ is a continuous function on $\mathbb{R}^q\setminus\{0\}$ and $V\subset L^{-1}(V)$), therefore $\mathrm{mes}(L^{-1}(V))>0$, which gives (A.16). If $\mathrm{mes}(B_A)>0$ then, by Lemma A.8, there exists some $\delta>0$ such that $\mu_\delta(B_A)>0$ and $\tilde{\mu}_\delta(B_A)>0$. Then (i) follows from (A.15). Next we show now that

$$\Lambda(A)=0\quad\Rightarrow\quad \mathrm{mes}(B_A)=0\,.\qquad\text{(A.17)}$$

Assume that $\mathrm{mes}(B_A)>0$. Then there exists an open set $V\subset B_A$ with $\mathrm{mes}(V)>0$. By definition of B_A the image $U=L(V)=\{L(y)\;\; y\in V\}\subseteq A$. We show that U is an open set in S. Indeed, for $z_0\in U$ there exists $y_0\in V$ such that $z_0=L(y_0)=y_0/|y_0|$. Since V is open there exists some $\delta>0$ such that $\{y\in\mathbb{R}^q\,:\,|y-y_0|<\delta\}\subset V$. Set $\varepsilon=\delta/|y_0|$ and take $z\in S$ such that $|z-z_0|<\varepsilon$. Note that for $y_z=|y_0|z$ we have $L(y_z)=z$ and

$$|y_z-y_0|=|y_0||z-z_0|<|y_0|\varepsilon=\delta\,.$$

Hence $y_z \in V$ and therefore $z \in U$, i.e. $\{z \in S : |z - z_0| < \varepsilon\} \subset U$. Consequently $U = L(V)$ is an open set in S. For $q \geq 2$ the Lebesgue measure of any open non-empty set in S is positive. This is a contradiction to $\Lambda(A) = 0$ and hence (A.17) holds. Furthermore, if $\mathrm{mes}(B_A) = 0$ then by Lemma A.5 and Corollary A.6,

$$\mathbf{P}_y(x_{2q+1} \in A) = \mathbf{E}\,\mathbf{P}(y'\Pi_{2q+1} \in B_A|\rho(y)) \;=\; \mathbf{E}\int_{B_A} p_1(z|\rho(y))\mathrm{d}z = 0\,.$$

$$\tilde{\mathbf{P}}_y(x_{q+1} \in A) = \tilde{\mathbf{E}}_y\,\tilde{\mathbf{P}}_y(y'\Pi_{2q+1} \in B_A|\rho(y)) = \tilde{\mathbf{E}}_y\int_{B_A} \tilde{p}_1(z|\rho(y))\mathrm{d}z = 0\,.$$

(b) Note that (i) immediatly implies Λ-irreducibility. From inequalities (A.15) we conclude then that every measurable subset in S is small. To prove aperiodicity suppose that there exist sets $D_1, \ldots, D_d$ in $\mathcal{B}(S)$ such that $\mathbf{P}_x(x_1 \in D_{i+1}) = 1$ for $x \in D_i$ and some $i \in \mathbb{N}$. (Recall that $D_i = D_{j(i)}$ for $j(i) \in \{1, \ldots, d\}$ and $j(i) = i \pmod d$; i.e. $(D_i\,,\,{}_{i\geq 1}) = \{D_1, \ldots, D_d, D_1, \ldots, D_d, \ldots\}$). Hence, if $x \in D_1$ then $\mathbf{P}_x(x_i \in D_{i+1}) = 1$ for all $i \in \mathbb{N}$. Therefore, denoting $m = 2q+1$ and $\mathbf{P}^i(x, D) = \mathbf{P}_x(x_i \in D)$ we get for all $i \in \mathbb{N}$

$$\mathbf{P}_x(x_{m+i} \in D_{m+i+1}) = \int_S \mathbf{P}^i(x, \mathrm{d}y)\mathbf{P}_y(x_m \in D_{m+i+1}) = 1\,.$$

Then (ii) implies $\Lambda(D_1) > 0, \ldots, \Lambda(D_d) > 0$. On the other hand, however, $\mathbf{P}_x(x_m \in D_{m+1}) = 1$ implies that $\mathbf{P}_x(x_m \in D_i) = 0$ for all $D_i \neq D_{m+1}$ and by (i) we obtain $\Lambda(D_i) = 0$. This contradicts $\Lambda(D_i) > 0$. Therefore $d = 1$ with respect to $\mathbf{P}$. Aperiodicity with respect to $\tilde{\mathbf{P}}$ is obtained by the same argument. This concludes the proof of Lemma A.10. □

Lemma A.11. *Assume that conditions* $\mathbf{D_1} - \mathbf{D_2}$ *hold,* $q \geq 2$ *and* $a_q^2 + \sigma_q^2 > 0$. *Then the Markov chain* $(x_n)_{n\geq 0}$ *with state space* S *is positive Harris recurrent and uniform geometric ergodic with respect to* $\mathbf{P}$ *(and* $\tilde{\mathbf{P}}$*). It has invariant measure* $\pi(\cdot)$ *(and* $\tilde{\pi}(\cdot)$*, respectively), which is equivalent to Lebesgue measure* $\Lambda(\cdot)$ *on* S.

Proof. Define $V : \mathbb{R}^q \to [1, \infty)$ by $V(y) = 1 + |<y>_1|$. Then

$$\mathbf{E}_x\,V(x_1) = 1 + \mathbf{E}\,\frac{|<x'A_1>_1|}{|x'A_1|} = L(x)\,V(x)\,,$$

where

$$L(x) = \frac{1}{V(x)}\left(1 + \mathbf{E}\frac{|\alpha_1(1) <x>_1 + <x>_2|}{|x'A_1|}\right)\,.$$

Since $a_q^2 + \sigma_q^2 > 0$ implies that $\alpha_q^2(1) > 0$ $\mathbf{P}$-a.s. we obtain

$$\lim_{|<x>_1|\to 1\,:\,x\in S} L(x) = \frac{1}{2}\left(1 + \mathbf{E}\,\frac{|\alpha_1(1)|}{|\alpha(1)|}\right) \leq \frac{1}{2}\left(1 + \mathbf{E}\,\frac{|\alpha_1(1)|}{\sqrt{\alpha_1^2(1) + \alpha_q^2(1)}}\right) < 1\,.$$

Similarly we obtain

$$\tilde{\mathbf{E}}_x V(x_1) = 1 + \tilde{\mathbf{E}}_x \frac{|< x'A_1 >_1|}{|x'A_1|} = \tilde{L}(x)V(x)\,,$$

where by (4.3)

$$\tilde{L}(x) = \frac{1}{V(x)}\left(1 + \tilde{\mathbf{E}}_x \frac{|< x'A_1 >_1|}{|x'A_1|}\right) = \frac{1}{V(x)}\left(1 + \mathbf{E}\,\varsigma(x)\frac{|< x'A_1 >_1|}{|x'A_1|}\right)\,,$$
$$\varsigma(x) = \frac{1}{h(x)}\,|x'A_1|^{\lambda}\,h(\overline{x'A_1})\,.$$

Since $\varsigma(\cdot)$ is a continuous positive function on S with $\mathbf{E}\,\varsigma(x) = 1$ for all $x \in S$, we find by the same argument

$$\lim_{|<x>_1|\to 1\,:\,x\in S} \tilde{L}(x) = \frac{1}{2}\left(1 + \tilde{\mathbf{E}}_{x_0}\frac{|\alpha_1(1)|}{|\alpha_1|}\right) = \frac{1}{2}\left(1 + \mathbf{E}\,\varsigma(x_0)\frac{|\alpha_1(1)|}{|\alpha_1|}\right) < 1\,,$$

where $x_0 = (1, 0, \ldots, 0)'$. Thus, for $\varepsilon > 0$ there exists $0 < r < 1$ such that

$$\sup_{|<x>_1|>r} L(x) < 1 - \varepsilon\,, \qquad \sup_{|<x>_1|>r} \tilde{L}(x) < 1 - \varepsilon\,,$$

and we obtain that the function $V(\cdot)$ satisfies the conditions of Lemma A.1 on the set $\Gamma = \{x \in S \;:\; |< x >_1| \le r\}$. By the second part of Lemma A.10 every subset of S is small and therefore, by Lemma A.1, $(x_n)_{n\ge 0}$ is uniform geometric ergodic with respect to both measures $\mathbf{P}$ and $\tilde{\mathbf{P}}$. It has stationary distributions $\pi(\cdot)$ and $\tilde{\pi}(\cdot)$, respectively.

Next we use Lemma A.10(a) to show that π respectively $\tilde{\pi}$ are equivalent to Lebesgue measure on S.
If $\pi(A) = \lim_{n\to\infty} \mathbf{P}_x(x_n \in A) = 0$ and $\Lambda(A) > 0$, then Lemma A.10(a,i) we obtain the following contradiction

$$\pi(A) = \lim_{n\to\infty} \mathbf{P}_x(x_{n+2q+1} \in A) = \lim_{n\to\infty} \int_S \mathbf{P}_y(x_{2q+1} \in A)\mathbf{P}^{(n)}(x, \mathrm{d}y) \ge \inf_{y\in S} \mathbf{P}_y(x_{2q+1} \in A) > 0\,.$$

Next, if $\Lambda(A) = 0$ then by Lemma A.10(a,ii)

$$\pi(A) = \lim_{n\to\infty} \mathbf{P}_x(x_{n+2q+1} \in A) = \lim_{n\to\infty} \int_S \mathbf{P}_y(x_{2q+1} \in A)\mathbf{P}^{(n)}(x, \mathrm{d}y) = 0\,.$$

Hence $\pi(\cdot)$ and $\Lambda(\cdot)$ are equivalent on S. In the same way we obtain the equivalence of $\tilde{\pi}(\cdot)$ and $\Lambda(\cdot)$ on S. $\square$

A5) A property of ψ_0.

Lemma A.12. *If conditions* $\mathbf{D_0}$ *and* $\mathbf{D_4}$ *hold, then the function* $\psi_0(x,u)$ *defined in (4.13) is non-negative and for all* $x = (<x>_1, \ldots, <x>_q)' \in S$ *with* $<x>_1 \neq 0$

$$mes(\{u \geq 0 \,:\, \psi_0(x,u) > 0\}) > 0\,, \tag{A.18}$$

where $mes(\cdot)$ *denotes Lebegues measure on* $\mathbb{R}$.

Proof. By definition we have

$$\psi_0(x,u) = \mathbf{P}(\tau_1 + \tau_2 > u) - \mathbf{P}(\tau_1 > u)$$

with $\tau_1 = x' A_1 Y_1$ and $\tau_2 = x'\zeta_1 =<x>_1 \xi_1$. If $<x>_1 = 0$, then $\tau_2 = 0$, and therefore $\psi_0(x,u) = 0$. We show that $\psi_0(x,u) \geq 0$ if $<x>_1 \neq 0$. By conditioning on τ_2 we get

$$\psi_0(x,u) = \int_0^\infty \left(\mathbf{P}(u - t < \tau_1 \leq u) - \mathbf{P}(u < \tau_1 \leq u + t)\right) p_{\tau_2}(t)\mathrm{d}t\,,$$

where $p_{\tau_2}(\cdot)$ is the density of τ_2, which is by $\mathbf{D_4}$ symmetric and non-increasing on $[0,\infty)$. Setting $\mathcal{A} = \sigma\{A_i\,, i \in \mathbb{N}\}$, again by $\mathbf{D_4}$, the conditional density $p_{\tau_1}(\cdot|\mathcal{A})$ of τ_1 is symmetric and non-increasing on $\mathbb{R}_+$. Hence for $0 \leq t \leq u$ we have

$$\begin{aligned}
f_x(u,t) &= \mathbf{P}(u - t \leq \tau_1 \leq u) - \mathbf{P}(u < \tau_1 \leq u + t) \\
&= \mathbf{E}(\mathbf{P}(u - t \leq \tau_1 \leq u|\mathcal{A}) - \mathbf{P}(u < \tau_1 \leq u + t|\mathcal{A})) \\
&= \mathbf{E}(\int_{u-t}^{u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a - \int_u^{u+t} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a) \\
&= \mathbf{E}\int_{u-t}^{u} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a+t|\mathcal{A}))\mathrm{d}a \geq 0\,.
\end{aligned}$$

On the other hand, for $t > u$ we get

$$\begin{aligned}
f_x(u,t) &= \mathbf{E}\left(\int_{u-t}^{u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a - \int_u^{u+t} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a\right) \\
&= \mathbf{E}\left(\int_{u-t}^{0} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a + \int_0^{u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a - \int_u^{u+t} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a\right) \\
&= \mathbf{E}\left(\int_0^{t-u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a + \int_0^{u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a - \int_u^{2u} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a - \int_{2u}^{u+t} p_{\tau_1}(a|\mathcal{A})\mathrm{d}a\right) \\
&= \mathbf{E}\left(\int_0^{t-u} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a+2u|\mathcal{A}))\mathrm{d}a + \int_0^{u} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a+u|\mathcal{A}))\mathrm{d}a\right) \geq 0\,,
\end{aligned}$$

again since $p_{\tau_1}(\cdot|\mathcal{A})$ is non-increasing on $\mathbb{R}_+$. This proves the first part of this lemma.

We show now (A.18). Let $x \in S$ with $<x>_1 \neq 0$. If for this $x \in S$

$$\mathrm{mes}(\{u \geq 0 \,:\, \psi_0(x,u) > 0\}) = 0\,,$$

then $\psi_0(x,u) = 0$ for all $u \geq 0$, because the function $\psi_0(x,u)$ is continuous on $u \geq 0$, i.e.

$$\int_0^\infty \psi_0(x,u)\,\mathrm{d}u = \int_0^\infty \int_0^\infty f_x(u,t)\,\mathrm{d}u\, p_{\tau_2}(t)\,\mathrm{d}t = 0\,.$$

Since $\mathrm{mes}\{u \geq 0 \,:\, p_{\tau_2}(u) > 0\} > 0$ there exists $0 < t_0 < \infty$ such that

$$\int_0^\infty f_x(u,t_0)\,\mathrm{d}u = 0$$

and taking into account that the function $f_x(\cdot, t_0)$ is nonnegative and continuous we obtain that for all $u \geq t_0$

$$f_x(u,t_0) = \mathbf{E}\int_{u-t_0}^{u} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a+t_0|\mathcal{A}))\mathrm{d}a = 0\,,$$

giving

$$\mathbf{E}\int_0^{+\infty} (p_{\tau_1}(a|\mathcal{A}) - p_{\tau_1}(a+t_0|\mathcal{A}))\mathrm{d}a = \int_0^{+\infty} (p_{\tau_1}(a) - p_{\tau_1}(a+t_0))\mathrm{d}a = 0\,.$$

By monotonicity of $p_{\tau_1}(\cdot) = \mathbf{E}\, p_{\tau_1}(\cdot|\mathcal{A})$ we get for every $t \in [0, t_0]$

$$p_{\tau_1}(a) = p_{\tau_1}(a+t_0) = p_{\tau_1}(a+t) \text{ for almost all } a \geq 0\,. \tag{A.19}$$

Since $\mathrm{mes}(\{a \geq 0 \,:\, p_{\tau_1}(a) > 0\}) > 0$ (recall that the function $p_{\tau_1}(\cdot)$ is monotone on $\mathbb{R}_+$ and $\int_{\mathbb{R}_+} p_{\tau_1}(a)\mathrm{d}a = 1/2$) there exists some $a_* \geq 0$ with $p_{\tau_1}(a_*) > 0$ for which (A.19) is satisfied; i.e. $p_{\tau_1}(a_*) = p_{\tau_1}(a)$ for all $a \in [a_*, a_* + t_0]$. Now set $a^* = \max\{a \geq a_* \,:\, p_{\tau_1}(a) = p_{\tau_1}(a_*)\}$ and note that $a^* < \infty$ since $\int_{\mathbb{R}} p_{\tau_1}(a)\mathrm{d}a < \infty$. Therefore (A.19) implies the existence of some $a_0 \in (a^* - \delta\,, a^*) \subset [a_*, a^*]$ (with $\delta = \min((a^* - a_*)/2\,, t_0/2)$) such that

$$p_{\tau_1}(a_*) = p_{\tau_1}(a_0) = p_{\tau_1}(a_0 + t_0)\,.$$

But this is a contradiction to the definition of a^*, because $a_0 + t_0 > a^*$. This implies (A.18). □

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