A possible counterexample to uniqueness of entropy solutions and Godunov scheme convergence

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Abstract

A particular case of initial data for the two-dimensional Euler equations is studied numerically. The results show that the Godunov method does not always converge to the physical solution, at least not on feasible grids. Moreover, they suggest that entropy solutions (in the weak entropy inequality sense) are not unique.

1 Introduction

Consider the Cauchy problem for a system of hyperbolic conservation laws,

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{f}(u)) = 0, \tag{1}$$

$$u(0, \cdot) = u_0, \tag{2}$$

where $u = u(t, \vec{x}) : \mathbb{R}^{d+1}_+ := (0, \infty) \times \mathbb{R}^d \to P \subset \mathbb{R}^m$ is the desired solution (P the set of physically reasonable values), $\vec{f} = (f^i)$, $f^i : P \to \mathbb{R}^m$, the (smooth) *flux* function, $u_0 : \mathbb{R}^d \to \mathbb{R}^m$ initial data. Here and in the sequel "$\nabla, \Delta, \cdot$" are meant with respect to $\vec{x}$.

An important example of hyperbolic systems of conservation laws are the (nonisentropic) compressible Euler equations:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \vec{v}) &= 0, \\ (\rho v^i)_t + \nabla \cdot (\rho v^i \vec{v}) + p_{x_i} &= 0 \qquad (i = 1, \ldots, d), \\ (\rho e)_t + \nabla \cdot ((\rho e + p)\vec{v}) &= 0. \end{aligned} \tag{3}$$

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Here, ρ is density, $\vec{v} = (v^i)$ velocity, e specific energy, which decomposes into

$$e = \frac{|\vec{v}|^2}{2} + q; \tag{4}$$

the first summand is specific kinetic energy, q is specific internal energy (in this case, specific heat). The pressure is a function of ρ, q; a common choice is the polytropic pressure law

$$p = (\gamma - 1)\rho q \tag{5}$$

($1 < \gamma \leq \frac{5}{3}$; for air, $\gamma = \frac{7}{5}$). The set of admissible values is

$$P = \{q > 0,\ \rho > 0\}.$$

It is well-known that (1) and (2) need not have a global smooth solution, even if the initial data u_0 is smooth. For this reason, one has to study *weak solutions*, defined as functions $u \in L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$ that satisfy

$$-\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \phi}{\partial t} + \vec{f}(u) \cdot \nabla \phi \ d\vec{x}\ dt = \int_{\mathbb{R}^d} u_0(\vec{x}) \phi(0, \vec{x})\ d\vec{x}, \tag{6}$$

for all test functions $\phi \in C_c^\infty(\overline{\mathbb{R}^{d+1}_+})$. Moreover, (6) can have more than one solution, so it is necessary to impose an additional condition, called *entropy condition*, to single out a unique weak solution (the *entropy solution*).

One definition of entropy solutions is the *vanishing viscosity* (VV) definition; it requires that u is the limit of the sequence $(u^\epsilon)_{\epsilon > 0}$ of solutions of

$$\frac{\partial u^\epsilon}{\partial t} + \nabla \cdot (\vec{f}(u^\epsilon)) = \epsilon \Delta u^\epsilon \qquad \text{in } \mathbb{R}^{d+1}_+, \tag{7}$$

$$u^\epsilon(0, \cdot) = u_0 \qquad \text{on } \{0\} \times \mathbb{R}^d. \tag{8}$$

The limit is taken in some suitable topology, usually as a boundedly almost everywhere limit. We call such a function u a *VV solution*.

Another definition uses *entropy/entropy flux* (EEF) pairs $(\eta, \vec{\psi})$, where $\eta : P \to \mathbb{R}$ is a smooth strictly convex function, called *entropy*, whereas $\vec{\psi} = (\psi^1, \dots, \psi^d)'$ with smooth $\psi^i : U \to \mathbb{R}$ is called *entropy flux*; η and $\vec{\psi}$ are required to satisfy

$$\frac{\partial \psi^i}{\partial u^\alpha} = \sum_{\beta=1}^m \frac{\partial \eta}{\partial u^\beta} \frac{\partial f^{i\beta}}{\partial u^\alpha} \qquad (i = 1, \dots, d,\ \alpha = 1, \dots, m). \tag{9}$$

By multiplying (7) from the left with $\eta'(u^\epsilon)$ and using (9), one obtains

$$\frac{\partial(\eta \circ u^\epsilon)}{\partial t} + \sum_{i=1}^d \frac{\partial(\psi^i \circ u^\epsilon)}{\partial x^i} = \epsilon \Delta(\eta \circ u^\epsilon) - \epsilon \sum_{i=1}^d \eta''(u^\epsilon) \frac{\partial u^\epsilon}{\partial x^i} \frac{u^\epsilon}{\partial x^i} \leq \epsilon \Delta(\eta \circ u^\epsilon). \tag{10}$$

(here, we used that η is convex). Upon multiplying the last equation with a *nonnegative* test function ϕ and integrating by parts, this yields

$$-\int_0^\infty \int_{\mathbb{R}^d} \eta(u^\epsilon)\frac{\partial\phi}{\partial t} + \vec{\psi}(u^\epsilon)\cdot\nabla\phi \; d\vec{x}\; dt \le \epsilon \int_0^\infty \int_{R^d} \eta(u^\epsilon)\Delta\phi \; d\vec{x}\; dt + \int_{\mathbb{R}^d} \eta(u_0)\; d\vec{x}. \tag{11}$$

If, as assumed above, $(u^\epsilon) \to u$ boundedly almost everywhere, then (11) implies

$$-\int_0^\infty \int_{\mathbb{R}^d} \eta(u)\frac{\partial\phi}{\partial t} + \vec{\psi}(u)\cdot\nabla\phi \; d\vec{x}\; dt \le \int_{\mathbb{R}^d} \eta(u_0)\; d\vec{x}. \tag{12}$$

Functions u that satisfy (12) for *all* EEF flux pairs are called *EEF solutions* (of (1)). As we have shown, VV solutions are necessarily EEF solutions.

In the literature, the term *entropy solution* is used to refer either to EEF or to VV solutions, often without explicit mention, because it has been assumed that the two definitions are equivalent for the Euler equations and many other physically relevant systems (see [Ser99] p. 101, [Daf00] p. 49, [GR96] p. 32; see the discussion in Section 5 for verified special cases). However, for the purposes of this paper it is necessary to distinguish the two notions, as we will discuss a possible numerical counterexample to their equivalence.

The (gas-dynamic) specific entropy s is defined as

$$s = \log q + (1-\gamma)\log\rho; \tag{13}$$

$$\eta := -\rho s, \quad \psi^i := -\rho s v^i \tag{14}$$

provides an EEF pair for the Euler equations.

A common simplification is to assume that s is constant in space and time. This yields the *isentropic* Euler equations

$$\begin{aligned} \rho_t + \nabla\cdot(\rho\vec{v}) &= 0, \\ (\rho v^i)_t + \nabla\cdot(\rho v^i \vec{v}) + p_{x_i} &= 0 \qquad (i = 1,\ldots,d) \end{aligned} \tag{15}$$

with

$$p(\rho) = \rho^\gamma. \tag{16}$$

In this case, $P = \{\rho > 0\}$. An EEF pair is provided by the specific energy e,

$$e = \frac{|\vec{v}|^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1},$$

with

$$\eta := \rho e, \quad \psi^i := (\rho e + p)v^i.$$

It is cumbersome to verify the EEF condition (12) directly, not to mention the VV condition. There are easier criteria for piecewise smooth functions, which we define in the following customized way:

Definition 1. 1. A point $(t, \vec{x}) \in \mathbb{R}^{d+1}_+$ is called *point of smoothness* if u is C^∞ in a small neighbourhood of $(t, \vec{x})$.

2. A point $(t, \vec{x}) \in \mathbb{R}^{d+1}_+$ is called *point of piecewise smoothness* of u if there is a C^∞ diffeomorphism Φ of a ball V around 0 in $\mathbb{R}^{d+1}$ onto a neighbourhood B of $(t, \vec{x}) = \Phi(0)$ so that $u \circ \Phi$ is C^∞ on B_- and on B_+ (where $B_\pm := \Phi(V_\pm)$, $V_\pm := \{y \in V : y_1 \gtrless 0\}$); for later use, let S be the surface $\Phi(V \cap (\{0\} \times \mathbb{R}^d))$, $n = (n^t, \vec{n}) \in \mathbb{R}^{d+1}$ a unit normal to S in $(t, \vec{x})$ pointing into B_+; let u_+, u_- be the one-sided limits of u in $(t, \vec{x})$ within B_- resp. B_+). We also require $\vec{n} \neq 0$.

3. u is called *piecewise smooth* if there is a set N of d-dimensional Hausdorff measure 0 so that all points in $\mathbb{R}^{d+1}_+ - N$ are points of piecewise smoothness.

Proposition 1. *Let u be piecewise smooth. u is an EEF solution of* (1) *if and only if*

1. *it is a (classical) solution of* (1) *in each point of smoothness,*

2. *$u(t, \cdot) \to u_0$ in L^1_{loc} as $t \downarrow 0$, and*

3. *in each point (t, x) of piecewise smoothness it satisfies the Rankine-Hugoniot conditions*

$$(u_+ - u_-)n^t + (\vec{f}(u_+) - \vec{f}(u_-)) \cdot \vec{n} = 0 \tag{17}$$

and (for all EEF pairs $(\eta, \vec{\psi})$)

$$(\eta(u_+) - \eta(u_-))n^t + (\vec{\psi}(u_+) - \vec{\psi}(u_-)) \cdot \vec{n} \leq 0. \tag{18}$$

Theorem 1 is well-known (see, for example, Section 11.1.1 in [Eva98]), as is the following property:

Proposition 2. *For the Euler equations* (3) *resp.* (15) *(with polytropic gas law* (5) *resp.* (16)*),* (18) *is equivalent to the simpler condition that the normal velocity does not increase across discontinuities:*

$$(\vec{v}_+ - \vec{v}_-) \cdot \vec{n} \leq 0.$$

The Cauchy problem for the Euler equations has several important symmetry properties, including the following:

Proposition 3. *Let $u = (\rho, \vec{v}', q)'$ be a weak solution for initial data $u_0 = (\rho_0, \vec{v}_0', \vec{q}_0)'$.*

1. *Change of inertial frame: For all $\vec{w} \in \mathbb{R}^d$, $(\rho(x + \vec{w}t), (\vec{v}(x + \vec{w}t, t) - \vec{w})', q(x + \vec{w}t))'$ is a weak solution for the same initial data u_0.*

2. *Self-similarity: a function $f : \mathbb{R}_+^{d+1} \to \mathbb{R}^m$ is called self-similar if $f(rt, r\vec{x}) = f(t, \vec{x})$ for all $r > 0$; same for functions on $\mathbb{R}^d$. If the initial data is self-similar, then for any $r > 0$, $u(r\vec{x}, rt)$ is a weak solution for the same initial data u_0.*

These symmetries remain true after replacing "weak" by "VV" or "EEF". Analogous symmetries hold for the isentropic case.

2 Example and numerical results

Consider the following set u_0 of initial data for (3) with $d = 2$ (see Figure 1): the data is symmetric under reflection across the x-axis and constant in each of four cones centered in the origin (in particular, constant along rays starting in the origin). In the origin, two weak shocks emanate into the first and fourth quadrant; the area on the left is supersonic inflow (parallel to the x-axis); the two areas on the other side of the shocks are denser and hotter gas, moving parallel to the contact discontinuities (see [CF48] Chapter IV C on choosing pre- and post-shock values that satisfy the Rankine-Hugoniot conditions; we choose the ones that yield the weaker shock). The gas in the stagnation area (enclosed by the contact discontinuities) has the same pressure as the post-shock gas on the other side, but velocity $\vec{v} = 0$. It is easy to check, using Propositions 1 and 2, that the steady solution $u(t, \vec{x}) = u_0(\vec{x})$ is an EEF solution of (3) resp. (15).

However, the numerical results in Figure 2 suggest that there is a second solution which is *not steady*, but *self-similar*. Adaptive refinement was used to achieve better resolution at same computational cost. To reduce numerical viscosity the grid was chosen so that near the right domain boundary the edges are aligned with the contact discontinuity and the shock. In order to capture self-similarity, the computations were done for a grid with moving vertices with coordinates $\vec{x} = t\vec{\xi}$ ($\vec{\xi}$ has the dimension of a velocity; its components are called *similarity coordinates*). The moving-edge modifications discussed in Section 2.1.6 of [Ell00] were used (the essential idea is to compute numerical fluxes across a moving edge by transforming to a steady edge, using invariance under change of inertial frame (Proposition 3), and to apply an arbitrary approximate Riemann solver to the transformed problem). The domain boundaries were chosen so that small perturbations on them propagate into the domain ($\vec{\xi} \cdot \vec{n}$ ($\vec{n}$ outer unit normal) in each boundary point is larger than the maximum of $|\vec{v} + c|$ in the domain). This allows to prescribe all components of the fluxes on the boundary.

Various modifications such as changing the numerical scheme (the experiments were repeated for the Godunov scheme [God59], the Osher-Solomon scheme [OS82], the ENO-RF scheme [SO89], and a second-order MUSCL code based on the first-order ENO-RF scheme), adding more numerical dissipation, refining uniformly rather than adaptively, using a Cartesian grid including origin and lower half-plane, or calculating in space rather than similarity coordinates, do not change the results significantly.

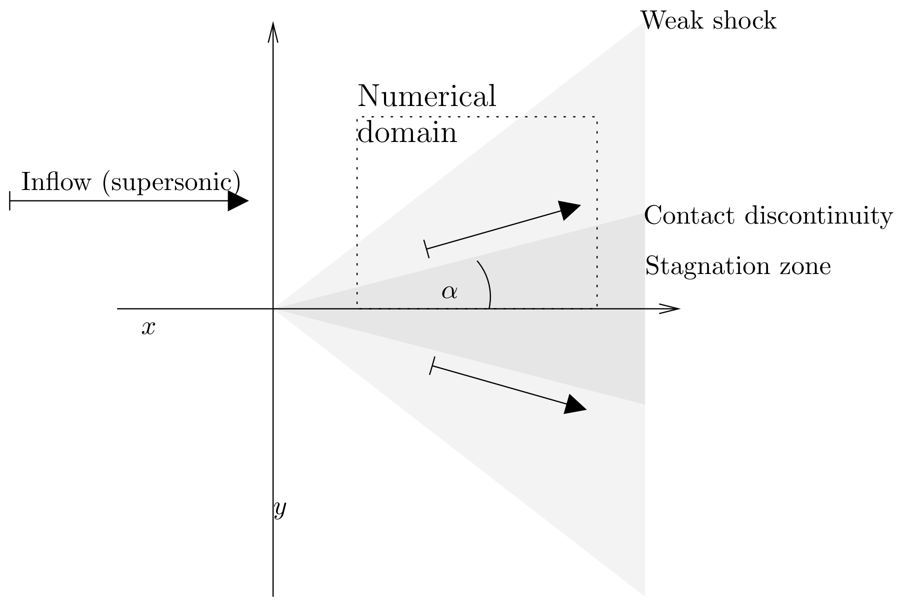

Figure 1: An example for the compressible Euler equations in 2D; this initial data is also a steady EEF solution.

3 Discussion

There are three possible explanations for the example and the numerical results: either numerical approximation is deceptive, or EEF solutions are not stable (in the sense of continuous dependence on initial data), or they are not unique.

3.1 Breakdown of numerical methods

It is possible that Figure 2 does not correspond to an EEF solution; in this case, our example provides an unprecedented case of failure of numerical methods. "Steadyness" or "self-similarity" are non-generic properties that will usually not be inherited to finite-accuracy numerical solutions (for example for a Riemann problem that is solved exactly by a single shock, most numerical schemes produce small additional waves and a slightly different shock). However, one would expect *almost* steady resp. self-similar numerical solutions, unlike Figure 2.

While in one space dimension, the conservation property of numerical schemes often guarantees accurate shock locations, in two or more dimensions numerical inaccuracy can significantly change the shape and location of shocks. It cannot be ruled out that this effect is responsible for the result in Figure 2. For example, the upwards deflection of the incoming flow by the high-pressure area in front of the stagnation region could be weaker in the numerical solution than in the exact solution; the additional pressure would cause the stagnation region to collapse. However, in this case, one would expect the results to depend strongly

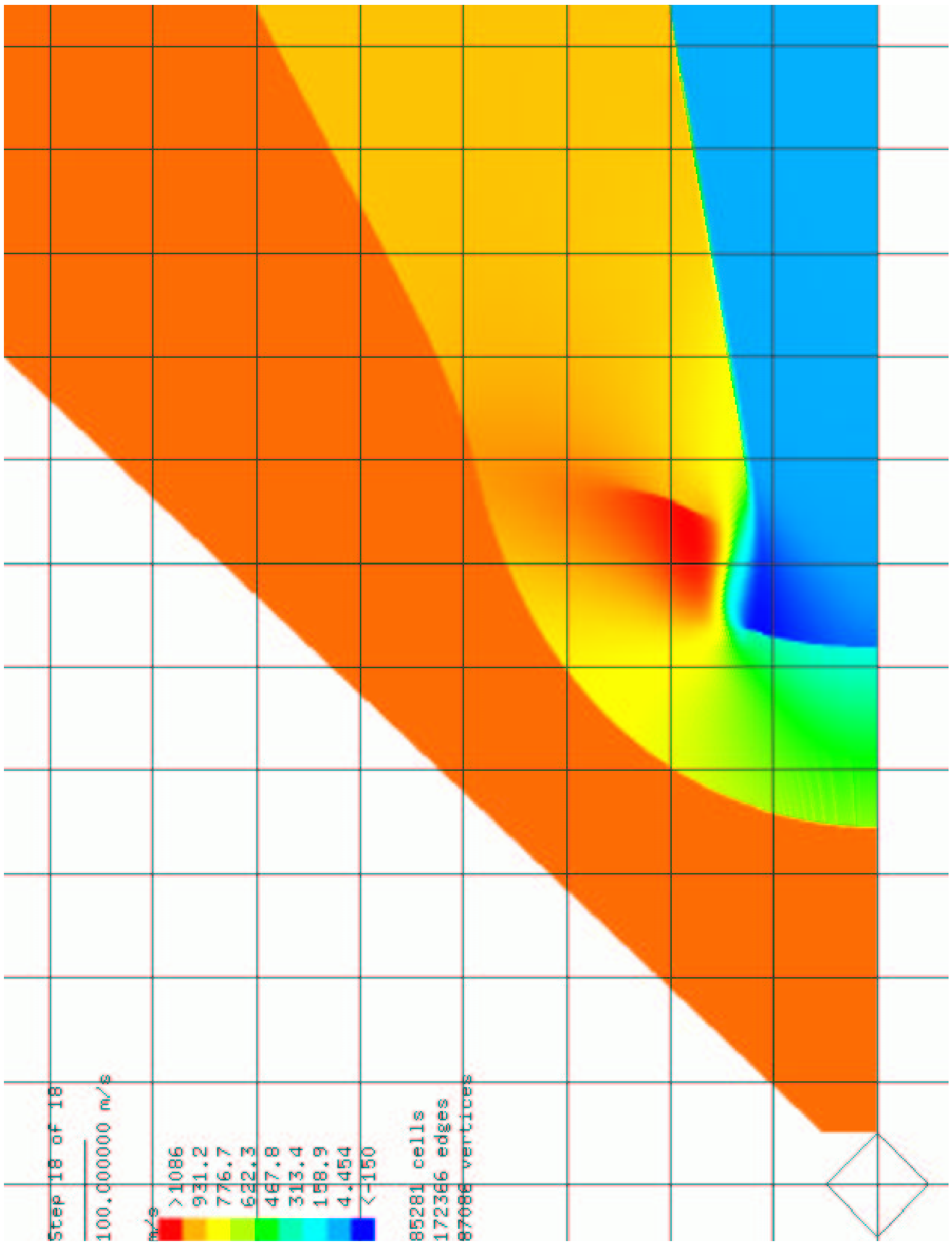

Figure 2: Plot of v^x (horizontal velocity), in similarity coordinates. Computation for the isentropic Euler equations with the Godunov scheme. Rotate clockwise by 90^o to align with Figure 1: Each square in the coordinate grid corresponds to a $100\ m/s \times 100\ m/s$ square in the $\vec{\xi}$ plane. The numerical domain is the non-black part of the plot (for better resolution, only the interesting part of the domain is displayed); the bottom coincides with the ξ_1-axis; the origin is marked by the diamond in the lower left corner. Data: $\gamma = 1.4$, $\alpha = 10^o$; inflow: $\rho = 1.19\ kg/m^3$, $v = 1000\ m/s$, $T = \vec{2}0^oC$. The solution differs significantly from Figure 1; the shock and contact discontinuity are perturbed and pushed away from the origin with speed $340\ m/s$ (= the value of ξ_1 where the first shock meets the ξ_1-axis). The results for nonisentropic Euler equations or other numerical schemes are similar.

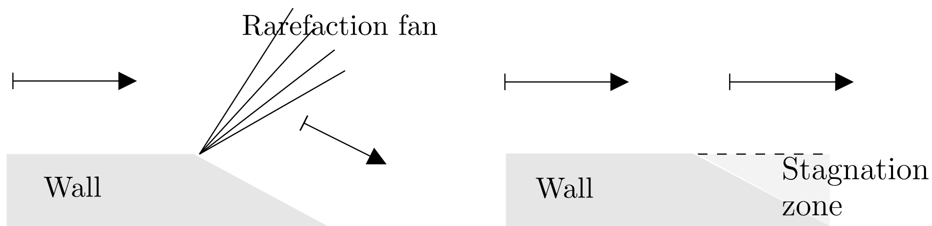

Figure 3: Supersonic inflow from the left reaches a corner. Left: physical solution. Right: solution with a stagnation area behind the corner; this solution is not observed in experiments.

on the choice of numerical method and parameters. This is not observed; rather all choices produce essentially the same results.

There is also a strong theoretical argument: the Lax-Wendroff theorem (see [LW60]; see also [GR96], [KRW96] and most generally [Ell03] for Lax-Wendroff-type theorems for irregular grids) states that if the numerical scheme is consistent and (like the Godunov scheme, for example) satisfies a discrete entropy inequality (see [HHL76, MO79, OC84, Tad84, Tad87, OT88]), then the limit of a boundedly almost everywhere converging sequence of numerical solutions is an EEF solution. (However, it is necessary to generalize the Lax-Wendroff theorem to cases with boundary conditions before its application is fully justified in this context; this is subject of ongoing research). Our numerical solutions do, on inspection, appear to converge quickly; this would imply that their limits are EEF solutions.

Finally, it seems unlikely that the solution in Figure 1 is steady under small perturbations at the origin (such as those from viscous terms in the VV limit). Note that a planar shock, with inflow state on one side and stagnation area state on the other side, would *not* be steady but move into the stagnation zone quickly — it seems unlikely that the example data, which has less mass and energy and more x-momentum in the $\{x > 0\}$ halfplane, would yield a steady pattern in the origin (however, the “maximum principle” implied in this argument is merely heuristic and may be wrong in some instances).

Figure 1 is closely related to the well-known Prandtl-Meyer problem of flow along solid walls with corners. The physically observed pattern, which coincides with the steady potential flow solution (which is in particular an EEF solution of the Euler equations) is on the left of Figure 3 (see Section 111 in [CF48] for discussion). On the right is another steady EEF solution; as in Figure 1, there is flow along a contact discontinuity with a stagnation area on the other side. This solution is not observed in physical or numerical experiments.

This analogy is further evidence that Figure 2 is indeed an approximation to the physically correct solution for the initial data in Figure 1.

3.2 Instability

Another possibility is that the numerical solutions are approximations to an unsteady EEF solution that evolves from a slight perturbation of the initial values (such perturbations are inevitable in most numerical computations due to inexact arithmetic, discretization error, artificial viscosity etc.) Such a solution might appear to be self-similar without being, e.g. because it is asymptotically self-similar as $t \to \infty$. Since Figure 2 is produced (up to minor differences) for any "perturbation" (i.e. for any choice of mesh, numerical method and parameters), it would indicate that Figure 1 constitutes a set of initial values where the Euler equations are not stable.

However, in light of the self-similarity of (3), it seems likely that asymptotically self-similar solutions indicate the presence of a self-similar solution nearby. Indeed, it is straightforward to establish this fact.

Theorem 1. *Let $u \in L^\infty(\mathbb{R}^{d+1}_+)$ be an EEF solution of* (6)*. Assume that u is asymptotically self-similar (see Definition 2), then its asymptotic limit w (a self-similar function) is an EEF solution as well.*

The proof of Theorem 1 and an analogous result for steady solutions are presented in the Appendix. On inspection in similarity coordinates, it appears that the numerical solutions for initial values in Figure (1) are bounded, converge quickly to the pattern in Figure 2 and remain steady (many orders of magnitude of time have been observed), so they suggest that there is an EEF solution that is at the least asymptotically self-similar (in fact, it appears to be self-similar, up to the available grid resolution). Theorem 1 shows that the asymptotic limit is a self-similar EEF solution (for the same initial values) which, being close to Figure 2, is clearly not Figure 1. Hence it would constitute a counterexample to uniqueness; instability seems to be an unsatisfactory explanation.

3.3 Nonuniqueness of EEF solutions

The third alternative is that Figures 1 and 2 provide a counterexample to uniqueness of EEF solutions.

The initial data is reminiscent of expansion shocks (the classical 1D example for unphysical weak solutions and nonuniqueness of weak solutions): in the origin, the gas in the upper and lower post-shock cones moves apart. Expansion shocks cannot appear in solutions to (uniformly) viscous perturbations of systems of conservation laws because the smoothing effect of viscosity smears the shock and the subsequent convection spreads the flow into a rarefaction fan; this happens for arbitrarily small (but positive) viscosity coefficient. It is likely that similarly, in Figure 1, slight smoothing from small viscous terms breaks up the shock pattern in the origin and causes the evolution into the self-similar solution in Figure 2. We believe that Figure 1 is not a VV solution.

In this context, it is interesting to point out that (under some mild restrictions) the EEF condition (12) is "insensitive" to sets with $(d-1)$-dimensional

Hausdorff measure 0 (such as a single point, for $d = 2$); e.g. if (12) is satisfied for $\phi \in C_c(\overline{\mathbb{R}_+^{d+1}} - \{0\})$, it is satisfied for *all* $\phi \in C_c(\overline{\mathbb{R}_+^{d+1}})$.

4 Conclusions about numerical methods

If we assume that the steady solution in Figure 1 is the physical solution, many — possibly most — popular numerical schemes suffer from unprecedented failure to converge to it. While it is still possible that, for some unknown reason, they ultimately converge to Figure 1 as $h \downarrow 0$, they approach Figure 2 for computationally accessible values of h — which is all that matters for practical purposes.

On the other hand, if we assume that Figure 2 is the correct solutions, there is a trivial *theoretical* example of misconvergence: consider a numerical scheme with Godunov fluxes, exact arithmetic and a grid whose edges are exactly aligned with the discontinuities in Figure 1: it would have that solution as steady state for all choices of grid parameter h.

Even in the latter case, which is less catastrophic for numerical analysts and appears more likely to us, we have to conclude that discrete entropy inequalities are not sufficient to avoid convergence to unphysical solutions on feasible grids. Hence they lose a bit of their value as design principles for numerical schemes, although they are still useful as easy-to-check necessary conditions that are most likely sufficient in the 1D case (as supported by the recent work on small total variation solutions described in Section 5).

5 Related work

For multidimensional scalar ($m = 1$) conservation laws with arbitrary f, [Kru70] (generalizing earlier work) shows that a global EEF solution exists, is unique and satisfies the VV condition as well, and that the solution operators form a monotone and L^1-contractive semigroup (hence EEF solutions are stable under L^1 perturbations of the initial data).

[Gli65] provides a famous existence proof for strictly hyperbolic *systems* with genuinely nonlinear fields and initial data with small total variation; the interaction functionals constructed in this paper are a crucial ingredient for all subsequent work. [Liu81] extends the result to systems with linearly and some nonlinearly degenerate fields. [BCP00] constructed the Standard Riemann Semigroup (SRS), an L^1-stable semigroup of EEF solutions for initial data with small total variation, for strictly hyperbolic systems with genuinely nonlinear or linearly degenerate fields (see also [LY99]). [BL97] showed that EEF solutions to 1D systems are unique and coincide with the SRS solutions, under certain smoothness assumptions including small total variation (see also [BG99]). [BB01] prove that for small TV initial data and strictly hyperbolic (but otherwise arbitrary) systems VV solutions exist and are stable under L^1 perturbations of the initial data, so for the class of solutions that are subject both to [BL97] and to [BB01], EEF and VV solutions are equivalent.

On the other hand, an EEF pair $(\eta, \vec{\psi})$ has to satisfy the condition (9) which is an overdetermined problem for $m \geq 3$, so for some systems no EEF pairs exist and the EEF condition is void. However, EEF pairs do exist for most physically relevant systems, even those with $m \geq 3$. More seriously, for certain 2×2 systems (with nonlinear degenerate fields) [CL81] construct a single weak shock that is an EEF solution but does not satisfy the Liu entropy condition (see [Liu74, Liu75]). By [BB01], there must be a VV solution (for the same initial data) that satisfies the Liu entropy condition as well — so it cannot be the aforementioned weak shock. Therefore the example in [CL81] also constitutes an example of a nonunique EEF solution, albeit for an "artificial" system with nonlinear degeneracy.

[Hop67] proposes the EEF condition for scalar conservation laws ($m = 1$), proves that it is implied by the VV condition under some circumstances and notes that there is a large set of convex entropies. Apparently independently, [Kru70] obtained analogous results for systems. [Lax71] contains the first use of the term "entropy condition" for the EEF condition. Various forms of the EEF condition had been known and in use for special systems such as the Euler equations for a long time (e.g. the *Clausius-Duhem inequality*), especially as shock relations; however, the above references seem to be the first to define the general notion of strictly convex EEF pairs, to propose the EEF condition as a mathematical tool for arbitrary systems of conservation laws and to formulate it in the weak form (12) rather than the special case (18).

[LZY98] provide an analytical and numerical discussion of 2D Riemann problems for various systems including the Euler equations. However, they focus on data constant in each of the four quadrants, so Figure 1 is not covered.

6 Final remarks

The example immediately applies to more than two dimensions, by constant extension in the other directions.

The total variation of the initial data cannot be made arbitrarily small because the oblique shock relations can be solved only for inflow velocity above some supersonic limit (which depends on α). It would be interesting to find modified examples with arbitrarily small total variation.

The results demonstrate that

the Godunov method need not converge to the physical solution (on feasible grids).

Assuming that EEF solutions are unique (which the author does not believe any more), they would demonstrate that

many popular numerical methods fail to converge to the correct solution (on feasible grids).

Moreover, they suggest the following conjecture:

EEF solutions to the Euler equations (in two or more dimensions) are not always unique.

If this conjecture was true, there would be far-reaching consequences. The EEF condition would not be sufficient as a selection principle for physical/unique solutions, except in special cases such as the ones described above. It would be necessary to find ways to use the (cumbersome) VV condition or to discover new entropy conditions.

Although the numerical results support the conjecture unambiguously, the question is so important that a rigorous proof is highly desirable. However, since the initial data has large vorticity at the contact discontinuity, it seems difficult to construct (or to prove results about) exact solutions. One possible line of attack is to derive novel entropy conditions from the VV condition and to check whether they are violated by the steady solution in Figure 1.

In any case, this paper motivates the investigation of multidimensional Riemann problems for systems; these appear to be very difficult and exhibit a large variety of phenomena (see [LL98, LZY98]). This goal requires techniques for proving existence of smooth steady or self-similar solutions to boundary-value problems for systems of nonlinear hyperbolic conservation laws; while there are methods for smooth solutions in hyperbolic regions, there are no tools for the elliptic and mixed case.

Finally, one could wonder whether Figure 1 is an example of a "generic" phenomenon or whether EEF solutions are unique for all initial data outside a "small" complement. This question is being investigated; we suspect that nonuniqueness is indeed generic.

Appendix: asymptotically steady and self-similar weak solutions

Remark: in Theorem 1 and in the following statements,

$$f(t,\cdot) \to g \qquad \text{in } L^1_{\text{loc}}(\Omega)$$

as $t \downarrow 0$ resp. $t \uparrow \infty$ is to be understood as: for all $\epsilon > 0$ and $K \Subset \Omega$ there is a $T = T(\epsilon) > 0$ so that for almost all $0 < t \leq T$ resp. $t \geq T$,

$$\|f(t,\cdot) - g\|_{L^1(K)} \leq \epsilon.$$

Lemma 1. *Let $u \in L^\infty(\mathbb{R}^{d+1}_+)$, $u_0 \in L^\infty(\mathbb{R}^d)$. If*

1. *$u(t,\cdot) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ as $t \downarrow 0$, and*
2. *u satisfies (6) for all $\phi \in C^\infty_c(\underline{(}0,\infty) \times \mathbb{R}^d)$,*

then u satisfies (6) for all $\phi \in C^\infty_c(\underline{[}0,\infty) \times \mathbb{R}^d)$.

Proof. Let $\theta \in C^\infty[0,\infty)$ so that $\theta = 1$ on $[0,1]$, $\theta = 0$ on $[2,\infty)$. For any $T > 0$, define $\theta_T(t) := \theta(T^{-1}t)$. Note that $|\theta_T| = O(1)$, $|\theta'_T| = O(T^{-1})$ (as $T \downarrow 0$). For given $\phi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$, split $\phi_1(t,x) := \theta_T(t)\phi(t,x)$ and $\phi_2 = \phi - \phi_1$.

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d} u\phi_t + \vec{f}(u)\cdot\nabla\phi \; dx\; dt \\ &= \int_0^{2T}\int_{\mathbb{R}^d} u\phi_{1t} + \vec{f}(u)\cdot\nabla\phi_1 \; dx\; dt + \int_T^\infty \int_{\mathbb{R}^d} u\phi_{2t} + \vec{f}(u)\cdot\nabla\phi_2 \; dx\; dt.\end{aligned}$$

Since $\phi_2 \in C_c([T,\infty)\times\mathbb{R}^d)$, the second summand vanishes by assumption. The first summand equals

$$\begin{aligned}&= O\left(T \sup_{0<t\le 2T} \|u(t,\cdot) - u_0\|_1 \cdot (T^{-1}+1)\right) + \int_0^{2T}\int_{\mathbb{R}^d} u_0\phi_{1t}\; dx\; dt \\ &+ O\left(T(\|u_0\|_1 + \sup_{0<t\le 2T}\|u(t,\cdot)-u_0\|_1\right) \\ &= O\left(\sup_{0<t\le 2T}\|u(t,\cdot)-u_0\|_1\right) - \int_{\mathbb{R}^d} u_0(x)\phi(0,x)\; dt \\ &+ O\left(T(\|u_0\|_1 + \sup_{0<t\le 2T}\|u(t,\cdot)-u_0\|_1\right).\end{aligned}$$

On taking $T \downarrow 0$, all O terms vanish; hence u satisfies (6). □

Remark: the converse of Lemma 1 (which is not needed) is not immediate because $u(t,\cdot) - u_0$ may be large for some t as long as the set of such t has small measure near 0.

Definition 2.

1. A function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+;\mathbb{R}^m)$ is called *asymptotically self-similar* if there is a function $w : \mathbb{R}^d \to \mathbb{R}^m$ so that

$$u(t, t^{-1}\cdot) \to w \qquad \text{in } L^1_{\mathrm{loc}}(R^d).$$

2. u is called *self-similar* if, for some w, $u(t,t^{-1}\cdot) = w$ for almost all $t > 0$.

Proof. (of Theorem 1) By Lemma 1, to show that w is a weak solution it is sufficient to check that

$$\int_0^\infty \int_{\mathbb{R}^d} w\left(\frac{x}{t}\right)\phi_t(t,x) + \vec{f}\left(w\left(\frac{x}{t}\right)\right)\cdot\nabla\phi(t,x)\; dx\; dt = 0 \tag{19}$$

for all $\phi \in C_c^\infty(]0,\infty))$. The essential idea is to scale coordinates to shift the support of ϕ into a large-t region and to use asymptotic convergence.

Let $0 < t_1 < t_2$ be such that $\operatorname{supp}\phi \subset [t_1,t_2]\times\mathbb{R}^d$. Let $\epsilon > 0$ be arbitrary, set $T = T(\epsilon)$ as in Definition 2. The change of coordinates $t = \frac{t_1}{T}\tau$, $x = \frac{t_1}{T}\xi$

changes the left-hand side of (19) into

$$\left(\frac{t_1}{T}\right)^{d+1} \int_T^{\frac{t_2 T}{t_1}} \int_{\mathbb{R}^d} w\left(\frac{\xi}{\tau}\right) \phi_t\left(\frac{t_1}{T}\tau, \frac{t_1}{T}\xi\right) + \vec{f}\left(w\left(\frac{\xi}{\tau}\right)\right) \cdot \nabla\phi(\tau,\xi)\ d\xi\ d\tau$$
$$= \left(\frac{t_1}{T}\right)^{d+1} \int_T^{\frac{t_2 T}{t_1}} \int_{\mathbb{R}^d} u(\tau,\xi)\phi_t\left(\frac{t_1}{T}\tau, \frac{t_1}{T}\xi\right) + \vec{f}(u(\tau,\xi)) \cdot \nabla_x\phi\left(\frac{t_1}{T}\tau, \frac{t_1}{T}\xi\right)\ d\xi\ d\tau$$
$$+ O\left(\left(\frac{t_1}{T}\right)^{d+1} \cdot T \cdot \epsilon T^d\right) \tag{20}$$

where O is with respect to $\epsilon \to \infty$. Note that the support of the scaled ϕ is in $[T,\infty) \times \mathbb{R}^d$. Also, the assumption that u is bounded is essential here. The second summand on the right-hand side equals

$$\int_T^{\frac{t_2 T}{t_1}} \int_{\mathbb{R}^d} u(\tau,\xi) \frac{T}{t_1} \phi\left(\frac{t_1}{T}\tau, \frac{t_1}{T}\xi\right)_\tau + \vec{f}(u(\tau,\xi)) \cdot \nabla_\xi(\phi(\tau,\xi))\ d\xi\ d\tau.$$

Since u is assumed to be a weak solution, this term vanishes. Taking $\epsilon \downarrow 0$ in (20) yields (19).

For the proof of the EEF part, replace u, w by $\eta(u), \eta(w)$ and $f(u), f(w)$ by $\psi(u), \psi(w)$ above. □

The same results as for self-similar weak solutions can be obtained for steady solutions:

Definition 3. 1. $u \in L^1_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ is called *steady* if, for some $w : \mathbb{R}^d \to \mathbb{R}^m$, $u(t,\cdot) = w$ for almost all $t > 0$.

2. $u \in L^1_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ is called *asymptotically steady* if there is a $w : \mathbb{R}^d \to \mathbb{R}^m$, so that

$$u(t,\cdot) \to w \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^d)$$

for almost all $t \geq T$.

Theorem 2. *If $u \in L^\infty(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ is an asymptotically steady and bounded weak solution, then w (as in Definition 3) is a weak solution as well. If u is an EEF solution, so is w.*

Proof. Let $\phi \in C_c^\infty((0,\infty) \times \mathbb{R}^d)$ be arbitrary. Let $\operatorname{supp}\phi \subset [0,\tau]$. For any $\epsilon > 0$,

$$\int_0^\infty \int_{\mathbb{R}^d} w(x)\phi_t(t,x) + \vec{f}(w(x)) \cdot \nabla\phi(t,x)\ dx\ dt$$
$$= \int_0^\infty \int_{\mathbb{R}^d} u(T+t,x)\phi_t(t,x) + \vec{f}(u(T+t,x)) \cdot \nabla\phi(t,x)\ dx\ dt + O(\tau\epsilon\|D\phi\|_\infty)$$
$$= O(\tau\epsilon\|D\phi\|_\infty)$$

because we can extend $\phi(\cdot - T, \cdot) \in C_c((T,\infty) \times \mathbb{R}^d)$ smoothly by 0 to a map $\tilde{\phi} \in C_c((0,\infty) \times \mathbb{R}^d)$ and use that u is a weak solution. Lemma 1 shows that w is a weak solution.

For the proof of the EEF part, replace u, w by $\eta(u), \eta(w)$ and $f(u), f(w)$ by $\psi(u), \psi(w)$ above. $\square$

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