Combined Competitive Flow Control and Routing in Multi-User Communication Networks with hard side-constraints

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Abstract

We consider in this paper the problem of combined flow control and routing in noncooperative setting, in which each user is faced with a multicriterion optimization problem which is formulated as the minimization of one criterion subject to constraint on others. We address here the basic questions of existence and uniqueness of equilibrium. We show that equilibria indeed exists but uniqueness may be destroyed due to the multi-criteria nature of the problem. We are able, however, to obtain uniqueness in some weaker sense under appropriate conditions, we show that the link utilizations are uniquely determined at equilibrium and the normalized Nash equilibrium is unique.

1 Introduction

Flow control and routing are two components of resource and traffic management in today's high-speed networks, such as the Internet and the ATM. Flow control is used by best-effort type traffic in order to adjust the input transmission rates (the instantaneous throughput of a connection) to the available bandwidth in the network. Routing decisions are taken to select paths with certain desirable properties, such as minimum delays. In real time applications, however, an application may have several criteria for quality of service. It might be sensitive to delays, to losses, or it might seek to minimize some cost imposed on the use of network resources. In the presence of several users each with several objectives, that determine the routes for flows they control, this gives rise to a noncooperative multicriteria game. As is often the case in todays networks, quality of service of an application is often given through an upper bound on some performance measure (delay, loss rate or jitter, see e.g. [4]). An appropriate framework for modeling this situation is that of noncooperative game theory.

Traditional noncooperative games combining flow and routing decisions have been studied in the past; see, for example, [9] and [15], and references therein. In particular, it is well known that, when the cost function of each player is the sum of link costs minus a reward which is a function of its throughput, then the underlying game can be transformed into one involving only routing decisions. Other recent papers that consider a combined flow control and routing game are [16, 17], where the utility of each player is related to the sum of powers over the links. (The power criterion is the ratio between some function of the throughput and the delay). The part of the utility in [16, 17] that corresponds to the delay is given by the sum of all link capacities minus all link flows, and in [16] it is further multiplied by some entropy function. Thus, the utility in this case does

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not directly correspond to the actual expected delay, but it has the advantage of leading to a computable Nash equilibrium in the case of parallel links. Yet another reference, [2], on the other hand, deals with the actual power criterion, i.e. the ratio between (some increasing function of) the total throughput of a user and the average delay experienced by traffic of that user. The paper introduces approximate Nash equilibrium corresponding to the case when the number of players is very large, establishes the existence of such an equilibrium and a characterization for it, and further shows that the solution need not be unique.

In this paper, we consider such constraints which can be expressed as bounds on the sum of some functions (such as delays or costs) along all links constituting a path from an origin to a destination. In mathematical terms, such problems are games where the users strategy sets are not independent but coupled. Games of this kind are called coupled constraint games, see [18] and we call the constrained (or the coupled) Nash equilibrium the corresponding solution concept.

In [10], we have investigated the special topology of parallel links without flow control. We showed in a simple example that there may be several equilibria, although in absence of side constraints there would be a single equilibrium [13]. We then showed the uniqueness of the normalized equilibrium concept of Rosen [18] for the constrained parallel links problem, and pointed on the application of this to pricing. Our objective is to extend to the case where the global throughput of each user may also controlled. This approach allows us to establish the uniqueness of normalized Nash equilibrium in the combined flow control and routing game with constraints of quality of service in a network of parallel links, and to obtain several qualitative characterizations of this equilibrium.

In the general topology case it is known that even without flow control and constraints there may be several equilibria; counter examples for the uniqueness are given in [13]. Yet in two special cases, uniqueness of Nash equilibrium has been established in general topology in the absence of side constraints [13]: 1. in the symmetrical case there is a unique Nash equilibrium which is itself symmetric. 2. Consider two equilibria each with the property that whenever a player sends a positive amount of flow to some link then all other players also do so. We call this assumption of "all positive flow". Then the total link under the two equilibria are equal. We focus in this paper on this case with flow control and additional side constraints, and show that the same type of uniqueness results that hold for the unconstrained case extend also to the constrained normalized equilibrium.

These specific results in this paper extend some previous results in [18] on the uniqueness of the Nash equilibrium in the context of noncooperative control of flow and routing without constraints.

The structure of the paper is as follows. In the next section we introduce the model and assumptions, In section 3 we establish the existence of coupled Nash equilibrium and normalized Nash equilibrium for general topology and motivate its use for decentralized pricing. In section 4 we consider a case of two nodes connected by a set of parallel links and we study the uniqueness of normalized Nash equilibrium. We also derive some properties of the equilibrium. Section 5, we extend the discussion to a general topology. We study the uniqueness of equilibria in the symmetrical framework or under the assumption of "all positive flows".

2 The model

We consider a network $(\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is a finite set of nodes and $\mathcal{L} = \{1, 2, .., L\}$ is a set of L links. We consider an extension of directional links (see [3]) where a link may carry traffic in both directions, but the direction for each user is fixed. For a node $v \in \mathcal{N}$, let $Out(v, i)$ is the set of outgoing links from node v available to user i, and $In(v, i)$ is the corresponding set of links in-going to node v available to user i. We consider a set $\mathcal{I}$ of I selfish users (players) who share the network. With each user $i \in \mathcal{I}$, we associate a unique pair, $(s(i), d(i))$, of source and destination nodes. Each user has to determine

the amount of flow $r^i \in R^i := [m^i, M^i]$ to ship between $s(i)$ and $d(i)$ and how to route it in the network.
Let f_l^i denote the amount of flow that user i sends over link l, which is constrained to be nonnegative, and satisfy the flow conservation law, i.e. for each node $v \notin (s(i), d(i))$,

$$f_l^i \geq 0, \qquad \sum_{l \in Out(v,i)} f_l^i = \sum_{l \in In(v,i)} f_l^i, \quad v \notin (s(i), d(i)), \tag{1}$$

and

$$r^i := \sum_{l \in Out(s(i),i)} f_l^i = \sum_{l \in In(d(i),i)} f_l^i \in [m^i, M^i], \tag{2}$$

Further define $\mathbf{f_l} := \{f_l^1, ..., f_l^I\}$, $f_l := \sum_{i=1}^I f_l^i$, $\mathbf{f^i} := \{f_l^i\}_{l \in \mathcal{L}}$, $\mathbf{f}^{-i} = \{\mathbf{f}^1, .., \mathbf{f}^{i-1}, \mathbf{f}^{i+1}, .., \mathbf{f}^I\}$, $\mathbf{f} := \{\mathbf{f_l}\}_{\mathbf{l} \in \mathcal{L}}$.
We consider a situation where extra side constraints are imposed. These may represent constraints on quality of service which may be user dependent. These are formulated as a set of flow restrictions of the form :

$$g_k(\mathbf{f}) \leq 0, \quad k \in \mathcal{K} \tag{3}$$

where $\mathcal{K}$ is a finite index set (e.g., formed by subsets of $\mathcal{I}$, $\mathcal{V}$, $\mathcal{L}$, $\mathcal{P}$), and $g_k : \mathbb{R}_+^{|\mathcal{L}| \times I} \to \mathbb{R}$, $k \in \mathcal{K}$.
Introduce the function $h : \mathbb{R}_+^{|\mathcal{L}| \times I} \to \mathbb{R}^m$ to describe the constraints (1)-(3), where m is the number of constraints. Hence admissible strategies will be limited by the requirement that $\mathbf{f}$ be selected from a set $\mathcal{R}$, where $\mathcal{R} = \{\mathbf{f}, h(\mathbf{f}) \leq 0\}$. We will say that $\mathcal{R}$ is a coupled constraint set. With $\mathbf{f^{-i}} := \{\mathbf{f^j}, j \in \mathcal{I};\ j \neq i\}$ fixed, we also introduce the set $\mathcal{R}^i(\mathbf{f^{-i}}) := \{\mathbf{f^i} : (\mathbf{f^i}, \mathbf{f^{-i}}) \in \mathcal{R}\}$. This is the set of allowable flows for user i with all other users flows fixed.

Example 2.1. The most immediate example of a set of flow restrictions is that of upper bounds on the end-to-end packet delay. For each $i \in \mathcal{I}$, let $w(i)$ be it's corresponding *O-D* pair, i.e., $w(i) = (s(i), d(i))$. Let $\mathcal{P}_{w(i)}$ be the set of routes connecting the *O-D* pair $w(i)$. The delay over link l can be given by $(c_l - f_l)^{-1}$ see Remark 2.1. In the framework of (3) such constraints are described by letting $\mathcal{K} = \{w(i)/w(i) = (s(i), d(i)), i \in \mathcal{I}\}$, and

$$\sum_{l \in \mathcal{L}} \frac{\delta_{lp}}{c_l - f_l} \leq D^i, \quad p \in \mathcal{P}_{w(i)}, \quad i \in \mathcal{I}. \tag{4}$$

Constraint (4) requires that for each *O-D* pair $(s(i), d(i))$, $i \in \mathcal{I}$, the end-to-end packet delay should be no larger that D^i. ∎

The performance objective of user i is quantified by means of a cost function $J^i(\mathbf{f})$. User i aims to find a strategy f^i that minimizes its cost. This optimization depends on the routing decisions of the other users, described by the strategy profile $\mathbf{f}^{-i}$, since J^i is a function of the system flow configuration $\mathbf{f}$, and the constraints (3) are coupled.

Definition 2.1. (Cost Functions and Nash equilibrium)
Let $J^i(\mathbf{f})$ be the cost for user i when the flows of all users are given by $\mathbf{f} \in \mathcal{R}$. A coupled Nash equilibrium of the routing game is a strategy profile from which no user it beneficial to unilaterally deviate. Then we seek for a Coupled Nash Equilibrium (CNE) $\tilde{\mathbf{f}}$, that is an $\tilde{\mathbf{f}} \in \mathcal{R}$ satisfying

$$J^i(\tilde{\mathbf{f}}) = \min_{(\mathbf{f^i}, \tilde{\mathbf{f}}^{-\mathbf{i}}) \in \mathcal{R}} J^i(\tilde{\mathbf{f}}^{-i}, \mathbf{f}^i) \text{ where} \tag{5}$$

$$J^i(\tilde{\mathbf{f}}^{-i}, \mathbf{f}^i) := J^i(\tilde{\mathbf{f}}^{\mathbf{1}}, ..., \tilde{\mathbf{f}}^{\mathbf{i-1}}, \mathbf{f^i}, \tilde{\mathbf{f}}^{\mathbf{i+1}}, ...\mathbf{f^I}).$$

We make the following assumptions on the cost function J^i for user i, which will be invoked throughout the paper:

G1 J^i is given as the sum of link costs $J^i_l(\mathbf{f}_l)$ minus the utility function $U^i(r^i)$: $J^i(\mathbf{f}) = \sum_{l\in\mathcal{L}} J^i_l(\mathbf{f}_l) - U^i(r^i)$.
G2 $J^i_l : [0,\infty]^I \to [0,\infty]$ is continuous and U^i is continuous in its argument.
G3 J^i_l is convex in f^i_l and g_k is convex in f^i_l and U^i is concave in its argument.
G4 J^i_l are continuously differentiable in f^i_l, g_k are continuously differentiable in f^i_l and U^i are continuously differentiable in its argument. We set $K^i_l := \partial J^i_l(\mathbf{f}_l)/\partial f^i_l$, $l \in \mathcal{L}$, and $K^i_0(r^i) := -\partial U^i(r^i)/\partial r^i$
G5 The feasible set of (1) and (3) is non-empty and contains a point that is strictly interior to every nonlinear constraints. Functions that comply with the above assumptions shall be referred to as *type*-**G** functions.

Our first set of assumptions is the following, only slightly different from those on p. 512 of [13].
A1. Assumptions G1-G5 are all satisfied, and J^i_l depends on the vector $\mathbf{f_l}$ only through user i's flow on link l and the total flow on that link. In other words, it can be written (with some abuse of notation) as $J^i_l(\mathbf{f_l}) = J^i_l(f^i_l, f_l)$.
A2. g_k is strictly increasing in each of its arguments, for each $k \in \mathcal{K}$.
A3. Viewing $K^i_l = K^i_l(f^i_l, f_l)$ now as a function of two arguments, whenever J^i_l is finite, $K^i_l(f^i_l, f_l)$, $l \in \mathcal{L}$, is increasing in each of its two arguments, and (due to **G3**)strictly increasing in the first one.
We refer to functions that meet the conditions of these three assumptions as *type*-**A** functions.

Typically, the performance of a link l is manifested through some function $T_l(f_l)$, which measures the cost per unit of flow on the link, and depends on the link's total flow. Thus, it is interest to consider cost functions of the following form:
B1. $J^i(\mathbf{f}) = \sum_{l\in\mathcal{L}} f^i_l T_l(f_l) - U(r^i)$.
B2. $T_l : [0,\infty) \to (0,\infty]$.
B3. $T_l(f_l)$ is continuously differentiable and $T_l{}'(f_l) = dT_l(f_l)/df_l$ is increasing in f_l, for all $l \in \mathcal{L}$.

We will refer to functions that meet the conditions of these three assumptions as *type*-**B** functions.

Remark 2.1. Cost functions used in real networks are either related to actual pricing, or they are related to some performance measure such as expected delay. In the first case, a frequently used cost is that of linear link costs, i.e. for each user i, $J^i(\mathbf{f}) = \sum_{i=1}^{2} f^i_l T_l(f_l)$ where $T_l(f_l) = a_l f_l + b_l$ [12]. When the costs represent delays they typically have the same form but with $T_l(f_l) = (c_l - f_l)^{-1} + d_l$. d_l represents the propagation delay related to link l, where as the first term represents queuing delay. This is the delay of an M/M/1 queue operating under the FIFO regime (packets are served at arrival order, see [13]) or of an M/G/1 queue operating under the processor sharing regime. c_l has the interpretation of the queuing capacity. Other more complicated costs can be found in [1].

3 Existence of equilibria and pricing

3.1 Characterization of equilibria and normalized Nash equilibria

If assumptions **G** hold, it follows that the minimization in (2.1) is equivalent to the following Kuhn-Tucker conditions: for every $i \in \mathcal{I}$, there exists a set of (Lagrange multipliers) $(\lambda^i_u)_{u\in\mathcal{N}}$, $(\beta^i_k)_{k\in\mathcal{K}}$, γ^i and μ^i such that, for every link $(u,v) \in \mathcal{L}$:

$$K^i_{uv}(\mathbf{f}_{uv}) + \lambda^i_v - \lambda^i_u + \sum_{k\in\mathcal{K}} \beta^i_k \frac{\partial g_k(\mathbf{f})}{\partial f^i_{uv}} = 0 \text{ if } f^i_{uv} > 0 \qquad (6)$$

$$K^i_{uv}(\mathbf{f}_{uv}) + \lambda^i_v - \lambda^i_u + \sum_{k\in\mathcal{K}} \beta^i_k \frac{\partial g_k(\mathbf{f})}{\partial f^i_{uv}} \geq 0 \text{ if } f^i_{uv} = 0 \tag{7}$$

$$K^i_0(r^i) - \lambda^i_{d(i)} + \lambda^i_{s(i)} + \gamma^i - \mu^i = 0 \tag{8}$$

$$\gamma^i(r^i - M^i) = 0,\ \mu^i(m^i - r^i) = 0,\ \beta^i_k g_k(\mathbf{f}) = 0 \tag{9}$$

$$r^i - M^i \leq 0,\ m^i - r^i \leq 0,\ g_k(\mathbf{f}) \leq 0,\ f^i_{uv} \geq 0 \tag{10}$$

$$\mu^i \geq 0,\ \gamma^i \geq 0,\ \beta^i_k \geq 0 \tag{11}$$

3.2 Normalized Nash equilibrium and Pricing

We consider a special kind of equilibrium such that each β^i_l is given by

Definition 3.1. *The coupled Nash equilibrium* $\mathbf{f}$ *is a normalized Nash equilibrium [18] if, for some vector* $\vec{\alpha} > \mathbf{0}$ *where* $\vec{\alpha} = (\alpha^1, .., \alpha^I)$ *and* $\mathbf{0}$ *is a vector of zeros, and constant* $\beta_k \geq 0$ $k \in \mathcal{K}$ *conditions (6)-(9) are satisfied where*

$$\beta^i_k = \beta_k/\alpha^i, \quad k \in \mathcal{K}, i \in \mathcal{I} \tag{12}$$

Notice that if a user's weight α^i is greater than those of his competitors, then his corresponding Lagrange multipliers are smaller.

The normalized Nash equilibrium can be used in relation to an appealing pricing scheme in which additional congestion costs are imposed by the network. Congestion pricing will allow us to relax the original constraints $g_k(\mathbf{f}) \leq 0$; yet the resulting equilibrium will have the following three appealing properties:

1. It will be a *CNE* for the original problem.
2. Non-zero congestion prices will only be imposed for saturated constraints: such constraints represent congestion, and in absence of congestion, no congestion cost is imposed.
3. The most interesting feature of this pricing is that congestion costs may be chosen to be user independent. This allows us to implement them in a decentralized way without requesting a per-flow information.

More precisely, assume that the utility of user i can be written as $-J^i(\mathbf{f}) - \frac{1}{a^i}\sum_{k\in\mathcal{K}} C_k(\mathbf{f})$. $C_k(\mathbf{f})$ is a cost function that all users are charged due to congestion related to the kth constraint. Let $(\beta^i_l)^*$ be Lagrange multipliers that correspond to a *CNE* induced by taking in (12) $\vec{\alpha} = (a^1, .., a^I)$. We set $C_k(\mathbf{f}) = \beta^*_k \cdot g_k(\mathbf{f_l})$. With this cost function we may now consider a competitive routing problem in which we ignore constraints (3). The obtained equilibrium is a *CNE* for the original constrained model, and the complementary slackness conditions imply that at the normalized equilibrium, no user actually pays any congestion cost. Under various conditions, there is a unique Nash Equilibrium [13] to the pricing game (where constraints (3) are removed) and the corresponding Kuhn-Tucker conditions obviously coincide with our original ones. We conclude that a simple pricing can replace the QoS (Quality of Service) constraints and yet force users to choose a *CNE* (so the constraints still hold). Since the pricing doesn't depend on the user, the charging can be performed in a distributed way without need for per flow information. The existence of such a pricing is equivalent to the existence of a Normalized Nash equilibrium.

3.3 Existence of Equilibria

Under assumption **G5**, the set $\mathcal{R}$ contains a point that is strictly interior to every nonlinear constraint. This is a sufficient condition for the Kuhn-Tucker constraint qualification [5]. Hence, the routing game (2.1) using the cost functions of *type*-**G** is equivalent to a convex game in the sense of [18] and, thus the existence of an *CNE* as well as a normalized equilibrium is guaranteed [18, Thm. 1] if the costs are finite for any strategy. Note that the proof of existence in [13] is based on [18] that restricted to finite costs by using the following assumption:" For every flow configuration $\mathbf{f}$, if not all cost are finite then at least one user with infinite cost ($J^i(\mathbf{X})$ can change its flow configuration to make its cost finite".

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Theorem 3.1. *Consider the cost function of type-*$\mathbf{G}$*. Then there exists a normalized Nash equilibrium point for every specified vector* $\vec{\alpha} > \mathbf{0}$ *(componentwise) where* $\vec{\alpha} = (\alpha^1, \alpha^2, .., \alpha^I)$*.*

We analyze the routing problem in two phases. First we consider a case of two nodes connected by a set of parallel links. Second, we extend some results to a general network, for the case of symmetric users and positive flows.

4 Parallel links

In this section, we consider the case of two nodes $\{1, 2\}$ connected by a set $\mathcal{L}$ of L links. Such a system of parallel links may represent a network in which resources are pre-allocated to various paths, or an internetworking in which each link models a different subnetwork. With each user i, we associate a unique pair, $(s(i), d(i))$, of source and destination nodes, where $s(i), d(i) \in \{1, 2\}$. For the parallel links we further impose a quality of service on each link. This is captured by the constraints (3), where $\mathcal{K} = \mathcal{L}$ and g_l depends on the flows only through total flow on link l. i.e.,

$$g_l(f_l) \leq 0. \tag{13}$$

Under assumption $\mathbf{A}$, the functions g_l are strictly increasing in f_l, and hence g_l^{-1} exists and the constraints (13) become

$$f_l \leq d_l\,, \quad l \in \mathcal{L}\,,$$

where $d_l = g_l^{-1}(0)$ (positive real).

Remark 4.1. *Equation (13) can be interpreted as capturing the link capacity constraints i.e., the total flow at each link l can not exceed the links capacity $d_l = c_l$. In many cases, however, performance measures (such as loss probabilities or delays) are monotone in the link load, which then implies that bounds on these measures are obtained by bounding the link load as in (13).*

In [10], the authors present an example which demonstrates that the above weak convexity conditions are not sufficient for uniqueness of a Nash equilibrium (without flow control). These indeed point at the complexity of the coupled constraint (13). This non-uniqueness of Nash equilibria will then motivate us to study normalized (*Rosen*) Nash equilibrium (defined below) in the parallel links topology, particularly its uniqueness and some of its characteristics.

Nonetheless, in the following we prove under some hypothesis that the link utilizations and the user demands are unique at each Nash equilibrium.

Theorem 4.1. *Consider the cost function of each user is of type-*$\mathbf{A}$*. Let* $\mathbf{f}$ *and* $\hat{\mathbf{f}}$ *be two coupled Nash equilibrium. Let* $(\beta_l^i, \mu^i, \gamma^i)$ *and* $(\hat{\beta}_l^i, \hat{\mu}^i, \hat{\gamma}^i)$ *be corresponding Lagrange multipliers. Assume that for each link* $l \in \mathcal{L}$, $\beta_l^i \leq \hat{\beta}_l^i, \quad \forall i \in \mathcal{I}$ *or* $\hat{\beta}_l^i \leq \beta_l^i, \quad \forall i \in \mathcal{I}$*. Then* $f_l = \hat{f}_l \quad \forall l \in \mathcal{L}$*,and* $r^i = \hat{r}^i \quad ,\forall i \in \mathcal{I}$ *(i.e., the link utilizations and the user demands are the same under* $\mathbf{f}$ *and* $\hat{\mathbf{f}}$*).*

Proof : Let $\mathbf{f}$ and $\hat{\mathbf{f}}$ be two Nash equilibria. Then we have from (6) and (7) :

$$K_l^i(f_l^i, f_l) + K_0^i(r^i) \geq \mu^i - \gamma^i - \beta_l^i; \quad K_l^i(f_l^i, f_l) + K_0^i(r^i) = \mu^i - \gamma^i - \beta_l^i; \text{ if } f_l^i > 0 \ \forall i, l \quad (14)$$
$$K_l^i(\hat{f}_l^i, \hat{f}_l) + K_0^i(\hat{r}^i) \geq \hat{\mu}^i - \hat{\gamma}^i - \hat{\beta}_l^i; \quad K_l^i(\hat{f}_l^i, \hat{f}_l) + K_0^i(\hat{r}^i) = \hat{\mu}^i - \hat{\gamma}^i - \hat{\beta}_l^i; \text{ if } \hat{f}_l^i > 0 \ \ \forall i, l \quad (15)$$

By using some procedure in the proof of [10, Thm. **3.1**] we have the following relations :
(i) $\{\hat{\beta}_l^i < \beta_l^i; \hat{f}_l \geq f_l\} \Longrightarrow \hat{f}_l = f_l$ moreover if $(\mu^i - \gamma^i \leq \hat{\mu}^i - \hat{\gamma}^i$ and $\hat{r}^i \leq r^i)$ then $\hat{f}_l^i \geq f_l^i$ and the last inequality is strict if $f_l^i > 0$.
(ii) $\{\hat{\beta}_l^i > \beta_l^i; \hat{f}_l \leq f_l\} \Longrightarrow \hat{f}_l = f_l$ moreover if $(\mu^i - \gamma^i \geq \hat{\mu}^i - \hat{\gamma}^i$ and $\hat{r}^i \geq r^i)$ then $\hat{f}_l^i \leq f_l^i$ and the last inequality is strict if $\hat{f}_l^i > 0$.
(iii) $\{\hat{\mu}^i - \hat{\gamma}^i \leq \mu^i - \gamma^i; \hat{r}^i \geq r^i; \hat{\beta}_l^i \geq \beta_l^i; \hat{f}_l \geq f_l\} \Longrightarrow \hat{f}_l^i \leq f_l^i$
(iv) $\{\hat{\mu}^i - \hat{\gamma}^i \geq \mu^i - \gamma^i; \hat{r}^i \leq r^i; \hat{\beta}_l^i \leq \beta_l^i; \hat{f}_l \leq f_l\} \Longrightarrow \hat{f}_l^i \geq f_l^i$

Let $\mathcal{L}_1 = \{l : \hat{f}_l > f_l\}$. Also, denote $\mathcal{I}_1 = \{i : \hat{r}^i \leq r^i; \hat{\mu}^i - \hat{\gamma}^i \geq \mu^i - \gamma^i\}$, $\mathcal{L}_2 = \{l : \hat{f}_l \leq f_l; \hat{\beta}_l^i \leq \beta_l^i\}$ and $\mathcal{L}_3 = \{l : \hat{f}_l \leq f_l; \hat{\beta}_l^i > \beta_l^i\}$. We observe that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Assume that $\mathcal{L}_1$ is nonempty, it follows by (iv) that for $i \in \mathcal{I}_1$:

$$\sum_{l \in \mathcal{L}_1} \hat{f}_l^i = \hat{r}^i - \sum_{l \in \mathcal{L}_2} \hat{f}_l^i - \sum_{l \in \mathcal{L}_3} \hat{f}_l^i \leq r^i - \sum_{l \in \mathcal{L}_2} f_l^i - \sum_{l \in \mathcal{L}_3} \hat{f}_l^i = \sum_{l \in \mathcal{L}_1} f_l^i + \sum_{l \in \mathcal{L}_3} (f_l^i - \hat{f}_l^i).$$

We now proceed to show that :

$$i \notin \mathcal{I}_1, \text{ implies that } \hat{r}^i \geq r^i \text{ and } \hat{\mu}^i - \hat{\gamma}^i \leq \mu^i - \gamma^i \quad (16)$$

Indeed, since $i \notin \mathcal{I}_1$, it follows that either $\hat{r}^i > r^i$ and $\hat{\mu}^i - \hat{\gamma}^i < \mu^i - \gamma^i$, we have either $\hat{\mu}^i - \hat{\gamma}^i \leq \mu^i - \gamma^i$ $(\hat{\mu}^i = \gamma^i = 0)$ and $\hat{r}^i \geq r^i$ (If $\hat{\mu}^i - \hat{\gamma}^i \geq 0$, then $\mu^i - \gamma^i > 0$, it follows that $r^i = m^i \leq \hat{r}^i$ $(\mu^i > 0)$; or $\hat{\mu}^i - \hat{\gamma}^i < 0$, then $\hat{\mu}^i < \hat{\gamma}^i$, it follows that $\hat{r}^i = M^i \geq r^i(\hat{\gamma}^i > 0)$). Noting that (i) implies that $\{l \in \mathcal{L}_1 / \hat{\beta}_l^i < \beta_l^i\} = \emptyset$, hence, if $l \in \mathcal{L}_1$ and $i \notin \mathcal{I}_1$, we have from (iii) $\hat{f}_l^i \leq f_l^i$. It follows that :

$$\begin{aligned}
\sum_{l \in \mathcal{L}_1} \hat{f}_l &= \sum_{l \in \mathcal{L}_1} \sum_{i \in \mathcal{I}_1} \hat{f}_l^i + \sum_{l \in \mathcal{L}_1} \sum_{i \notin \mathcal{I}_1} \hat{f}_l^i \\
&\leq \sum_{l \in \mathcal{L}_1} \sum_{i \in \mathcal{I}_1} f_l^i + \sum_{l \in \mathcal{L}_3} \sum_{i \in \mathcal{I}_1} (f_l^i - \hat{f}_l^i) + \sum_{l \in \mathcal{L}_1} \sum_{i \notin \mathcal{I}_1} f_l^i \\
&= \sum_{l \in \mathcal{L}_1} f_l + \sum_{l \in \mathcal{L}_3} \sum_{i \in \mathcal{I}_1} (f_l^i - \hat{f}_l^i) \\
&= \sum_{l \in \mathcal{L}_1} f_l + \sum_{l \in \mathcal{L}_3} (f_l - \hat{f}_l) - \sum_{l \in \mathcal{L}_3} \sum_{i \notin \mathcal{I}_1} (f_l^i - \hat{f}_l^i) \\
&\leq \sum_{l \in \mathcal{L}_1} f_l
\end{aligned} \quad (17)$$

The last inequality follows from (ii), since for $l \in \mathcal{L}_3$, $f_l = \hat{f}_l$ and for $l \in \mathcal{L}_3$ and $i \notin \mathcal{I}_1$, $f_l^i \geq \hat{f}_l^i$.
Hence, the inequality (17) and the definition of $\mathcal{L}_1$ are contradictory, which implies that $\mathcal{L}_1$ is an empty set. By symmetry, it may also be concluded that the set $\mathcal{L}_1{}' = \{l : \hat{f}_l < f_l\}$ is empty. Therefore we conclude that, $\hat{f}_l = f_l$, $\forall l \in \mathcal{L}$.

We now proceed to show that $r^i = \hat{r}^i, \forall i \in \mathcal{I}$.
To this end, let $\mathcal{I}_1 = \{i : \hat{r}^i > r^i\}$. Also, denote $\mathcal{L}_1 = \{l : \hat{\beta}_l^i \leq \beta_l^i\}$, $\mathcal{I}_2 = \{i : \hat{r}^i \leq r^i; \hat{\mu}^i - \hat{\gamma}^i \geq \mu^i - \gamma^i\}$ and $\mathcal{I}_3 = \{i : \hat{r}^i \leq r^i; \hat{\mu}^i - \hat{\gamma}^i < \mu^i - \gamma^i\}$. We observe that $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$. Assume that $\mathcal{I}_1$ is nonempty, it follows by (iv) that for $l \in \mathcal{L}_1$:

$$\sum_{i\in\mathcal{I}_1} \hat{f}_l^i = \hat{f}_l - \sum_{i\in\mathcal{I}_2} \hat{f}_l^i - \sum_{i\in\mathcal{I}_3} \hat{f}_l^i \leq f_l - \sum_{i\in\mathcal{I}_2} f_l^i - \sum_{i\in\mathcal{I}_3} \hat{f}_l^i = \sum_{i\in\mathcal{I}_1} f_l^i + \sum_{i\in\mathcal{I}_3} (f_l^i - \hat{f}_l^i).$$

Noting that from (16), the relation (iii) implies that $\hat{f}_l^i \leq f_l^i$ for $i \in \mathcal{I}_1$ and $l \notin \mathcal{L}_1$, it follows that :

$$\begin{aligned}
\sum_{i\in\mathcal{I}_1} \hat{r}^i &= \sum_{i\in\mathcal{I}_1}\sum_{l\in\mathcal{L}_1} \hat{f}_l^i + \sum_{i\in\mathcal{I}_1}\sum_{l\notin\mathcal{L}_1} \hat{f}_l^i \\
&\leq \sum_{i\in\mathcal{I}_1}\sum_{l\in\mathcal{L}_1} f_l^i + \sum_{i\in\mathcal{I}_3}\sum_{l\in\mathcal{L}_1} (f_l^i - \hat{f}_l^i) + \sum_{i\in\mathcal{I}_1}\sum_{l\notin\mathcal{L}_1} f_l^i \\
&= \sum_{i\in\mathcal{I}_1} r^i + \sum_{i\in\mathcal{I}_3}\sum_{l\in\mathcal{L}_1} (f_l^i - \hat{f}_l^i) \\
&= \sum_{i\in\mathcal{I}_1} r^i + \sum_{i\in\mathcal{I}_3} (r^i - \hat{r}^i) - \sum_{i\in\mathcal{I}_3}\sum_{l\notin\mathcal{L}_1} (f_l^i - \hat{f}_l^i) \\
&\leq \sum_{i\in\mathcal{I}_1} r^i \qquad (18)
\end{aligned}$$

The last inequality follows from (i), since for $i \in \mathcal{I}_3$, $\hat{r}^i = r^i$ and for $i \in \mathcal{I}_3$ and $l \notin \mathcal{L}_1$, $\hat{f}_l^i \leq f_l^i$.
Hence, the inequality (18) and the definition of $\mathcal{I}_1$ are contradictory, which implies that $\mathcal{I}_1$ is an empty set. By symmetry, it may also be concluded that the set $\mathcal{I}_1{}' = \{i : \hat{r}^i < r^i\}$ is empty. Therefore we conclude that, $\hat{r}^i = r^i$, $\forall i \in \mathcal{I}$. ∎

4.1 Uniqueness of the normalized Nash equilibrium

A set of sufficient conditions for uniqueness of the normalized Nash equilibrium has been established by Rosen in [18] under some strict diagonal convexity conditions. These conditions may not be satisfied in our case, and hence we need to prove uniqueness in some other way.

Remark 4.2. *Let $\vec{\alpha}$ and $\vec{\hat{\alpha}}$ be two positive vectors such that $\vec{\hat{\alpha}} = a\vec{\alpha}$ for some positive real a. Let $\mathcal{A}(\vec{\alpha})$ and $\mathcal{A}(\vec{\hat{\alpha}})$ be corresponding normalized Nash equilibria sets. Then $\mathcal{A}(\vec{\alpha}) = \mathcal{A}(\vec{\hat{\alpha}})$*

The following result shows that the parallel-links network also has a unique normalized Nash equilibrium for every specified vector $\vec{\alpha} > 0$.

Theorem 4.2. *In a network of parallel links where the cost function of each user is of type-**A**, the normalized Nash equilibrium for every specified $\vec{\alpha} > 0$ is unique.*

Proof: The hypothesis of theorem 4.1 are verified in the normalized Nash equilibrium, then we have the uniqueness for link utilizations and each user have the same demand under all normalized Nash equilibria. Then the theorem follows directly from [10, Thm. **4.1**] (i.e., the case where the demands are fixed). ∎

Corollary 4.1. *In a network of parallel links where the cost function of each user is of type-**A**, and in the absence of the side constraints (i.e., $\beta_l = 0, \forall l \in \mathcal{L}$) there would be a single Nash Equilibrium.*

The above result can be considered an extension of [13, Thm **1**] in noncooperative flow control and routing games.

4.2 Properties of the normalized Nash equilibrium

Here, we assume that the cost functions of all users are symmetrically identical, i.e., $J_l^i \equiv J_l$ and $U^i \equiv U$ for all $i \in \mathcal{I}$ and $l \in \mathcal{L}$. And let $\eta^i = \mu^i - \gamma^i, \forall i \in \mathcal{I}$.

Lemma 4.1. *Assume that all users have the same weight and assume that the condition* $f_{\hat{l}}^i > f_{\hat{l}}^j$ *holds for some link* $\hat{l}$ *and some users* i *and* j*. Then* $f_l^i \geq f_l^j$ *for all* $l \in \mathcal{L}$*; moreover, the inequality is strict if* $f_l^j > 0$.

Proof: See Appendix.

Proposition 4.1. *Consider the identical type-***A** *cost functions and the identical user weights. Assume that* $m^i \geq m^j$ *and* $M^i \geq M^j$*. Then* $r^i \geq r^j$ *and* $f_l^i \geq f_l^j$ *for all links* $l \in \mathcal{L}$*. If* $m^i = m^j$ *and* $M^i = M^j$*, then* $f_l^i = f_l^j$ *for all* $l \in \mathcal{L}$.

Proof.- See Appendix.
The next proposition shows that, for identical type-**A** cost functions, and identical interval demands (i.e.,$m^i = m$ and $M^i = M, \forall i \in \mathcal{I}$), there is a monotonicity among users in their demands, i.e., a user with a higher weight sends more demand.

Proposition 4.2. *Assume that all users have the same type-***A** *cost function. Then for some vector* $\vec{\alpha} = (\alpha^1, \alpha^2, ..., \alpha^I) > 0$*, we have :*

$$\alpha^i > \alpha^j \Longrightarrow r^i \geq r^j, \quad \forall i, j \in \mathcal{I}.$$

Proof.- See Appendix.

4.3 Application: Virtual Path allocation with Level QoS

An interesting application of our model (parallel links) is virtual path allocation with level QoS. The system of parallel links may represent a network in which the resources are pre-allocated to various paths. Each user reserve some of the resource capacity in order to establish a virtual path level, i.e., that find the corresponding virtual path full upon arrival, can be assumed to either be lost or else to be accommodated through an alternative virtual circuit scheme. In the latter case, the user faces the need to consume processing resources and waste time on call setup. Thus, in either case, blocking at the virtual path level leads to performance degradation.
To that end, the user minimizes a cost function. This function should account for the following tradeoff. On the one hand, each user try to minimize the blocking probability of its incoming calls at the virtual path level, which is a decreasing function of the reserved capacity of the user's virtual paths. On the other hand, reserving capacity becomes more difficult as the system's resources are less available. According to our model, users make requests for the VC service at the boundaries of the network.
Let B_l^i the amount of capacity reserved on link l by user i, which is constrained to be nonnegative and not exceed the capacity C_l. Further define $\mathbf{B_l} = (B_l^1, B_l^2, .., B_l^I)$, $\mathbf{B^i} = (B_1^i, B_2^i, .., B_L^i)$, $B_l = \sum_{i \in \mathcal{I}} B_l^i$ and $B^i = \sum_{l \in \mathcal{L}} C_l^i$.
The cost function for user i, denoted by J^i, is of the following form :

$$J^i(\mathbf{B}) = \sum_l F_l^i(B_l^i, B_l) + G^i(B^i), \tag{19}$$

F_l^i accounts for the cost of reserving capacity for a user on link l, as perceived by that user, whereas the function G^i accounts for the effect that the amount of reserved capacity has on the performance of that user.
This VP network is transparent to the users in which the users calculate the routes and capacities of virtual paths in the network such that the following requirement are satisfied

1. *Capacity constraints* : The sum of **VP** capacities on each link does not exceed its capacity, i.e.,

$$B_l = \sum_{i \in \mathcal{I}} B_l^i \leq C_l, \quad \forall l \in \mathcal{L} \tag{20}$$

2. *Constraints of quality of service* : The blocking constraint used by user i is denoted by δ^i. The blocking constraint enforce QoS at the call level. The blocking probability P_i is the percentage of call attempts of user i that are denied service due to the unavailability of resources. We must always have that

$$P_i \leq \delta^i \tag{21}$$

The blocking probability P_i is a function of two variables, the total of capacity allowed by user i (B^i) and the total arrival rate of user i that we assume that is fixed. Moreover, this function is decreasing with respect to B^i. Then (21) is equivalent to :

$$m^i \leq B^i \quad ; \quad \forall i \in \mathcal{I} \,, \tag{22}$$

where $m^i = P_i^{-1}(B^i)$.
The following result establishes the uniqueness of normalized Nash equilibrium for the **VP** allocation game.

Corollary 4.2. *Consider the cost functions of type-***A***. Then the normalized Nash equilibrium of* **VP** *allocation game, is unique.*

Proof.- Follows directly from Theorem 4.2.

Remark 4.3. *In the absence of the capacity constraint (20), we use the property* $F5$ *in [14], then corollary 4.1 implies that the Nash equilibrium that corresponds to the* **VP** *allocation game is unique.*

5 General topology

In this section, we study an extension to a general network. We assume that all users have the same source and destination (s, d). For the general topology, we further impose a quality of service on each path. The goals are formulated as a set of flow restrictions of the form :

$$\sum_{l \in p} g_l(f_l) \leq d_p, \quad p \in \mathcal{P} \tag{23}$$

where $\mathcal{P}$ is set of paths connecting the *O-D* pair (s, d) and $g_l : \mathbb{R}_+ \to \mathbb{R}$, $l \in \mathcal{L}$.
Constraints (23) requires that for each path connected the source s and destination d, the end-to-end packet cost (delay) should be no larger that d_p.

Lemma 5.1. *Consider the identical type-***A** *cost functions and identical intervals for demand (i.e.,* $m^i = m$ *and* $M^i = M, \forall i \in \mathcal{I}$*). Let a vector* $\vec{\alpha}$ *such that* $\vec{\alpha} = (\alpha^1, \alpha^2, .., \alpha^I) > 0$*. Then for* $i, j \in \mathcal{I}$ *such that* $\alpha^i = \alpha^j$*, then* $f_l^i = f_l^j$ *for all* $l \in \mathcal{L}$*. Moreover if* α^i*,* $i \in \mathcal{I}$ *are the same then* $f_l^i = \frac{f_l}{I} \;\; \forall\, i, l$*.*

Proof.- See Appendix.

Theorem 5.1. *Consider the identical type-***A** *cost functions and all users have the same interval for demand (i.e.,* $m^i = m$ *and* $M^i = M, \forall i \in \mathcal{I}$*) and the same weight (i.e.,*$\alpha^i = \alpha, \forall i \in \mathcal{I}$*). In a network with symmetrical users has a unique normalized Nash equilibrium for every* $\alpha > 0$*. Moreover* $f_l^i = \frac{f_l}{I}$*,* $\forall i, l$*, where* $\mathbf{f}$ *is the unique normalized Nash equilibrium.*

Proof.- From Remark 4.2, it suffices to show this theorem with $\alpha = 1$. We suppose by contradiction there are two normalized equilibria $\tilde{\mathbf{f}}$ and $\hat{\mathbf{f}}$. The first step is to establish that $\hat{f}_l = \tilde{f}_l \ \forall l \in \mathcal{L}$.
¿From the Kuhn-Tucker conditions (6) and (7), for $\tilde{\mathbf{f}}$ and $\hat{\mathbf{f}}$ we have :

$$\begin{array}{ll} K_{uv}(\hat{f}^i_{uv}, \hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}^i_v \geq \hat{\lambda}^i_u; & K_{uv}(\hat{f}^i_{uv}, \hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}^i_v = \hat{\lambda}^i_u \text{ if } \hat{f}^i_{uv} > 0 \\ K_{uv}(\tilde{f}^i_{uv}, \tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}^i_v \geq \tilde{\lambda}^i_u; & K_{uv}(\tilde{f}^i_{uv}, \tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}^i_v = \tilde{\lambda}^i_u \text{ if } \tilde{f}^i_{uv} > 0 \end{array} \tag{24}$$

where $\tilde{\beta}_l = \sum_p \delta_{lp} \tilde{\beta}_p$ and $\hat{\beta}_l = \sum_p \delta_{lp} \hat{\beta}_p$. From lemma 5.1, we have that, for all $i \in \mathcal{L}$ and $l \in \mathcal{L}$, $\hat{f}^i_l = \frac{\hat{f}_l}{I}$ and $\tilde{f}^i_l = \frac{\tilde{f}_l}{I}$. Thus (24) become :

$$\begin{array}{ll} K_{uv}(\frac{\hat{f}_{uv}}{I}, \hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}^i_v \geq \hat{\lambda}^i_u; & K_{uv}(\frac{\hat{f}_{uv}}{I}, \hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}^i_v = \hat{\lambda}^i_u \text{ if } \hat{f}_{uv} > 0 \\ K_{uv}(\frac{\tilde{f}_{uv}}{I}, \tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}^i_v \geq \tilde{\lambda}^i_u; & K_{uv}(\frac{\tilde{f}_{uv}}{I}, \tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}^i_v = \tilde{\lambda}^i_u \text{ if } \tilde{f}_{uv} > 0 \end{array}$$

We define the function G_l for $l \in \mathcal{L}$ by $G_l(f_l) = K_l(\frac{f_l}{I}, f_l)$. Summing each of these equation over i, we get :

$$\begin{array}{ll} G_{uv}(\hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}_v \geq \hat{\lambda}_u; & G_{uv}(\hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}_v = \hat{\lambda}_u \text{ if } \hat{f}_{uv} > 0 \\ G_{uv}(\tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}_v \geq \tilde{\lambda}_u; & G_{uv}(\tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}_v = \tilde{\lambda}_u \text{ if } \tilde{f}_{uv} > 0 \end{array} \tag{25}$$

where $\tilde{\lambda}_u = \frac{1}{I} \sum_i \tilde{\lambda}^i_u$ and $\hat{\lambda}_u = \frac{1}{I} \sum_i \lambda^i_u$. Note that these equation are very similar to the Kuhn-Tucker conditions for a single-user optimization problem of link flow, with respect to a modified link (convex) cost function with derivative G_l and the following constraints.

$$\begin{aligned} f_l \geq 0, \quad \sum_{l \in Out(v)} f_l &= \sum_{l \in In(v)} f_l, \quad v \notin (s,d), \\ r = \sum_{l \in Out(s)} f_l &= \sum_{l \in In(d)} f_l \ \in [m, M], \\ \sum_{l \in p} g_l(f_l) &\leq d_p, \quad p \in \mathcal{P}. \end{aligned}$$

Since the link cost function $\int G_l$ is convex (see Assumption **A**) then the uniqueness of their solution is actually a consequence of standard convex programming results. It has thus established that $\hat{f}_l = \tilde{f}_l$ for all $l \in \mathcal{L}$,this implies by lemma 5.1 that $\hat{f}^i_l = \tilde{f}^i_l$ for every l, i, and uniqueness of the normalized Nash equilibrium is thus proved. ∎

Corollary 5.1. *Consider the identical type-***A** *cost functions and all users have the same interval demands (i.e., $m^i = m$ and $M^i = M, \forall i \in \mathcal{I}$). Then in the absence of the QoS constraints (23), the Nash equilibrium is unique.*

5.1 Positive Flows

In this paragraph, we suppose that all users use the type-**B** cost functions. The following result establishes a uniqueness of equilibria among those that satisfy the so called *"all-positive flow"* assumption : whenever a player sends a positive amount of flow to some link then all other players also do so.

Theorem 5.2. *Consider cost functions of type-***B***, and let* $\tilde{\mathbf{f}}$ *and* $\hat{\mathbf{f}}$ *be two Nash equilibria. Assume that there exists a set* $\mathcal{L}_1$ *of links,* $\mathcal{L}_1 \subset \mathcal{L}$*, such that* $\{\tilde{f}_l^i > 0 \ \text{ and } \ \hat{f}_l^i > 0, \ \ i \in \mathcal{I}\}$ *for* $l \in \mathcal{L}_1$*, and* $\{\tilde{f}_l^i = \hat{f}_l^i = 0, \ \ i \in \mathcal{I}\}$ *for* $l \notin \mathcal{L}_1$*. Then,* $\tilde{f}_l = \hat{f}_l, \ \forall l \in \mathcal{L}$*.*

Proof.- By using same procedure as in proof of Theorem 5.1 with assumption positive flows, we show that the Kuhn-Tucker conditions for Nash equilibria $\hat{\mathbf{f}}$ and $\tilde{\mathbf{f}}$ implies the following conditions :

$$\begin{array}{ll} G_{uv}(\hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}_v \geq \hat{\lambda}_u; & G_{uv}(\hat{f}_{uv}) + \hat{\beta}_{uv} g'_{uv}(\hat{f}_{uv}) + \hat{\lambda}_v = \hat{\lambda}_u \ \text{ if } \ \hat{f}_{uv} > 0 \\ G_{uv}(\tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}_v \geq \tilde{\lambda}_u; & G_{uv}(\tilde{f}_{uv}) + \tilde{\beta}_{uv} g'_{uv}(\tilde{f}_{uv}) + \tilde{\lambda}_v = \tilde{\lambda}_u \ \text{ if } \ \tilde{f}_{uv} > 0 \end{array} \tag{26}$$

where $\lambda_u = \sum_i \lambda_u^i$, $\beta_{uv} = \sum_i \beta_{uv}^i$, and $G_{uv}(f_{uv}) = f_{uv} T'_{uv}(f_{uv}) + I.T_{uv}(f_{uv})$. This proof can be done proceeding as in the proof of theorem 5.1 (starting from equation (25)) it may be inferred that $\hat{f}_l = \tilde{f}_l, \ \ l \in \mathcal{L}$. ∎

The next result shows in particular that there exists at most one normalized Nash equilibrium with *"all-positive flow"* assumption.

Theorem 5.3. *Consider cost functions of type-***B***. For some vector* $\vec{\alpha} > 0$*, let* $\tilde{\mathbf{f}}$ *and* $\hat{\mathbf{f}}$ *be two normalized Nash equilibria. Assume that there exists a set* $\mathcal{L}_1$ *of links,* $\mathcal{L}_1 \subset \mathcal{L}$*, such that* $\{\tilde{f}_l^i > 0 \ \text{ and } \ \hat{f}_l^i > 0, \ \ i \in \mathcal{I}\}$ *for* $l \in \mathcal{L}_1$*, and* $\{\tilde{f}_l^i = \hat{f}_l^i = 0, \ \ i \in \mathcal{I}\}$ *for* $l \notin \mathcal{L}_1$*. Then* $\tilde{\mathbf{f}}_\mathbf{l} = \hat{\mathbf{f}}_\mathbf{l}$*.*

Proof.- Assume that for some $\vec{\alpha}$ we have two normalized Nash equilibrium points $\hat{\mathbf{f}}$ and $\tilde{\mathbf{f}}$. Then we have from (6) and (7) :

$$\begin{aligned} K_{uv}^i(\tilde{f}_{uv}^i, \tilde{f}_{uv}) + \frac{\tilde{\beta}_{uv}}{\alpha^i} g'_{uv}(\tilde{f}_{uv}) &= \tilde{\lambda}_u^i - \tilde{\lambda}_v^i \ \text{ if } \ \tilde{f}_{uv}^i > 0 \\ K_{uv}^i(\hat{f}_{uv}^i, \hat{f}_{uv}) + \frac{\hat{\beta}_{uv}}{\alpha^i} g'_{uv}(\hat{f}_{uv}) &= \hat{\lambda}_u^i - \hat{\lambda}_v^i \ \text{ if } \ \hat{f}_{uv}^i > 0 \end{aligned}$$

By contradiction, assume that there exists $(l_0, i) \in \mathcal{L} \times \mathcal{I}$ such that $\hat{f}_{l_0}^i \neq \tilde{f}_{l_0}^i$, and without loss of generality assume that $\hat{f}_{l_0}^i < \tilde{f}_{l_0}^i$.

In the sequel, we consider two cases :

Case 1 : I is even.

Since $\hat{f}_{l_0}^i < \tilde{f}_{l_0}^i$ and $\tilde{f}_l = \hat{f}_l, l \in \mathcal{L}$, then it's easy to show that there exists two disjoint sets $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, $|\mathcal{I}_1| = |\mathcal{I}_2| = \frac{I}{2}$ and

$$\tilde{f}_{l_0,1} := \sum_{i \in \mathcal{I}_1} \tilde{f}_{l_0}^i > \hat{f}_{l_0,1} := \sum_{i \in \mathcal{I}_1} \hat{f}_{l_0,1}^i, \ \text{ and } \ \tilde{f}_{l_0,2} := \sum_{i \in \mathcal{I}_2} \tilde{f}_{l_0}^i < \hat{f}_{l_0,2} := \sum_{i \in \mathcal{I}_2} \hat{f}_{l_0,2}^i$$

Now, we construct a directed network $(\mathcal{N}', \mathcal{L}')$, where $\mathcal{N}' = \mathcal{N}$ and the set of links $\mathcal{L}'$ is constructed as follows :

1. For each link $l = (u, v) \in \mathcal{L}$, such that $\tilde{f}_{l,1} \geq \hat{f}_{l,1}$, we have a link $l' = (u, v) \in \mathcal{L}'$; to such a link l' we assign a (flow) value $z_{l'} = \tilde{f}_{l,1} - \hat{f}_{l,1}$.

2. for each link $l = (u, v) \in \mathcal{L}$, such that $\tilde{f}_{l,1} < \hat{f}_{l,1}$, we have a link $l' = (v, u) \in \mathcal{L}'$; to such a link we assign a (flow) value $z_{l'} = \hat{f}_{l,1} - \tilde{f}_{l,1}$.

It is easy to verify that the value $z_{l'}$ constitutes a nonnegative, directed flow in the network. Let $\hat{r}_1 = \sum_{i \in \mathcal{I}_1} \hat{r}^i$ and $\tilde{r}_1 = \sum_{i \in \mathcal{I}_1} \tilde{r}^i$, if $\hat{r}_1 = \tilde{r}_1$ and since $\tilde{f}_{l_0,1} > \hat{f}_{l_0,1}$, then there exists a cycle $\mathbf{D}$ such that $z_{l'} > 0, \ \forall l \in \mathbf{D}$, else (i.e., $\hat{r}_1 \neq \tilde{r}_1$) $z_{l'}$ must carry some flow (the amount of $|\hat{r}_1 - \tilde{r}_1|$) from the source s to the destination d, this implies that there exists a path p^* from s to d, such that $z_{l'} > 0$ for all $l' \in p^*$. Let $\mathbf{S}$ is a set that represent $\mathbf{D}$ if $\hat{r}_1 = \tilde{r}_1$ and p^* otherwise.

Consider now a link $l' = (u, v) \in \mathbf{S}$. Since $z_{l'} > 0$, either $\tilde{f}_{uv,1} > \hat{f}_{uv,1}$ or $\hat{f}_{vu,1} > \tilde{f}_{vu,1}$. With assumption positive flows, we shows that the Kuhn-Tucker conditions for Nash equilibria $\tilde{\mathbf{f}}$ and $\hat{\mathbf{f}}$ implies the following conditions :

$$\tilde{\lambda}_{u,1} - \tilde{\lambda}_{v,1} = \tilde{f}_{uv,1} T'(\tilde{f}_{uv}) + \frac{I}{2} T_{uv}(\tilde{f}_{uv}) + \delta_1 \frac{I}{2} \tilde{\beta}_{uv} g'(\tilde{f}_{uv}) \text{ if } \tilde{f}_{uv} > 0 \tag{27}$$

$$\hat{\lambda}_{u,1} - \hat{\lambda}_{v,1} = \hat{f}_{uv,1} T'(\hat{f}_{uv}) + \frac{I}{2} T_{uv}(\hat{f}_{uv}) + \delta_1 \frac{I}{2} \hat{\beta}_{uv} g'(\hat{f}_{uv}) \text{ if } \hat{f}_{uv} > 0 \tag{28}$$

where $\lambda_{uv,1} = \sum_{i \in \mathcal{I}_1} \lambda^i_{uv}$ and $\delta_1 = \sum_{i \in \mathcal{I}_1} \frac{1}{\alpha^i}$.
In the case where $\tilde{f}_{uv,1} > \hat{f}_{uv,1}$, we have :

$$\begin{aligned} \tilde{\lambda}_{u,1} - \tilde{\lambda}_{v,1} &= \tilde{f}_{uv,1} T'(\tilde{f}_{uv}) + \frac{I}{2} T_{uv}(\tilde{f}_{uv}) + \delta_1 \frac{I}{2} \tilde{\beta}_{uv} g'(\tilde{f}_{uv}) \\ &> \hat{f}_{uv,1} T'(\hat{f}_{uv}) + \frac{I}{2} T_{uv}(\hat{f}_{uv}) + \delta_1 \frac{I}{2} \tilde{\beta}_{uv} g'(\hat{f}_{uv}) \\ &= \hat{\lambda}_{u,1} - \hat{\lambda}_{v,1} + \delta_1 \frac{I}{2} g'(\tilde{f}_{uv})(\tilde{\beta}_{uv} - \hat{\beta}_{uv}) \end{aligned}$$

Thus

$$\tilde{\lambda}_{u,1} - \tilde{\lambda}_{v,1} > \hat{\lambda}_{u,1} - \hat{\lambda}_{v,1} + \delta_1 \frac{I}{2} g'(\tilde{f}_{uv})(\tilde{\beta}_{uv} - \hat{\beta}_{uv}) \tag{29}$$

Since $\tilde{f}_{uv,1} > \hat{f}_{uv,1}$ then $\tilde{f}_{uv,2} < \hat{f}_{uv,2}$, similarly we have :

$$\hat{\lambda}_{u,2} - \hat{\lambda}_{v,2} > \tilde{\lambda}_{u,2} - \tilde{\lambda}_{v,2} + \delta_2 \frac{I}{2} g'(\tilde{f}_{uv})(\hat{\beta}_{uv} - \tilde{\beta}_{uv}) \tag{30}$$

where $\lambda_{uv,2} = \sum_{i \in \mathcal{I}_2} \lambda^i_{uv}$ and $\delta_2 = \sum_{i \in \mathcal{I}_2} \frac{1}{\alpha^i}$.

If $\hat{f}_{vu,1} > \tilde{f}_{vu,1}$, we have by symmetry

$$\hat{\lambda}_{v,1} - \hat{\lambda}_{u,1} > \tilde{\lambda}_{v,1} - \tilde{\lambda}_{u,1} + \delta_1 \frac{I}{2} g'(\hat{f}_{vu})(\hat{\beta}_{vu} - \tilde{\beta}_{vu})$$

Thus

$$\tilde{\lambda}_{u,1} - \tilde{\lambda}_{v,1} > \hat{\lambda}_{u,1} - \hat{\lambda}_{v,1} + \delta_1 \frac{I}{2} g'(\hat{f}_{vu})(\hat{\beta}_{vu} - \tilde{\beta}_{vu}) \tag{31}$$

Since $\tilde{f}_{vu,1} < \hat{f}_{vu,1}$ then $\tilde{f}_{vu,2} > \hat{f}_{vu,2}$, similarly we have :

$$\hat{\lambda}_{u,2} - \hat{\lambda}_{v,2} > \tilde{\lambda}_{u,2} - \tilde{\lambda}_{v,2} + \delta_2 \frac{I}{2} g'(\tilde{f}_{vu})(\tilde{\beta}_{vu} - \hat{\beta}_{vu}) \tag{32}$$

Summing each inequalities (29) over $uv \in \{\mathbf{S} \cap \mathcal{L}\}$ and each inequalities (31) over $uv \in \{\mathbf{S} - \mathcal{L}\}$, we obtain :

$$\sum_{uv \in \{\mathbf{S} - \mathcal{L}\}} g'(\hat{f}_{vu})(\tilde{\beta}_{vu} - \hat{\beta}_{vu}) > \sum_{uv \in \{\mathbf{S} \cap \mathcal{L}\}} g'(\tilde{f}_{uv})(\tilde{\beta}_{uv} - \hat{\beta}_{uv}) \tag{33}$$

and by summing each inequalities (30) or (32), we obtain :

$$\sum_{uv\in\{\mathbf{S}-\mathcal{L}\}} g'(\tilde{f}_{vu})(\hat{\beta}_{vu}-\tilde{\beta}_{vu}) > \sum_{uv\in\{\mathbf{S}\cap\mathcal{L}\}} g'(\hat{f}_{uv})(\hat{\beta}_{uv}-\tilde{\beta}_{uv}) \tag{34}$$

contradiction between inequalities (33) and (34).
Case 2 : I is odd.
Since $\hat{f}^i_{l_0} < \tilde{f}^i_{l_0}$ and $\tilde{f}_l = \hat{f}_l, l \in \mathcal{L}$, then it's easy to show that there exists a user i_0 and two disjoint sets $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \{i_0\} = \mathcal{I}$, $|\mathcal{I}_1| = |\mathcal{I}_2| = (I-1)/2$ and

$$\begin{aligned} \tilde{f}_{l_0,1} &:= \sum_{i\in\mathcal{I}_1} \tilde{f}^i_{l_0} + \frac{\tilde{f}^{i_0}_{l_0}}{2} > \hat{f}_{l_0,1} := \sum_{i\in\mathcal{I}_1} \hat{f}^i_{l_0,1} + \frac{\hat{f}^{i_0}_{l_0}}{2}, \text{ and} \\ \tilde{f}_{l_0,2} &:= \sum_{i\in\mathcal{I}_2} \tilde{f}^i_{l_0} + \frac{\tilde{f}^{i_0}_{l_0}}{2} < \hat{f}_{l_0,1} := \sum_{i\in\mathcal{I}_2} \hat{f}^i_{l_0,2} + \frac{\hat{f}^{i_0}_{l_0}}{2} \end{aligned}$$

If we continue with same procedure as in the case where I is even, we obtain analogous results. ∎

Corollary 5.2. *Consider cost functions of type-***B***. Then in the absence of the QoS constraints (23) and under "all-positive flow" assumption, the Nash equilibrium is unique.*

Appendix

Proof of Lemma 4.1 : From Remark **4.2** we shall only prove lemma for $\alpha^i = 1, \forall i \in \mathcal{I}$. Choose an arbitrary link l. If $f^j_l = 0$, then the implication is trivial. Otherwise, i.e., if $f^j_l > 0$, from the Kuhn-Tucker conditions we have that

$$\eta^j = \beta_l + K_l(f^j_l, f_l) + K_0(r^j) \le \beta_{\hat{l}} + K_{\hat{l}}(f^j_{\hat{l}}, f_{\hat{l}}) + K_0(r^j)$$

and since $f^i_{\hat{l}} > f^j_{\hat{l}}$ implies $f^i_{\hat{l}} > 0$, we have

$$\eta^i = \beta_{\hat{l}} + K_{\hat{l}}(f^i_{\hat{l}}, f_{\hat{l}}) + K_0(r^i) \le \beta_l + K_l(f^i_l, f_l) + K_0(r^i)$$

Thus, we have

$$\beta_l + K_l(f^j_l, f_l) \le \beta_{\hat{l}} + K_{\hat{l}}(f^j_{\hat{l}}, f_{\hat{l}}) < \beta_{\hat{l}} + K_{\hat{l}}(f^i_{\hat{l}}, f_{\hat{l}}) \le \beta_l + K_l(f^i_l, f_l)$$

i.e., $K_l(f^j_l, f_l) < K_l(f^i_l, f_l)$, which implies $f^j_l < f^i_l$. ∎

Proof of Proposition 4.1: Note that $r^i \ge r^j$ holds trivially if $r^j = m^j$. Otherwise, if $r^j > m^j$, by contradiction assume that $r^i < r^j$; then, $\eta^i \ge \eta^j$. Since $r^j > r^i$, then there must be at least one link $\hat{l}$ for which $f^i_{\hat{l}} < f^j_{\hat{l}}$. From the Kuhn-Tucker conditions, we have that

$$\eta^j - \beta_{\hat{l}} = K_{\hat{l}}(f^j_{\hat{l}}, f_{\hat{l}}) + K_0(r^j) > K_{\hat{l}}(f^i_{\hat{l}}, f_{\hat{l}}) + K_0(r^i) \ge \eta^i - \beta_{\hat{l}}$$

We then have a contradiction since $\eta^i \ge \eta^j$.
Now we show that $f^i_l \ge f^j_l$ for all links $l \in \mathcal{L}$. Assume that to the contrary $f^i_{\hat{l}} < f^j_{\hat{l}}$ for some $\hat{l}$. Then, by the Lemma 4.1 we have $f^i_l \le f^j_l$ on all other links, which upon

summation yields $r^i < r^j$, which contradict $r^i \geq r^j$. ∎

Proof of Proposition 4.2: Let i, j such that $\alpha^i > \alpha^j$. By contradiction, assume that $r^i < r^j$, implies that exist a link $l \in \mathcal{L}$ such that $f_l^i < f_l^j$. Hence (14) together with it's assumptions imply that :

$$\begin{aligned} \eta^j - \beta_l/\alpha^j = K_l(f_l^j, f_l) + K_0(r^j) &> K_l(f_l^i, f_l) + K_0(r^i) \geq \eta^i - \beta_l/\alpha^i \\ \eta^j - \eta^i + \beta_l(\frac{1}{\alpha^i} - \frac{1}{\alpha^j}) &> 0 \end{aligned}$$

Since $\alpha^i > \alpha^j$, it follows from the last inequality :

$$\eta^j > \eta^i \tag{35}$$

However, $r^i < r^j$ implies that $\eta^j \leq \eta^i$, which contradict (35). ∎

Proof of Lemma 5.1.- We first show that $r^i = r^j$ for $i, j \in \mathcal{I}$ such that $\alpha^i = \alpha^j$. By contradiction we assume that $r^i \neq r^j$, and without loss of generality we assume that $r^i > r^j$.

Now we construct a directed network $(\mathcal{N}', \mathcal{L}')$, where $\mathcal{N}' = \mathcal{N}$ and the set of links $\mathcal{L}'$ is constructed as follows :
1. For each link $l = (u, v) \in \mathcal{L}$, such that $f_l^i \geq f_l^j$, we have a link $l' = (u, v) \in \mathcal{L}'$; to such a link l' we assign a (flow) value $z_{l'} = f_l^i - f_l^j$.
2. for each link $l = (u, v)$, such that $f_l^i < f_l^j$, we have a link $l' = (v, u) \in \mathcal{L}'$; to such a link we assign a (flow) value $z_{l'} = f_l^j - f_l^i$.
It is easy to verify that the value $z_{l'}$ constitutes a nonnegative, directed flow in the network. Since $r^i > r^j$, $z_{l'}$ must carry some flow (the amount of $r^i - r^j$) from the source s to the destination d, this implies that there exists a path p^* from s to d, such that $z_{l'} > 0$ for all $l' \in p^*$.

Consider now a link $l' = (u, v) \in p^*$. Since $z_{l'} > 0$ either $f_{uv}^i > f_{uv}^j$ or $f_{vu}^j > f_{vu}^i$. In the case where $f_{uv}^i > f_{uv}^j$, we have :

$$\begin{aligned} \alpha^i(\lambda_u^i - \lambda_v^i) = \alpha^i K_{uv}(f_{uv}^i, f_{uv}) + \beta_{uv} g'(f_{uv}) &> \alpha^j K_{uv}(f_{uv}^j, f_{uv}) + \beta_{uv} g'(f_{uv}) \\ &\geq \alpha^j(\lambda_u^j - \lambda_v^j) \end{aligned}$$

Thus

$$\lambda_u^i - \lambda_v^i > \lambda_u^j - \lambda_v^j \tag{36}$$

If $f_{vu}^j > f_{vu}^i$, we have by symmetry that $\alpha^j(\lambda_v^j - \lambda_u^j) > \alpha^i(\lambda_v^i - \lambda_u^i)$ thus we obtain (36). Define more precisely the path p^*, by $p^* = (s, u_1, u_2, ..., u_{n^*}, d)$, where u_k, $k = 1, 2, .., n^*$, is the k^{th} node after the source s on the path p^* and n^* is the number of nodes between the source s and the destination d. Hence, from (36) we have :

$$\lambda_s^i - \lambda_s^j > \lambda_{u_1}^i - \lambda_{u_1}^j > .... > \lambda_{u_{n^*}}^i - \lambda_{u_{n^*}}^j > \lambda_d^i - \lambda_d^j. \tag{37}$$

In the other hand, we have $m \leq r^j < r^i \leq M$, it follows that $\mu^i = \gamma^j = 0$, and from (8) we have :

$$K_0(r^i) - \lambda_d^i + \lambda_s^i \leq K_0(r^j) - \lambda_d^j + \lambda_s^j$$

Since K_0 is strictly increasing and $r^i > r^j$, we have from the last inequality that $-\lambda_d^i + \lambda_s^i < -\lambda_d^j + \lambda_s^j$, which contradict (37).

We now proceed to show that $f_l^i = f_l^j$, for all $l \in \mathcal{L}$. By contradiction we assume that $f_{l_0}^i > f_{l_0}^j$ for some $l_0 \in \mathcal{L}$. Since $r^i = r^j$, by using the same procedure as previously we can show that there exists a cycle $\mathbf{S} = (u_0, u_1, u_2, ..., u_{n^*}, u_0)$, such that $z_{u_{k-1}u_k} > 0$ and $z_{u_{n^*}u_0} > 0$, where u_k, $k = 1, 2, .., n^*$, is the k^{th} node after the node u_0 on the cycle $\mathbf{S}$ and n^* is the number of nodes in the cycle. Similarly we have :

$$\lambda^i_{u_0} - \lambda^j_{u_0} > \lambda^i_{u_1} - \lambda^j_{u_1} > .... > \lambda^i_{u_{n^*}} - \lambda^j_{u_{n^*}} > \lambda^i_{u_0} - \lambda^j_{u_0}$$

Which is contradiction. ■

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