Proceedings VII European Signal Processing Conference (EUSIPCO'94), vol. 3, pp. 1582-1585, Edinburgh, 13-16 Sept. 1994

High Order Transformations for Flexible IIR Filter Design

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Abstract. Extensions of popular transformations for IIR filters are given which employ high order mapping filters. Easy control of prototype transfer function features in multiband renditions is demonstrated. A wider interpretation of transformation is also suggested which permits "less-than-N-band replication" (at a cost in dimensionality , phase fidelity and attention to stability enforcement) that is believed to be of considerable benefit in practical design situations.

1. Introduction

Conversion of an existing FIR or IIR filter design to a modified IIR form is often done by means of allpass transformations. Although the resulting designs are considerably more expensive, in terms of dimensionality, than the original prototype, the ease of use (in fixed or variable application) is a big advantage.

Up to now the definitive mapping equations are those put forward by Constantinides [1] and since adopted as "industry standard". These well-known equations are geared up to map lowpass to bandpass and several other highly stylised combinations. They were the culmination of preceding work [2]-[4] which pioneered departure from the earliest transformation work by Broome, where a simple modulation approach (suffering from severe aliasing) was used [5]. Recent work [6], [7] has further strengthened the general utility of both of these methods. Here we pursue only an extension of the Constantinides approach. The basic form of mapping in common use is:

$$H_r(z) = H_p\left[H_2(z)\right] \tag{1}$$

where $H_r(z)$ is the resulting filter when a prototype filter $H_p(z)$ is acted upon by a second-order mapping filter:

$$H_2(z) = \pm\frac{z^2 + \beta_1 z + \beta_2}{\beta_2 z^2 + \beta_1 z + 1} \tag{2}$$

The two degrees of freedom providable by β_1 and β_2 choices are under-used by the usual restrictive set of "flat-top" classical mappings like lowpass to bandpass. Instead, any two transfer function features can be migrated to (almost) any two other frequency locations if β_1 and β_2 are chosen so as to keep the poles of $H_2(z)$ strictly outside the unit circle (since $H_2(z)$ is substituted for **z** in the original prototype transfer function). Moreover, as first pointed out by Constantinides, the selection of outside sign influences whether the original feature at zero can be moved (the minus sign, a condition we refer to as "DC mobility") or whether the Nyquist frequency can be migrated (the "Nyquist mobility" case - as we call it - arising when the leading sign is positive). Unfortunately there is not total freedom in re-deployment of any pair of frequency constraint points; in Section 2 below we outline both the generalised second-order β_1, β_2 selection relation and the forbidden combinations, through a pair of simple inequalities.

In this paper we treat also the case of transformations of higher order than two. Though the enhanced design flexibility this offers is readily evident, there has been little work reported in this area. Although Mullis and colleagues have given one very useful multiband solution to the general mapping problem [8]-[11], it seems that scant application experience of that (or any other) method has been related in IIR design literature. Here we present an N^{th} order generalization of our second-order linear equation solution method as an alternative to the Mullis approach and demonstrate its utility in the context of several examples. Our method permits migration of selected constraint points, as well as embracing the multiband possibilities of Mullis.

2. Extensions to Second-Order Mappings

We like to think of (2) as relating "old" and "new" z-domain images:

$$z_{old} = \pm\frac{z_{new}^2 + b_1 z_{new} + b_2}{b_2 z_{new}^2 + b_1 z_{new} + 1} \tag{3}$$

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where the upper (+) sign is applicable for "Nyquist mobility", while the lower one is for the "DC mobility". We can independently specify two distinct migrations $z_{old1} \rightarrow z_{new1}$, $z_{old2} \rightarrow z_{new2}$ and use these to solve the two simultaneous equations which arise from eq.(3):

$$b_1 = \frac{EC - FB}{AE - BD} \quad \text{and} \quad b_2 = \frac{AF - CD}{AE - BD} \tag{4}$$

where we shorten "old" to "o" and "new" to "n" in the subscripts utilized in these terms:

$$A = z_{n1}(1 \pm z_{o1}) \quad D = z_{n2}(1 \pm z_{o2})$$
$$B = 1 \pm z_{o1} z_{n1}^2 \quad E = 1 \pm z_{o2} z_{n2}^2 \tag{5}$$
$$C = -(z_{o1} \pm z_{n1}^2) \quad F = -(z_{o2} \pm z_{n2}^2)$$

Putting in an explicit unit-circle form, we specialize "old" locations z_o to be called $e^{j2\pi\nu_o}$ and append suitable subscripts. This is likewise done for "new" unit-circle locations z_n. Here ν is the normalized frequency variable. This yields the more fullsome (DC mobility) expressions:

$$b_1 = \frac{\cos\pi(2\nu_{n1} + \nu_{o1}) \cdot \cos\pi(2\nu_{n2} - \nu_{o2}) - \cos\pi(2\nu_{n1} - \nu_{o1}) \cdot \cos\pi(2\nu_{n2} + \nu_{o2})}{\cos\pi\nu_{o1} \cdot \cos\pi(2\nu_{n2} + \nu_{o2}) - \cos\pi\nu_{o2} \cdot \cos\pi(2\nu_{n1} + \nu_{o1})} \tag{6}$$

and

$$b_2 = \frac{\cos\pi\nu_{o2} \cdot \cos\pi(2\nu_{n1} - \nu_{o1}) - \cos\pi\nu_{o1} \cdot \cos\pi(2\nu_{n2} - \nu_{o2})}{\cos\pi\nu_{o1} \cdot \cos\pi(2\nu_{n2} + \nu_{o2}) - \cos\pi\nu_{o2} \cdot \cos\pi(2\nu_{n1} + \nu_{o1})} \tag{7}$$

and similar manifestations of eq.(5) apply for Nyquist mobility.

In the well-known Constantinides formulas "old" features are usually cast as bandedges whose "new" images are also bandedges. For instance, selection of $-\nu_{o1}=\nu_{o2}=\nu_p$ (where ν_p is the edge of the passband of a prototype lowpass filter), along with corresponding images ν_1 and ν_2, will deliver a bandpass resulting filter which is "live" over the positive-frequency band $\nu \in (\nu_1, \nu_2)$. However, as Constantinides took care to point out [1], the DC frequency gain of the prototype does not map to $\nu = (\nu_1 + \nu_2)/2$, but rather is warped to a location which requires calculation of a supplementary equation.

It is quite easy to modify the standard Constantinides result, for instance, to explicitly control the movement of the DC feature and one bandedge; selection of $\nu_{o1}=0$, $\nu_{n1} = (\nu_1 + \nu_2)/2$, $\nu_{o2}= \nu_p$ and $\nu_{n2}= \nu_2$ for use in eq.(6) and eq.(7) immediately results in a useful "DC plus upper bandedge-controlled" alternative to the standard Constantinides lowpass-to-bandpass equation usually invoked. Always, however, we will be limited to explicit control of only two features as long as we employ only a second-order mapping filter.

It is worthwhile reflecting upon whether any two (z_{o1}, z_{n1}), (z_{o2}, z_{n2}) combinations will be legitimate when using eq.(4). If it is our intention to map a stable prototype $H_p(z)$ to a stable $H_r(z)$ purely through use of eq.(1), taking $H_2(z)$ to have poles strictly outside the unit circle (a condition that is easily seen to be sufficient to guarantee such stability inheritance), then we find that certain restrictions in mapping pairs must be observed. First we note that a minimum-phase numerator in eq.(2) requires:

$$|b_2| < 1 \tag{8}$$

and eq.(7) begins to reveal the interplay of allowable "old" and "new" locations if these are specified in unit-circle forms (as is most often of interest to the filter designer). Continuing in this way and demanding DC movement to frequency ν_d, we have this relation:

$$b_1 = -(1 + b_2)\cos 2\pi\nu_d \tag{9}$$

so that

$$|b_1| < 2|\cos 2\pi\nu_d| \tag{10}$$

and eq.(6) finishes (for DC mobility) the story on allowable frequency specification combinations.

3. Higher-Order Mappings

Usage of a "transformation filter" like $H_2(z)$ is often inadequate. It is frequently required to have greater control - at other frequencies besides the (almost) arbitrary pair under direct control - than that delivered as a consequence of the warping attending this simple transformation. A typical example is the need to transport a lowpass filter to bandpass, simultaneously retaining a capability of precise placement of upper bandedge and lower bandedge frequencies, along with a couple of specified intervening frequency features. This requires extension to higher-order versions of eq.(1) and eq.(2). In particular, a mapping filter $H_N(z)$ would replace $H_2(z)$ in eq.(1).

Of course such enhanced transfer function control can only be achieved at a cost in complexity, as can be seen clearly from the escalated dimensionality of the pole-zero patterns characteristic of such designs. Nevertheless, practical goals such as rapid re-design in tuneable filtering scenarios is one of the good reasons practitioners might wish to absorb this cost and has provided the motivation for a noteworthy body of earlier work (e.g. [12],[13]). The approach is indeed flexible; the reciprocal of

$$H_N(z) = \pm \frac{z^N + \sum_{i=1}^{N} b_i z^{N-i}}{1 + \sum_{i=1}^{N} b_i z^i} \tag{11}$$

would be used to replace each delayor in a prototype filter structure. This gives four distinct opportunities for influencing the overall filtering operation:

a) Choice of the structure and order for the prototype filter

b) Selection of coefficients for the prototype filter

c) Choice of the structure and order N for mapping filter $H_N(z)$

d) Selection of the β_i in $H_N(z)$

Real-time design in items (b) and (d), in particular, give a nice way of achieving nested variability. There is, moreover, scope for driving these changes (including even dynamic change of N) in an <u>adaptive</u> IIR arrangement. This adds greatly to the appeal of the whole approach, and has motivated our development of a general matrix solution equation for the N β_i in eq.(11) in terms of (almost) arbitrary frequency migration specifications. Considering eq.(11) as a mapping function $z_{old}=H_N(z_{new})$ then it can be rewritten using:

$$\sum_{i=1}^{N} b_i \left(z_{old} z_{new}^{i} \mp z_{new}^{N-i} \right) = -\left(z_{old} \mp z_{new}^{N} \right) \tag{12}$$

for each of **N** basic mappings $z_{old} \rightarrow z_{new}$, giving a set of **N** linear equations:

$$[\beta] \cdot [\mathbf{A}] = [\mathbf{b}] \tag{13}$$

where:

$$\begin{aligned} A_{i,j} &= z_{old_i} z_{new_i}^{j} \mp z_{new_i}^{N-j} \\ & \qquad\qquad i, j = 1,\ldots,N \\ b_i &= -\left(z_{old_i} \mp z_{new_i}^{N} \right) \end{aligned} \tag{14}$$

The designer need only (in principle) specify the N pairs (z_{old}, z_{new}), assemble $H_N(z)$ by solving eq.(13) for the β_i values and then map with:

$$H_r(z) = H_p\left[H_N(z) \right] \tag{15}$$

The only difficulty lies in selecting allowable mapping point pairs at the outset of the procedure.

4. Mapping Examples

Figure 1 shows a fourth-order lowpass filter which we will take as $H_p(z)$ for all our examples. We will try to identify how its distinctive passband and stopband characteristics translate into other mapping results.

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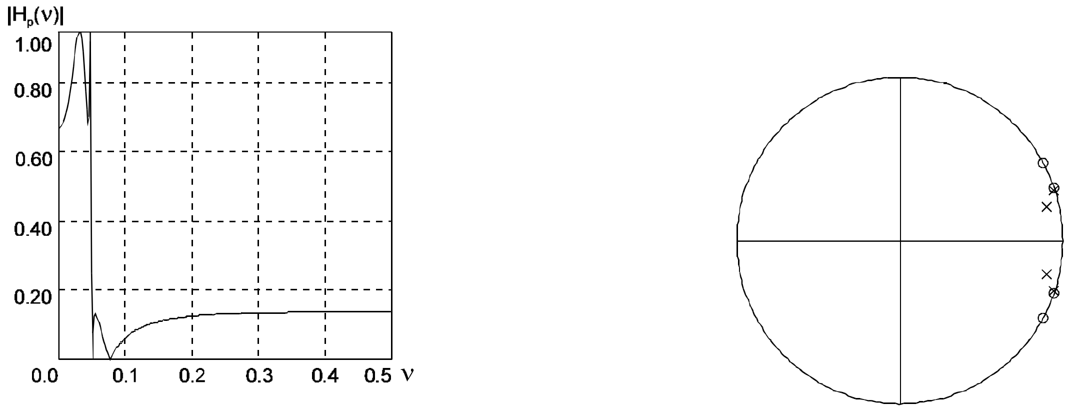

Figure 1. Our 4th Order Lowpass Prototype.

If we choose $H_N(z)=H_5(z)$ and form a particular mapping filter by solving eq.(13), we can try the migration (with DC mobility):

	# 1	# 2	# 3	# 4	# 5
ν_{old}	-0.05	0.05	-0.05	0.05	-0.05
ν_{new}	0.10	0.20	0.30	0.40	0.48

Table 1. A Five-point Five-band Mapping Requirement.

This gives us five replicates laid onto the unit z circle. Note that two of the positive-frequency replicas are highly symmetric about ν=0.25, while the portion seen of the third is warped very differently.

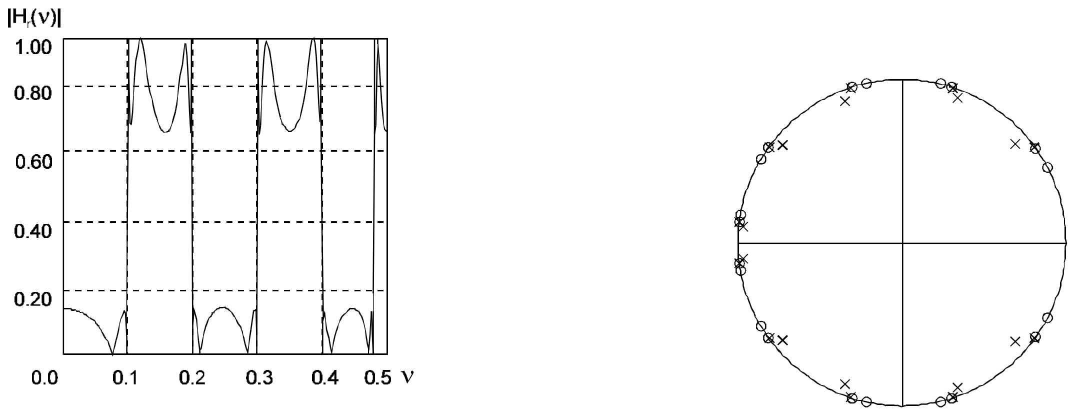

Figure 2. Result of the 5th Order Mapping of Table 1.

Separated replicas are easy to create, but do not exercise all the possibilities we want in practice. There is intense interest in control of versions of the prototype filter which number less than N, but where several selected features of the prototype transfer function are deployed through judicious selection of the β_i.

It is at this juncture that we must be prepared to sacrifice complete transformability of the prototype filter in order to assure stability of the resulting $H_r(z)$. It is often acceptable to deliver only the magnitude response, $|H_p(e^{j2\pi\nu})|$, over targetted spectral bands. If we have such freedom then the notion of strict transformation can be modified to this more liberal procedure:

(i) Create an allpass $H_N(z)$ that might have some or all poles inside the unit circle.

(ii) Utilize eq.(15), getting $H_r(z)$ with not all poles interior to the unit circle.

(iii) Obtain a final, stable result $H_r(z)$ by flipping each exterior pole inside the unit circle and scaling appropriately.

This "stability-forced transformation" method - though requiring root-finding and associated numerical intervention at stage (iii) - vastly expands the utility of the mapping idea. We show this by employing a 4th order transformation to take our prototype lowpass filter to a bandpass version using these specified constraints:

	# 1	# 2	# 3	# 4
ν_{old}	0.0	-0.051683	0.051683	0.078687
ν_{new}	0.2	0.1	0.3	0.335

Table 2. A Four-point Two-band Mapping Requirement.

Here we have taken fairly precise values of the locations of $H_p(z)$'s zeroes and transported them (along with the DC feature) to clearly identifiable locations which Figure 3 exhibits:

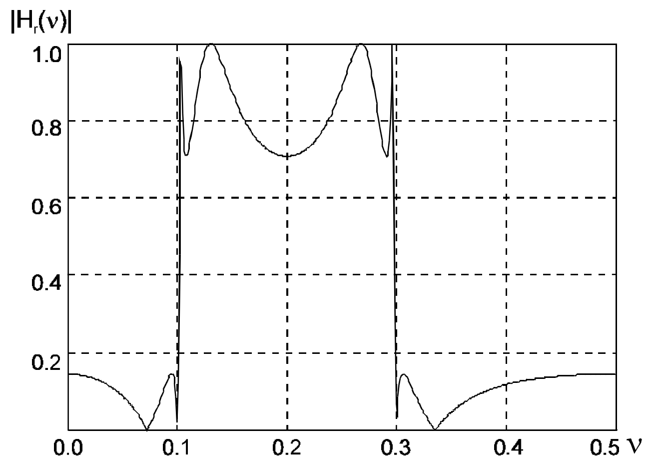

Figure 3 Result of the 4th Order Mapping of Table 2.

But it should not be thought that this approach is without its difficulties. It is no accident that 0.335 has been specified as an image frequency target in Table 2; even a small offset (say to 0.34) from that "favourable value" causes upset to the integrity of the mapping.

We have found that we can often counter the extreme sensitivity of such degenerate "Less than N-band" mapping by raising N and specifying more points still ! The objective in doing this is to provide a pole/zero pattern sufficiently rich to encourage alignments that can maintain a phase plot (for a mapping filter) free of discontinuities. This phase plot is the key to good unit-circle mapping and must be a one-to-one curve (for positive frequencies), for instance , if a single-band mapping is sought. Figure 4 is a waterfall plot of four filter gain plots which (by exploiting symmetric disposition about ν=0.25) have been produced by lowpass-to-bandpass transformation with complete freedom in placement of three pairs of constraint points. Here, just for ease of display, bandedges and zeroes coincide in all four plots, while the "ridge" arising from movement of a single peak feature (with its frequency changing between 0.13 and 0.23) mainly distinguishes them. This flexibility comes at the cost of employing a 6th order filter to carry out the mapping task.

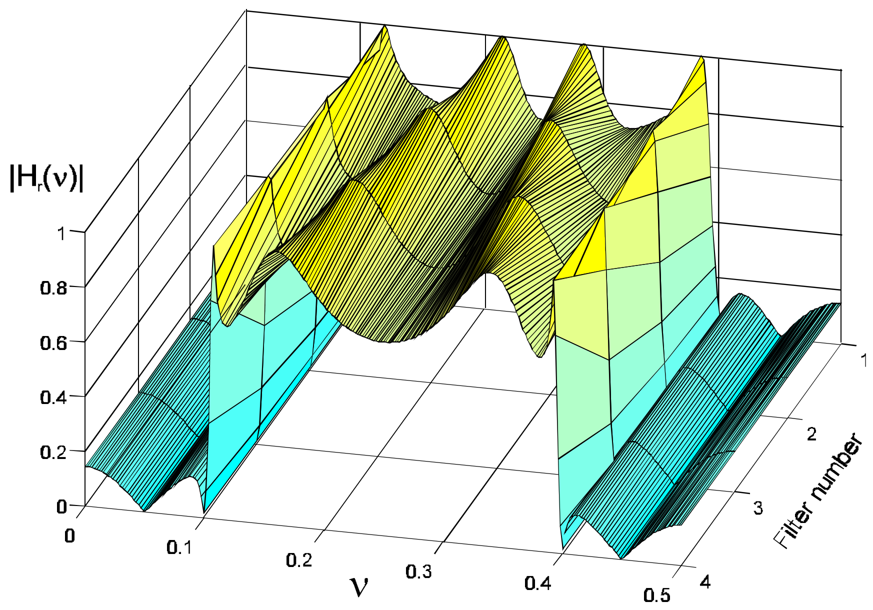

Figure 4. Resulting magnitude responses with ±0.051683 bandedges moved to (0.1, 0.4) while zeroes at ±0.07868 are anchored at (0.05, 0.45); in addition movements of peak features originally at ν=±0.0357 are shown.

It is felt that the great flexibility such "dimensionality overkill" offers to the filter designer is, in many circumstances, a legitimate and attractive alternative to direct IIR design and has the considerable advantage of permitting precise deployment of familiar gain features. Interim relaxation of stability requirements and abandonment of phase fidelity offers enormous scope in a general design setting.

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