Process Algebra with Probabilistic Choice*

Suzana Andova

Department of Computing Science
Eindhoven University of Technology
P.O.Box 513, 5600 Eindhoven, The Netherlands
suzana@win.tue.nl

Abstract. Published results show that various models may be obtained by combining parallel composition with probability and with or without non-determinism. In this paper we treat this problem in the setting of process algebra in the form of *ACP*. First, probabilities are introduced by an operator for the internal probabilistic choice. In this way we obtain the Basic Process Algebra with probabilistic choice *prBPA*. Afterwards, *prBPA* is extended with parallel composition to ACP_π^+. We give the axiom system for ACP_π^+ and a complete operational semantics that preserves the interleaving model for the dynamic concurrent processes. Considering the PAR protocol, a communication protocol that can be used in the case of unreliable channels, we investigate the applicability of ACP_π^+. Using in addition only the priority operator and the pre-abstraction operator we obtain a recursive specification of the behaviour of the protocol that can be viewed as a Markov chain.

1 Introduction

Due to the increasing complexity and the number of components of real-life parallel systems, the probability that a system or some of its components will be subject to failure during the work is increased, as well. This means that very often it is desirable or even necessary to "predict" chances of failure occurring in the system. Therefore, it is insufficient to assume that the system is reliable and to specify it under this assumption, but there is a need to describe the probabilistic behaviour of the components and the system as a whole. For the last ten years various traditional specification formalisms have been extended with a notion of probabilistic behaviour for different models of probabilistic processes.

Besides this new, probabilistic approach in modelling concurrent systems, non-determinism still has an essential role specially due to interleaving of activities of independent components of a system. In treating non-determinism mainly two different approaches have been followed, one approach which allows both non-deterministic and probabilistic choices (e.g. concurrent Markov chains [16], the alternating model [12]), and one where only probabilistic choice is allowed ([14,10,11,13,6,9]).

* Research is supported by PROMACS project, SION 612-10-000 of the Netherlands Organisation for Scientific Research (NWO)

J.-P. Katoen (Ed.): ARTS'99, LNCS 1601, pp. 111–129, 1999.

The objective of this paper is to introduce a probabilistic version of *ACP* ([3,7]) where non-determinism and probability are combined.

Following the idea of *ACP*-like process algebra for interleaving parallel composition, we first investigated a probabilistic version of *ACP*, *prACP*, where the axiom $x \parallel y = x \mathbin{\|\!\!\lfloor} y + y \mathbin{\|\!\!\lfloor} x + x \mid y$ holds for arbitrary processes x and y. This axiom leads to a situation where processes that depend on each other in their probabilistic behaviour are involved in merging atomic actions of x with those of y. In Section 3 we give an example of merge of parallel processes and point out an unwanted outcome that occurs. So we rejected this approach, as it is not suitable for specification of some concurrent systems such as for example PAR protocol.

Thus, we propose in this paper a new variant of the extension of *prBPA* by parallel composition. We still keep the idea of the interleaving model but this time only for dynamic processes (processes that do only trivial probabilistic transition with probability 1). This novel process algebra has a more complex axiom system than *prACP* in [1]. But an advantage here is a simple and intuitively clear operational semantics. We use an extra quaternary operator $]\!\![$, called merge with memory, which helps in axiomatising the merge of dynamic processes. This operator is not necessary in the sense that an equivalent algebra, called ACP_π, can be obtained by adding new axioms without any extra operators. These two process algebras, ACP_π and the presented ACP_π^+ are equivalent but only for processes that do not contain the $\mathbin{\|\!\!\lfloor}$, $\mid$ and $]\!\![$ operators. This version of combining probabilities and parallel composition in the framework of interleaving approach is proposed in [9] where the authors use bundle probabilistic transition systems.

The operational semantics of ACP_π^+ is based on the alternating model of [12] and it is defined by a term deduction system of which the signature contains an extended set of constants (each atomic action has a dynamic counterpart) and of which the deduction rules include two transition types: probabilistic and action transition. The probability associated to a probabilistic transition is determined by the value of a probability distribution function. In the construction of the term models we use probabilistic bisimulation as proposed by Larsen and Skou ([14]) and we show soundness and completeness of the term model with respect to proposed axiom systems.

Dealing with the PAR protocol ([15]) a communication protocol used in cases of unreliable channels, we investigate the applicability of ACP_π^+. We give a specification in ACP_π^+ of the constituent processes of the protocol and of the whole system. In order to do performance analysis, non-determinism has to be resolved. Using in addition only the priority operator and the pre-abstraction operator [2] we obtain a recursive specification of the behaviour of the protocol that can be viewed as a Markov chain.

2 Basic Process Algebra

We give a brief introduction of Basic Process Algebra with probabilistic choice and a complete operational semantics.

The signature of Basic Process Algebra with probabilistic choice, *prBPA*, consists of a (finite) set of constants $A = \{a, b, c, \dots\}$, a special constant $\delta \notin A$ (we usually denote $A_\delta = A \cup \{\delta\}$) and the binary operators: $+$ (non-deterministic choice), $\cdot$ (sequential composition) and $\underline{+}_\pi$ (probabilistic choice) for each $\pi \in \langle 0, 1\rangle$. The probabilistic choice operator is modeled after the partial choice operator of [5]. Intuitively, process $x \underline{+}_\pi y$ behaves like x with probability π and behaves like y with probability $1-\pi$. The choice is already made, and cannot be influenced by the environment. We can observe the outcome, and the probability distribution of the possible outcomes. The axioms for $+$ and $\cdot$ are standard axioms for BPA_δ ([3]) (Table 1, $a \in A$), except that axiom $A3$ ($x + x = x$) is restricted to atomic actions. $A3$ is restricted, because it does not hold anymore for processes that contain the new choice operator. In our intuition about combining non-determinism and probabilistic choice, the "top" operator is the probabilistic choice, that is we consider that the probabilistic choice is made first and later non-deterministic choice. In such a way in the process $(a \underline{+}_\pi b) + (a \underline{+}_\pi b)$ non-deterministic choice between actions a and b is possible with a certain probability which is not a case in the process $a \underline{+}_\pi b$. The axioms for the new operators are shown in Table 2 ($\pi \in \langle 0, 1\rangle$).

$x + y$	$= y + x$	$A1$
$(x + y) + z$	$= x + (y + z)$	$A2$
$a + a$	$= a$	$AA3$
$(x + y) \cdot z$	$= x \cdot z + y \cdot z$	$A4$
$(x \cdot y) \cdot z$	$= x \cdot (y \cdot z)$	$A5$
$x + \delta$	$= x$	$A6$
$\delta \cdot x$	$= \delta$	$A7$

Table 1. BPA_δ with restricted $A3$.

$x \underline{+}_\pi y$	$= y \underline{+}_{1-\pi} x$	$PrAC1$
$x \underline{+}_\pi (y \underline{+}_\rho z)$	$= (x \underline{+}_{\frac{\pi}{\pi+\rho-\pi\rho}} y) \underline{+}_{\pi+\rho-\pi\rho} z$	$PrAC2$
$x \underline{+}_\pi x$	$= x$	$PrAC3$
$(x \underline{+}_\pi y) \cdot z$	$= x \cdot z \underline{+}_\pi y \cdot z$	$PrAC4$
$(x \underline{+}_\pi y) + z$	$= (x + z) \underline{+}_\pi (y + z)$	$PrAC5$

Table 2. Additional axioms for *prBPA*.

We introduce abbreviations in order to deal with probabilistic sums of several arguments:

$x \underline{+}_\pi y \underline{+}_\rho z \equiv x \underline{+}_\pi (y \underline{+}_{\frac{\rho}{1-\pi}} z)$ $\quad (\pi + \rho < 1)$

$x \underline{+}_\pi y \underline{+}_\rho z \underline{+}_\sigma w \equiv x \underline{+}_\pi (y \underline{+}_{\frac{\rho}{1-\pi}} z \underline{+}_{\frac{\sigma}{1-\pi}} w)$ $\quad (\pi + \rho + \sigma < 1)$, etc.

Example 1. By this example we show the interpretation of non-determinism when it is combined with probabilistic choice. In Figure 1 the transition systems for the processes are shown.

$$(a \boxplus_{\frac{1}{2}} b) + (c \boxplus_{\frac{1}{3}} d) = (a + c) \boxplus_{\frac{1}{6}} (a + d) \boxplus_{\frac{1}{3}} (b + c) \boxplus_{\frac{1}{6}} (b + d).$$

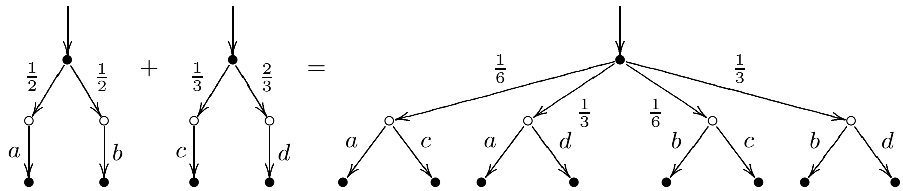

Fig. 1. An example of non-deterministic choice between probabilistic processes

In [5], the authors propose a method for verification which is based on a partial ordering of processes. They introduce the realization axiom $x \leq x \boxplus y$, which says that x has less static non-determinism than $x \boxplus y$. By the following proposition we show that this approach cannot be followed in the framework of *prBPA* because such a partial ordering of processes cannot be defined when probabilisties are involved.

Proposition 1. *If prBPA* $\vdash p = q \boxplus_{\pi} p$ *for some probability* $\pi \in \langle 0, 1\rangle$*, then prBPA* $\vdash p \approx q$*, where* $p \approx q$ *denotes the probability of* p *be equal to* q *has a limit of* 1. □

We define basic terms as representatives of classes of closed terms. Theorem 2, the Elimination theorem, shows that each closed term can be reduced to a basic term. We distinguish two types of basic terms: terms that are constants or that have a non-deterministic choice or a sequential composition as the outermost operator (we denote a set of these terms by $\mathcal{B}_+$) and the basic terms of the second type are such that have a probabilistic choice as the outermost operator. The precise definition of basic *prBPA* terms is given in [1].

Remark. If we consider terms that only differ in the order of the summands to be identical (i.e. we work modulo axioms $A1$, $A2$, $PrAC1$ and $PrAC2$) we have that the basic terms are exactly the terms of the form

$$x \equiv x_1 \quad \text{or} \tag{1}$$

$$x \equiv x_1 \boxplus_{\pi_1} x_2 \boxplus_{\pi_2} x_3 \ldots x_{n-1} \boxplus_{\pi_{n-1}} x_n \text{ and } \; n > 1 \tag{2}$$

where for each $i, 1 \leq i \leq n$, $x_i \equiv \sum\limits_{j<l_i} a_{ij} t_{ij} + \sum\limits_{k<m_i} b_{ik}$ for certain atomic actions a_{ij} and b_{ik}, basic terms t_{ij} and $n, m_i, l_i \in \mathbb{N}$. We have the convention that: $\sum\limits_{j<0} s_j \equiv \delta$.

Theorem 2 *(Elimination theorem) Let p be a closed prBPA term. Then there is a basic prBPA term q such that $prBPA \vdash p = q$.* □

Further, by $\mathcal{SP}$ (the set of static processes) we will denote the set of all closed terms over the signature of *prBPA*, Σ_{prBPA}. By **D** we denote a set of closed *prBPA* terms of which an associated basic term, which exists by the Elimination theorem, is a term from $\mathcal{B}_+$.

2.1 Structured Operational Semantics of *prBPA*

The operational semantics consists of two types of transition rules, probabilistic transitions and action transitions and it is based on the alternating model as it is proposed in [12]. Each process in our model may make either probabilistic transitions or atomic transitions, but not both. Action transitions are labelled with atomic actions: $\xrightarrow{a}$. Although in the presentation of processes as probabilistic transition systems we will use labelled probabilistic transitions as $\xrightarrow{\pi}$, in the formal description of the operational semantics probabilistic transitions are unlabelled: $\rightsquigarrow$. If process p may do a probabilistic transition to process x, there is a non-zero probability with which process p may behave as process x. The probability that this transition may happen is determined by a probability distribution function $\mu(p, x)$.

In order to distinguish processes that may do a probabilistic transition and processes that may do an action transition we consider a term deduction system with a signature different from the signature of *prBPA* by the addition of new constants. If A is the set of atomic actions of *prBPA* then we define the set of dynamic atomic actions $\breve{A}_\delta = \{\breve{a} \mid a \in A_\delta\}$. By a symbol $\breve{a}$, $(a \neq \delta)$ we denote a process that can successfully terminate by executing a. Further we will write $\breve{\Sigma}_{prBPA}$ for $(A_\delta \cup \breve{A}_\delta, +, \cdot, \uplus_\pi)$.

Definition 3 *We define the set of dynamic processes $\mathcal{DP}$ (processes that may do an action transition) in the following way:*

1. $\breve{A}_\delta \subseteq \mathcal{DP}$;
2. $s \in \mathcal{DP}, t \in \mathcal{SP} \Rightarrow s \cdot t \in \mathcal{DP}$;
3. $t, s \in \mathcal{DP} \Rightarrow t + s \in \mathcal{DP}$.

By $\mathcal{PR}$ we denote the set of all static and dynamic processes, that is $\mathcal{PR} = \mathcal{SP} \cup \mathcal{DP}$. Moreover, there is a bijection from **D** to $\mathcal{DP}$.

The operational semantics of *prBPA* is given by the term deduction system $\breve{T} = (\breve{\Sigma}_{prBPA}, D)$ induced by the deduction rules shown in Table 3 where a is a variable that ranges over the set A, and the probability distribution function as it is defined in Definition 4.

$$
\begin{array}{ccc}
a \rightsquigarrow \breve{a} & \delta \rightsquigarrow \breve{\delta} & \dfrac{p \rightsquigarrow x}{p \cdot q \rightsquigarrow x \cdot q} \\[2ex]
\dfrac{p \rightsquigarrow x, q \rightsquigarrow y}{p + q \rightsquigarrow x + y} & \dfrac{p \rightsquigarrow x}{p \sqcup_\pi q \rightsquigarrow x, q \sqcup_\pi p \rightsquigarrow x} & \\[2ex]
\breve{a} \xrightarrow{a} \surd & \dfrac{x \xrightarrow{a} x'}{x \cdot y \xrightarrow{a} x' \cdot y} & \dfrac{x \xrightarrow{a} \surd}{x \cdot y \xrightarrow{a} y} \\[2ex]
\dfrac{x \xrightarrow{a} x'}{x + y \xrightarrow{a} x', y + x \xrightarrow{a} x'} & \dfrac{x \xrightarrow{a} \surd}{x + y \xrightarrow{a} \surd, y + x \xrightarrow{a} \surd} &
\end{array}
$$

Table 3. Deduction rules of *prBPA*.

Definition 4 *(Probability distribution function) We define a probability distribution function* $\mu : \mathcal{PR} \times \mathcal{PR} \to [0,1]$ *as follows: for each* $x \in \mathcal{PR}$

$$
\begin{array}{ll}
\mu(a, \breve{a}) & = 1, \\
\mu(\delta, \breve{\delta}) & = 1, \\
\mu(p \cdot q, x' \cdot q) & = \mu(p, x'), \\
\mu(p + q, x' + x'') & = \mu(p, x')\mu(q, x''), \\
\mu(p \sqcup_\pi q, x) & = \pi\mu(p, x) + (1 - \pi)\mu(q, x), \\
\mu(p, x) & = 0 \quad \textit{otherwise.}
\end{array}
$$

Because in the construction of the term model we use the Larsen-Skou probabilistic bisimulation relation (Definition 7) we need to extend the probability distribution function to the power set of $\mathcal{PR}$.

Definition 5 *We define the map* $\mu^* : PR \times 2^{\mathcal{PR}} \to [0,1]$ *as: for each* $M \subseteq \mathcal{PR}$

$$\mu^*(p, M) = \sum_{x \in M} \mu(p, x).$$

Proposition 6. *The map* μ^* *is well defined.* □

From now on we will denote $\mu^*(p, M)$ simply by $\mu(p, M)$.

Definition 7 *Let* R *be an equivalence relation on the set of processes* $\mathcal{PR}$. R *is a probabilistic bisimulation if the following four clauses are satisfied:*

1. *If* pRq *and* $p \rightsquigarrow s$, *then there is a term* t *such that* $q \rightsquigarrow t$ *and* sRt;
2. *If* sRt *and* $s \xrightarrow{a} p$ *for some* $a \in A$, *then there is a term* q *such that* $t \xrightarrow{a} q$ *and* pRq;

3. *If sRt and $s \xrightarrow{a} \surd$, then $t \xrightarrow{a} \surd$;*
4. *If pRq, then $\mu(p, M) = \mu(q, M)$ for each $M \in \mathcal{PR}/R$.*

We say that p is probabilistically bisimilar to q, denote $p \underline{\leftrightarrow} q$, if there is a probabilistic bisimulation R such that pRq.

Below we give some obtained technical results. The detailed proofs of these propositions are given in [1].

Proposition 8. *If p is a $\mathcal{SP}$ term and $p \rightsquigarrow x$, then $x \in \mathcal{DP}$.* □

Proposition 9. *If x is a $\mathcal{DP}$ term and $x \xrightarrow{a} y$ for some $a \in A$, then $y \in \mathcal{SP}$.* □

Remark. From Proposition 8 and 9 it follows easily that we may consider:

1. $\rightsquigarrow \subseteq \mathcal{SP} \times \mathcal{DP}$,
2. $\xrightarrow{a} \subseteq \mathcal{DP} \times \mathcal{SP}$,
3. $\xrightarrow{a} \surd \subseteq \mathcal{DP}$,
4. for every probabilistic bisimulation R we have $R \subseteq \mathcal{SP} \times \mathcal{SP} \cup \mathcal{DP} \times \mathcal{DP}$.

Theorem 10 *$\underline{\leftrightarrow}$ is a congruence relation on prBPA.* □

Theorem 11 *(Soundness) Let p and q be closed prBPA terms. If prBPA $\vdash p = q$ then $p \underline{\leftrightarrow} q$.* □

Proposition 12. *Let p be a closed prBPA term and $a \in A$ and let op(p) be the number of operators of p. Then:*

i. if $p \rightsquigarrow \breve{x}$ then prBPA $\vdash p = x$ and $\mu(p, \breve{x}) = 1$ and $op(x) \leq op(p)$ or prBPA $\vdash p = x \boxplus_{\mu(p,\breve{x})} q$ for some $q \in \mathcal{SP}$ and $\mu(p, \breve{x}) < 1$;
ii. if $\breve{p} \xrightarrow{a} \surd$ then prBPA $\vdash p = a + p$;
iii. if $\breve{p} \xrightarrow{a} q$ then prBPA $\vdash p = a \cdot q + p$. □

Lemma 13 *If p, q and r are closed prBPA terms and $\pi \in \langle 0, 1\rangle$ such that $p \boxplus_{\pi} q \underline{\leftrightarrow} p \boxplus_{\pi} r$, then $q \underline{\leftrightarrow} r$.* □

Theorem 14 *(Completeness) Let p and q be closed prBPA terms. If $p \underline{\leftrightarrow} q$ then prBPA $\vdash p = q$.* □

3 Extension with Merge and Communication

Published results concerning design of probabilistic concurrent systems show various possibilities for combining parallel composition with probability with or without non-determinism. Different approaches lead to various formalisms and theories that treat this problem as well as various semantics. Following the idea of *ACP*-like process algebra for interleaving parallel composition, we studied a probabilistic version of *ACP*, *prACP* in [1], where the choice of the process which executes the next action is considered to be non-deterministic choice, and where according to this, the axiom $x \parallel y = x \mathbin{|\!\!\lfloor} y + y \mathbin{|\!\!\lfloor} x + x \mid y$ holds for arbitrary processes x and y. In this way we obtain a theory with very simple set of axioms (if we do not consider the axioms for the new operator, these axioms are in essence the same as those of *ACP*), but unfortunately it is not the case with the associated complete operational semantics which defines the term model of this process algebra where the crucial deduction rule (for parallel composition) is $\dfrac{p \rightsquigarrow x, q \rightsquigarrow y, p \rightsquigarrow x', q \rightsquigarrow y'}{p \parallel q \rightsquigarrow x \mathbin{|\!\!\lfloor} q + y \mathbin{|\!\!\lfloor} p + x' \mid y'}$. In [1] we give an example of an application of this process algebra for specification and performance analysis of concurrent systems, considering the Alternating Bit Protocol. But we realised that this approach to parallel composition does not give the anticipate results for some concurrent probabilistic processes, as the following example shows.

Let us consider the processes $P \equiv send_1$ and $Q \equiv read_1 \boxplus_{\pi} fail$. The process P executes the action $send_1$ which may be treated as "send a datum at the port 1" and the process Q executes the action $read_1$ with probability π, that is with probability π it reads the datum at the port 1, or executes the action $fail$ with probability $1-\pi$, that is it fails with probability $1-\pi$ and no further communication with the process P is possible. We remark that this situation is a realistic one when an unreliable transmission channel is designed (Section 4). We define a communication action $comm_1 = send_1 \mid read_1$. By intuition we expect that the behaviour of the whole system $\partial_H(P \parallel Q)$, for $H = \{send_1, read_1\}$, is described by the process $comm_1 \boxplus_{\pi} fail \cdot \delta$. But we obtain the following equation:

$$prACP \vdash \partial_H(P \parallel Q) = comm_1 \boxplus_{\pi^2} fail \cdot \delta \boxplus_{(1-\pi)^2} (fail \cdot \delta + comm_1) \boxplus_{(1-\pi)\pi} \delta.$$

As a consequence of the axiom mentioned above and the interpretation given of the non-deterministic choice between probabilistic processes, a possibility to combine probabilistically dependent processes $fail$ and $comm_1$ in a non-deterministic choice arises. Moreover, there is a non-zero probability with which deadlock may occur. It is obvious that this process does not satisfy our intuition about the behaviour of the parallel composition given above.

In order to overcome this difficulty, we propose in this paper a new variant of the extension of *prBPA* by parallel composition. Again, we want to keep the idea of the interleaving model but only for dynamic processes. For instance, let us consider the processes $X = a \boxplus_{\frac{1}{2}} b$ and $Y = c \boxplus_{\frac{1}{3}} d$. Since X may execute a with probability $\frac{1}{2}$ and Y may execute c with probability $\frac{1}{3}$, the probability of merging a and c in the parallel composition $X \parallel Y$ is exactly the product of the separate probabilities, so it is $\frac{1}{6}$. After the first action occurrence, for instance if a has been performed, an outcome among the actions c and d has not been known yet, so each of these two actions may be performed in accordance with the given probabilities. In Figure 2 the transition system of $X \parallel Y$ is shown, where $a \,|\, c = e$, $a \,|\, d = f$, $b \,|\, c = g$ and $b \,|\, d = h$. This version of combining probabilities and parallel composition with the interleaving reasoning is proposed in [9] where authors use bundle probabilistic transition systems.

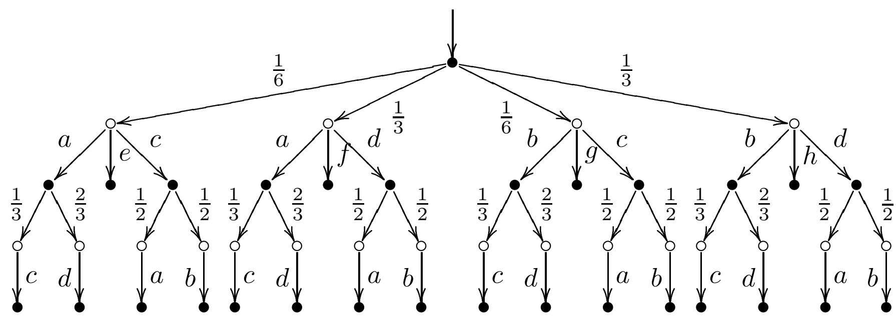

Fig. 2. An example of parallel composition of probabilistic processes.

3.1 Process Algebra

Below we give a definition (the signature and the axiom system) of the probabilistic version of *ACP*, called Algebra of communicating processes with probabilistic choice ACP_π^+, where + stands for the extra merge operator added to this algebra. We remark that this probabilistic version of *ACP* has a more complex axiom system than the algebra proposed in [1]. But an advantage here is a simple and intuitively clear operational semantics.

The signature of ACP_π^+ consists of the operators of *prBPA*, three binary operators: $\parallel$ (merge), $\lfloor\!\lfloor$ (left merge) and $|$ (communication merge), a unary operator ∂_H (encapsulation) where $H \subseteq A$ and a quaternary operator $]\![$ (merge with memory). The axioms of the new operators are given in Table 4 together with the conditional axioms given in Table 5 and 6.

The idea behind the merge with memory operator is to delay a merge of two concurrent processes (the first and the third arguments) as long as at least one of them has a possibility for nontrivial probabilistic choice (the axioms $PrMM2$ and $PrMM3$). If none of them has a possibility for a nontrivial probabilistic choice (the condition $x = x + x$ & $y = y + y$), then the processes may be merged. The other two auxiliary arguments (the second and fourth) help in the realisation of the interleaving model as it was mentioned in the previous example. Actually, they contain the two processes which have started parallel composition, because in further derivation these processes may get lost.

$x \parallel y$	$= (x,x)]\!\![(y,y)$		$PrMM1$
$(x \sqcup\!\!\!\!+_{\pi} x', z)]\!\![(y,w)$	$= (x,z)]\!\![(y,w) \sqcup\!\!\!\!+_{\pi} (x',z)]\!\![(y,w)$		$PrMM2$
$(x,z)]\!\![(y \sqcup\!\!\!\!+_{\pi} y', w)$	$= (x,z)]\!\![(y,w) \sqcup\!\!\!\!+_{\pi} (x,z)]\!\![(y',w)$		$PrMM3$
$a \mathbin{\|\!\!\lfloor} x$	$= a \cdot x$		$CM2$
$a \cdot x \mathbin{\|\!\!\lfloor} y$	$= a \cdot (x \parallel y)$		$CM3$
$(x+y) \mathbin{\|\!\!\lfloor} z$	$= x \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} z$		$CM4$
$(x \sqcup\!\!\!\!+_{\pi} y) \mathbin{\|\!\!\lfloor} z$	$= x \mathbin{\|\!\!\lfloor} z \sqcup\!\!\!\!+_{\pi} y \mathbin{\|\!\!\lfloor} z$		$PrCM1$
$a \mid b \cdot x$	$= (a \mid b) \cdot x$		$CM5$
$a \cdot x \mid b$	$= (a \mid b) \cdot x$		$CM6$
$a \cdot x \mid b \cdot y$	$= (a \mid b) \cdot (x \parallel y)$		$CM7$
$(x \sqcup\!\!\!\!+_{\pi} y) \mid z$	$= x \mid z \sqcup\!\!\!\!+_{\pi} y \mid z$		$PrCM6$
$x \mid (y \sqcup\!\!\!\!+_{\pi} z)$	$= x \mid y \sqcup\!\!\!\!+_{\pi} x \mid z$		$PrCM7$
$\partial_H(a)$	$= a$	if $a \notin H$	$D1$
$\partial_H(a)$	$= \delta$	if $a \in H$	$D2$
$\partial_H(x+y)$	$= \partial_H(x) + \partial_H(y)$		$D3$
$\partial_H(x \cdot y)$	$= \partial_H(x) \cdot \partial_H(y)$		$D4$
$\partial_H(x \sqcup\!\!\!\!+_{\pi} y)$	$= \partial_H(x) \sqcup\!\!\!\!+_{\pi} \partial_H(y)$		$PrD4$

Table 4. Additional axioms for ACP_{π}^{+}.

$x = x + x, y = y + y \Rightarrow (x,z)]\!\![(y,w) = x \mathbin{\|\!\!\lfloor} w + y \mathbin{\|\!\!\lfloor} z + x \mid y$

Table 5. Merge for Dynamic processes (DyM).

The axiom system contains three conditional axioms. All they have a condition of form: $p = p{+}p$. In the terms of equalities (in the theory) $ACP_{\pi}^{+} \vdash p = p{+}p$ holds for all terms which have as a basic term a term from $\mathcal{B}_{+}$. This condition guarantees that communication (in the case of DyPR) and merge (in the case of DyM) will not occur before all possibilities of applying axioms $PrCM6$, $PrCM7$ and $PrMM2$, $PrMM3$, respectively, have been exhausted. Moreover in the model, Lemma 22 shows that the property $p \underline{\leftrightarrow} p + p$ is fulfilled by all processes which cannot do probabilistic transitions to different equivalent classes.

Elimination of $\parallel$, $\mathbin{\|\!\!\lfloor}$, $\mid$, ∂_H and $]\!\![$ operators from closed ACP_{π}^{+} terms is guaranteed by the following theorem:

Theorem 15 *(Elimination theorem of ACP_{π}^{+}) Let p be a closed ACP_{π}^{+} term. Then there is a closed prBPA term q such that $ACP_{\pi}^{+} \vdash p = q$.* □

$$z = z + z \Rightarrow (x + y) \,|\, z = x \,|\, z + y \,|\, z$$
$$z = z + z \Rightarrow z \,|\, (x + y) = z \,|\, x + z \,|\, y$$

Table 6. Communication merge for Dynamic Processes (DyPR).

3.2 Structured Operational Semantics of ACP_π^+

In ACP_π^+, as in *prBPA*, we need to distinguish static from dynamic processes. Indeed, we obtain the term model of ACP_π^+ as an extension of the term model of *prBPA*, that is, by extension of the signature and the set of deduction rules of the term deduction system and the probability distribution function given in Section 2.1. We consider the signature: $\breve{\Sigma}_{ACP_\pi^+} = (\, A_\delta \cup \breve{A}_\delta\,, +, \cdot, \boxplus_\pi, \,\|\,, \,⌊\!⌊\,, \,|\,, \partial_H, \,]\![\,)$.

Analogously, we extend the sets of static and dynamic processes as follows:

Definition 16 *A set of static processes $\mathcal{SP}(ACP_\pi^+)$ in ACP_π^+ is the set of all closed terms over the signature of ACP_π^+, Σ_{prACP}.*

A set of dynamic processes $\mathcal{DP}(ACP_\pi^+)$ over the signature $\breve{\Sigma}_{ACP_\pi^+}$ is defined inductively as follows:

1. $\breve{A}_\delta \subseteq \mathcal{DP}(ACP_\pi^+)$;
2. $s, t \in \mathcal{DP}(ACP_\pi^+) \Rightarrow s + t, s \,|\, t, \partial_H(s) \in \mathcal{DP}(ACP_\pi^+)$;
3. $s \in \mathcal{DP}(ACP_\pi^+), t \in \mathcal{SP}(ACP_\pi^+) \Rightarrow s \cdot t, s ⌊\!⌊\, t \in \mathcal{DP}(ACP_\pi^+)$.

The operational semantics is defined by the deduction rules for the new operators in ACP_π^+ given in Table 7, where a, b, c range over A and $H \subseteq A$, and the probability distribution function (Definition 4 and Definition 17).

Definition 17 *The probability distribution function $\mu : \mathcal{PR}(prACP) \times \mathcal{PR}(prACP) \to [0, 1]$ is defined with the equalities given in Definition 4 and the following:*

$$\begin{aligned}
\mu(p \,\|\, q, x' ⌊\!⌊\, q + x'' ⌊\!⌊\, p + x' \,|\, x'') &= \mu(p, x')\mu(q, x''),\\
\mu(p ⌊\!⌊\, q, x' ⌊\!⌊\, q) &= \mu(p, x'),\\
\mu(p \,|\, q, x' \,|\, x'') &= \mu(p, x')\mu(q, x''),\\
\mu(\partial_H(p), \partial_H(x')) &= \mu(p, x'),\\
\mu((p, z) \,]\![\, (q, w), x' ⌊\!⌊\, w + x'' ⌊\!⌊\, z + x' \,|\, x'') &= \mu(p, x')\mu(q, x'').
\end{aligned}$$

3.3 Soundness and Completeness

Definition 18 *The probabilistic bisimulation in ACP_π^+ is defined in the same way as in prBPA.*

$$\frac{p \rightsquigarrow x, q \rightsquigarrow y}{p \parallel q \rightsquigarrow x \mathbin{\lfloor\!\lfloor} q + y \mathbin{\lfloor\!\lfloor} p + x \mid y} \qquad \frac{p \rightsquigarrow x, q \rightsquigarrow y}{(p,z) \mathbin{]\![} (q,w) \rightsquigarrow x \mathbin{\lfloor\!\lfloor} w + y \mathbin{\lfloor\!\lfloor} z + x \mid y}$$

$$\frac{p \rightsquigarrow x}{p \mathbin{\lfloor\!\lfloor} q \rightsquigarrow x \mathbin{\lfloor\!\lfloor} q} \qquad \frac{p \rightsquigarrow x, q \rightsquigarrow y}{p \mid q \rightsquigarrow x \mid y} \qquad \frac{p \rightsquigarrow x}{\partial_H(p) \rightsquigarrow \partial_H(x)}$$

$$\frac{x \xrightarrow{a} p}{x \mathbin{\lfloor\!\lfloor} y \xrightarrow{a} p \parallel y} \qquad \frac{x \xrightarrow{a} \surd}{x \mathbin{\lfloor\!\lfloor} y \xrightarrow{a} y} \qquad \frac{x \xrightarrow{a} p, y \xrightarrow{b} q, \gamma(a,b) = c}{x \mid y \xrightarrow{c} p \parallel q}$$

$$\frac{x \xrightarrow{a} p, y \xrightarrow{b} \surd, \gamma(a,b) = c}{x \mid y \xrightarrow{c} p} \qquad \frac{x \xrightarrow{a} \surd, y \xrightarrow{b} q, \gamma(a,b) = c}{x \mid y \xrightarrow{c} q} \qquad \frac{x \xrightarrow{a} \surd, y \xrightarrow{b} \surd, \gamma(a,b) = c}{x \mid y \xrightarrow{c} \surd}$$

$$\frac{x \xrightarrow{a} p, a \notin H}{\partial_H(x) \xrightarrow{a} \partial_H(p)} \qquad \frac{x \xrightarrow{a} \surd, a \notin H}{\partial_H(x) \xrightarrow{a} \surd}$$

Table 7. Operational semantics of ACP_π^+.

Theorem 19 *$\underline{\leftrightarrow}$ is a congruence relation on ACP_π^+.* □

Lemma 20 *If $p \in \mathcal{SP}(ACP_\pi^+)$ and $p \rightsquigarrow x$ for some closed term x over the signature $\breve{\Sigma}_{ACP_\pi^+}$, then $x \in \mathcal{DP}(ACP_\pi^+)$.* □

Lemma 21 *If $x \in \mathcal{DP}(ACP_\pi^+)$ then $x \underline{\leftrightarrow} x + x$.* □

Lemma 22 *Let p be a closed ACP_π^+ term such that $p \underline{\leftrightarrow} p + p$. Then if $p \rightsquigarrow x'$ and $p \rightsquigarrow x''$ for some $x', x'' \in \mathcal{DP}(ACP_\pi^+)$, then $x' \underline{\leftrightarrow} x''$.* □

Theorem 23 *(Soundness) Let p and q be closed ACP_π^+ terms.*
If $ACP_\pi^+ + DyPR + DyM \vdash p = q$ then $p \underline{\leftrightarrow} q$. □

In order to prove completeness we use the method given in [17], [8], [4]. By a trivial check of the conditions in Theorem 4.8 in [8] we can prove the following result:

Theorem 24 *(Operational conservative extension) The term deduction system determined by the signature and operational rules of ACP_π^+ is an operational conservative extension of the one for prBPA.* □

As we need to get an operational conservative extension up to the probabilistic bisimulation, we need to check if this relation is defined "in terms of predicate and relation symbols". Besides the fourth clause in Definition 7, the probabilistic bisimulation relates terms level by level, that is, transition by transition. Using the previous theorem for operational conservative extension we obtain that for each closed *prBPA* term s, its term-relation-predicate diagrams (in our terminology it is a transition system) in both *prBPA* and ACP_π^+ are the same. Because $\underline{\leftrightarrow}$ is defined in the same way for transitions in *prBPA* and ACP_π^+ and the term-transition diagrams of s and t ($s, t \in \mathcal{SP}$) are the same in both term deduction systems, we have that $prBPA \vdash s \underline{\leftrightarrow} t \Leftrightarrow ACP_\pi^+ \vdash s \underline{\leftrightarrow} t$.

Theorem 25 *(Completeness) Let p and q be closed ACP_π^+ terms. If $p \underline{\leftrightarrow} q$ then $ACP_\pi^+ + DyPR + DyM \vdash p = q$.* □

3.4 An Equivalent Axiomatization

In this section we give an axiom system that can be considered as equivalent one to ACP_π^+ in the sense that a equation of terms that do not contain $\mathbin{\|\!\!\lfloor}$, $|$ and $]\!\![$ operators holds in one theory if and only if it holds in the other theory. The main idea for proposing a new axiom system is to find an appropriate theory which does not have any extra operators. As it has been already mentioned, in order to obtain a complete axiom system of the term model determined by the deduction rules in Table 3 and Table 7 the merge with memory operator has been added to ACP_π^+.

We denote the new process algebra by ACP_π. The signature of ACP_π consists of the operators of *prBPA*, three binary operators: $\|$ (merge), $\mathbin{\|\!\!\lfloor}$ (left merge) and $|$ (communication merge) and an unary operator ∂_H (encapsulation) where $H \subseteq A$. The axioms of these operators are given in Table 8 together with the axioms $D1 - D4$ and $PrD4$ in Table 4 and the conditional axioms given in Table 6.

$$
\begin{aligned}
x \| y &= x \mathbin{\|\!\!\lfloor} y + y \mathbin{\|\!\!\lfloor} x + x \,|\, y \\
(x \sqcup\!\!\!+_\pi x') \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} w + (x \sqcup\!\!\!+_\pi x') \,|\, y &= (x \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} w + x \,|\, y) \sqcup\!\!\!+_\pi (x' \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} w + x' \,|\, y) \\
x \mathbin{\|\!\!\lfloor} z + (y \sqcup\!\!\!+_\pi y') \mathbin{\|\!\!\lfloor} w + x \,|\, (y \sqcup\!\!\!+_\pi y') &= (x \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} w + x \,|\, y) \sqcup\!\!\!+_\pi (x \mathbin{\|\!\!\lfloor} z + y' \mathbin{\|\!\!\lfloor} w + x \,|\, y') \\
\\
a \mathbin{\|\!\!\lfloor} x &= a \cdot x \\
a \cdot x \mathbin{\|\!\!\lfloor} y &= a \cdot (x \| y) \\
(x + y) \mathbin{\|\!\!\lfloor} z &= x \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} z \\
\\
a \,|\, b \cdot x &= (a \,|\, b) \cdot x \\
a \cdot x \,|\, b &= (a \,|\, b) \cdot x \\
a \cdot x \,|\, b \cdot y &= (a \,|\, b) \cdot (x \| y)
\end{aligned}
$$

Table 8. Additional axioms for ACP_π.

It can be noticed that the distribution laws $PrCM1$, $PrCM6$ and $PrCM7$ are not included in this axiom system. As a result of this, if we consider this theory as an extension of *prBPA* then the Elimination theorem does not hold anymore.

4 PAR Protocol

In this section we consider the PAR protocol (Positive Acknowledgement with Retransmission protocol) as it is described in [15]. We give a specification in ACP_π^+ of the constituent processes of the protocol and of the whole system. In order to do a performance analysis of the system non-determinism has to be resolved. Using only a partial ordering of the set of atomic actions and pre-abstraction we derive the recursive specification of the behaviour of the protocol which can be viewed as a Markov chain.

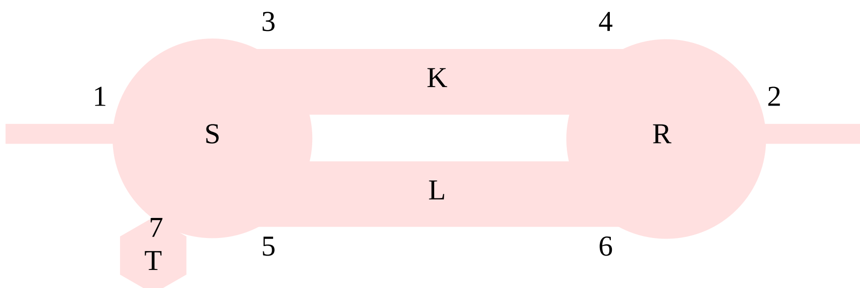

Fig. 3. Components of the protocol

The protocol is modeled as five processes, one sender process S, which is equipped with the timer T, one receiver R and two communication channels K and L, Figure 3. The sender sends a message to the receiver via a communication channel, starts the timer and after that it waits for an acknowledgement. After having received a message the receiver writes the message at the output port and sends an acknowledgement to the sender. A control bit is used in order to avoid multiple writing of a message at the output port. If a message or acknowledgement is sent via the (unreliable) channels K or L three situations can happen: 1) the message is transmitted correctly 2) the message is damaged in transit 3) the message gets lost in the channel. If the sender receives a damaged acknowledgement it sends a duplicate of the sent message. If the sent message or the acknowledgement has been lost in the channel no other action can be performed except the time-out communication action between T and S. When the sender gets the time-out message from the timer it sends a duplicate of the sent message. Unreliability of the channel K (in the similar way it is specified for the channel L) is specified by the probabilistic choice operator: correct transmission of a message with probability π, corruption of a message with probability σ and loss of a message with probability $1-\pi-\sigma$.

Let D be a finite set of data. The set of atomic actions A contains read, send and communication action and k and l actions which present loss of a message and acknowledgement, respectively. We use the standard read/send communication function given by $r_k(x) \mid s_k(x) = c_k(x)$ for communication port k and message x. The specifications of the five processes are given by the recursive equations in Figure 4.

Sender :

$$\begin{array}{ll} S &= S_0 \\ S_b &= \sum_{d\in D} r_1(d)\cdot S_b^d \qquad\qquad (b=0,1) \\ S_b^d &= s_3(db)\cdot s_7(st)\cdot WS_b^d \\ WS_b^d &= r_5(ack)\cdot S_{1-b} + (r_5(\perp)+r_7(to))\cdot S_b^d \quad (b=0,1,d\in D) \end{array}$$

Receiver :

$$\begin{array}{ll} R &= R_0 \\ R_b &= r_4(\perp)\cdot R_b + \sum_{d\in D} r_4(d\;1-b)\cdot SR_b + \sum_{d\in D} r_4(db)\cdot s_2(d)\cdot SR_{1-b} \quad (b=0,1) \\ SR_b &= s_6(ack)\cdot R_b \qquad\qquad (b=0,1) \end{array}$$

Timer :

$$\begin{array}{ll} T &= r_7(st)\cdot T^r \\ T^r &= r_7(st)\cdot T^r + s_7(to)\cdot T \end{array}$$

Channels :

$$\begin{array}{ll} K &= \sum_{d\in D, b\in\{0,1\}} r_3(db)\cdot (s_4(db) \boxplus_\pi s_4(\perp) \boxplus_\sigma k)\cdot K \\ L &= r_6(ack)\cdot (s_5(ack) \boxplus_\rho s_5(\perp) \boxplus_\eta l)\cdot L \end{array}$$

Fig. 4. Specification of the five components of the protocol.

As it is proposed in [15], in order to verify the protocol we use a unary priority operator Θ in the given specification, as well. In Table 9 and Table 10 we give the axioms of the priority operator and axioms of the auxiliary unless operator $\triangleleft$. We note that the Elimination theorem of the priority operator holds for closed terms, but it is not the case with the unless operator because distribution laws with probabilistic choice are missing. But in the theory we do not consider this as a problem, because in specification of processes this operator appears only as an auxiliary operator of the priority operator and conditional axiom $DyTH3$ guarantees that this operator does not appear between probabilistic processes.

On the set of atomic actions A the following partial ordering is defined:

1. $a < c_7(st)$, for each $a \in A \setminus \{c_7(st)\}$;
2. $c_7(to) < a$, for each $a \in A \setminus \{c_7(to)\}$.

The action $c_7(to)$ has a lower priority than every other atomic action because a premature time-out can disturb the functioning of the protocol. In order to express that immediately after sending a message the timer is started the action $c_7(st)$ has been given a higher priority than the other actions. (This assumption is very realistic because in such a system a communication between the sender and the timer is usually faster than a communication between other processes in the system.)

$\Theta(a)$	$= a$	$TH1$
$\Theta(x \cdot y)$	$= \Theta(x) \cdot \Theta(y)$	$TH2$
$\Theta(x \sqcup\!\sqcup_\pi y)$	$= \Theta(x) \sqcup\!\sqcup_\pi \Theta(y)$	$PrTH4$

$x = x + x, y = y + y \Rightarrow \Theta(x + y) = \Theta(x) \triangleleft y + \Theta(y) \triangleleft x$	$DyTH3$

Table 9. Axioms for the priority operator.

$a \triangleleft b$	$= a$	if $\neg(a < b)$	$P1$
$a \triangleleft b$	$= \delta$	if $a < b$	$P2$
$x \triangleleft (y \cdot z)$	$= x \triangleleft y$		$P3$
$x \triangleleft (y + z)$	$= (x \triangleleft y) \triangleleft z$		$P4$
$x \cdot y \triangleleft z$	$= (x \triangleleft z) \cdot y$		$P5$
$(x + y) \triangleleft z$	$= (x \triangleleft z) + (y \triangleleft z)$		$P6$

Table 10. Axioms for the unless operator.

The behaviour of the protocol is obtained by composition of the five processes:

$$PAR = t_I \circ \Theta \circ \partial_H(S \parallel T \parallel K \parallel L \parallel R),$$

where
$H = \{r_i(x), s_i(x) | i \in \{3,4,5,6,7\}, x \in (D \times \{0,1\}) \cup \{ack, \perp, st, to\}\}$
is the set of encapsulated atomic actions and t_I is the pre-abstraction operator ([2]), that renames all internal action from the set
$I = \{c_i(x) | i \in \{3,4,5,6,7\}, x \in (D \times \{0,1\}) \cup \{ack, \perp, st, to\}\} \cup \{k, l\}$
into t.

Shortly, we will describe how non-determinism is resolved in the derivation of the recursive specification given below. First of all, non-determinism which occurs as a result of conditional axiom DyM is resolved by using the encapsulation operator. Then, by merge of atomic actions of processes in the parallel composition, two sub-processes which contain non-determinism between processes are obtained. In the first case, we obtain the following sub-process: $c_7(st) \cdot Q + (c_4(db) \cdot X \sqcup\!\sqcup_\pi c_4(\perp) \cdot Y \sqcup\!\sqcup_\sigma k \cdot Y)$, for some processes Q, X and Y. Then, applying the distribution laws and the axioms of Θ operator, by taking into account the partial ordering of the set of atomic actions, we obtain that

$\Theta(c_7(st) \cdot Q + (c_4(db) \cdot X \sqcup\!\!\!\!+_{\pi} c_4(\bot) \cdot Y \sqcup\!\!\!\!+_{\sigma} k \cdot Y)) = c_7(st) \cdot \Theta(Q)$. (This situation corresponds to the state of the system in which in parallel the timer might be started or the message might be delivered to the receiver and as a result of the interleaving model non-determinism occurs. Under the assumption that the timer is started immediately after sending the message from the sender, it follows that this non-deterministic choice actually is deterministic.)

In the second situation we obtain non-deterministic choice between $c_7(to) \cdot R$ process and some other process P which may be in a form $a \cdot Z \sqcup\!\!\!\!+_{\rho} b \cdot U \sqcup\!\!\!\!+_{\eta} c \cdot V$ or $a \cdot Z$ for certain processes R, Z, U, V and atomic actions a, b and c. (There are more variants where non-determinism with $c_7(to) \cdot R$ occurs and we consider all of them in general.) Again, using the axioms of Θ operator and axioms of ACP_{π}^{+} and the partial ordering defined on the set A we obtain that $\Theta(c_7(to) \cdot R + P) = \Theta(P)$ and P does not have non-determinism as a top operator.

We can derive the following recursive specification for PAR:

$$
\begin{aligned}
P_0 &= \sum_{d \in D} r_1(d) \cdot P_1 & P_4 &= t \cdot t \cdot P_6 \\
P_1 &= t \cdot t \cdot (t \cdot P_2 \sqcup\!\!\!\!+_{\pi} t \cdot t \cdot P_1) & P_5 &= t \cdot t \cdot t \cdot P_6 \\
P_2 &= s_2(d) \cdot P_3 & P_6 &= t \cdot P_3 \sqcup\!\!\!\!+_{\pi} t \cdot P_5 \\
P_3 &= t \cdot (t \cdot P_0 \sqcup\!\!\!\!+_{\rho} t \cdot P_4 \sqcup\!\!\!\!+_{\eta} t \cdot P_5) & &
\end{aligned}
$$

The behaviour of the whole process is depicted in Figure 5. In order to obtain a clearer transition system we omit the labels that present probability 1 and we join a probabilistic and an action transition into one edge. If we abstract from the content of message sent from the environment this transition system can be considered as a labelled Markov chain. Further using Markov chain analysis various results for the behaviour of the protocol can be obtained. We can prove liveness of the protocol by showing that state P_0 in Figure 5 is a recurrent state. Moreover, because t_I operator does not reduce the number of internal actions we can compute the mean number of actions that are executed between reading of two successive data (between two read actions from environment) of the protocol by computing the mean recurrence time of state P_0. For example, for $\pi = 0.95$, $\rho = 0.92$ and $\eta = 0.07$ this mean number of atomic actions is 7.71.

5 Conclusions and Future Work

The objective of this paper is to introduce a probabilistic version of ACP where non-determinism and probability are combined. The presented probabilistic process algebra ACP_{π}^{+} improves the variant of probabilistic process algebra proposed in [1]. In order to get a more effective axiom system we have proposed a new variant of an extension of *prBPA* with parallel composition. Following the idea of ACP-like process algebras for the interleaving model we have given the axiom system where only parallel dynamic processes are merged. In order to realise this concept we have added an extra quaternary operator, $]\!|\![$ called merge with memory.

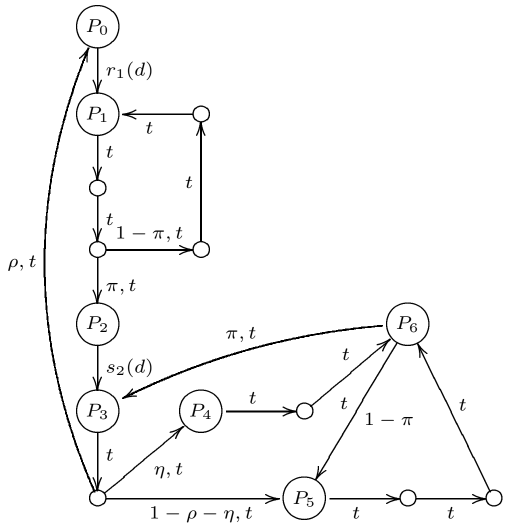

Fig. 5. The behaviour of the whole system.

The operational semantics of ACP_π^+ is based on the alternating model and it has been defined by a term deduction system of which the signature contains an extended set of constants (each atomic action has a dynamic counterpart) and of which the deduction rules include two transition types: probabilistic and action transition. Instead of labelled probabilistic transitions we have defined the probability distribution function which gives a probability with which one probabilistic transition may occur. In the construction of the term models we have used probabilistic bisimulation and we have shown soundness and completeness of the term model with respect to the proposed axiom systems.

Dealing with the PAR protocol with unreliable channels we have investigated the applicability of ACP_π^+. We have given a specification in ACP_π^+ of the constituent processes of the protocol and of the whole system. In order to do a performance analysis non-determinism has to be resolved. Using in addition only the priority operator and the pre-abstraction operator we have obtained a recursive specification of the behaviour of the protocol that can be viewed as a Markov chain. Our results indicate that for more complex protocols where pre-abstraction does not help, in order to verify some properties of a system, non-determinism can be resolved by standard methods used in ACP, for instance abstraction, applied only for sub-processes where non-determinism occurs. In that sense one of the directions in our further research is to investigate possibilities of combining abstraction and probability.

We mention as a possible option for future work the integration of a timed and probabilistic version of ACP.

Acknowledgments. I would like to thank Jos Baeten, Kees Middelburg and Michel Reniers for help during the work on this paper. I also thank Chris Verhoef and Pedro D'Argenio for fruitful discussion.

References

1. S. Andova, *Process algebra with interleaving probabilistic parallel composition*, Eindhoven University of Technology, CSR 99-xx, 1999.
2. J.C.M. Baeten, J. A. Bergstra, *Global renaming operators in concrete process algebra*, Information and Computation 78, pp. 205-245, 1988.
3. J.C.M. Baeten, W. P. Weijland, *Process algebra*, Cambridge University Press, 1990.
4. J.C.M. Baeten, C. Verhoef, *Concrete process algebra*, Handbook of Logic in Computer Science, volume 4: "Semantic Modelling", Oxford University Press, 1995.
5. J.C.M. Baeten, J.A. Bergstra, *Process algebra with partial choice*, Proc. CONCUR '94, Uppsala, B. Jonsson & J. Parrow, eds., LNCS 836, Springer Verlag, pp. 465-480, 1994.
6. J.C.M. Baeten, J.A. Bergstra, S.A. Smolka, *Axiomatizing probabilistic processes: ACP with generative probabilities*, Information and Computation 121(2), pp. 234-255, Sep. 1995.
7. J.A. Bergstra, J.W. Klop, *Process algebra for synchronous communication*, Information and Control 60, pp. 109-137, 1984.
8. P.R. D'Argenio, C. Verhoef, *A general conservative extension theorem in process algebra with inequalities*, Theoretical Computer Science 177, pp. 351-380, 1997.
9. P.R. D'Argenio, H. Hermanns, J.-P. Katoen *On generative parallel composition*, Preliminary Proc. of PROBMIV'98, Indianopolis, USA, C. Baier & M. Huth & M Kwiatkowska & M. Ryan ed., pp. 105-121, 1998.
10. A. Giacalone, C.-C. Jou, S. A. Smolka, *Algebraic reasoning for probabilistic concurrent systems*, Proc. Working Conference on Programming Concepts and Methods, IFIP TC 2, Sea of Galilee, Israel, M. Broy & C.B. Jones ed., pp. 443-458, 1990.
11. R. J. van Glabbeek, S. A. Smolka, B. Steffen, C. M. N. Tofts, *Reactive, generative and stratified models of probabilistic processes*, Proc. of 5th Annual IEEE Symp. on Logic in Computer Science, Philadelphia, PA, pp. 130-141, 1990.
12. H. Hansson, *Time and probability in formal design of distributed systems*, Ph.D. thesis, DoCS 91/27, University of Uppsala, 1991.
13. C.-C. Jou, S. A. Smolka *Equivalences, congruences and complete axiomatizations for probabilistic processes*, Proc. CONCUR '90, LNCS 458, Springer Verlag, Berlin, pp. 367-383, 1990.
14. K.G.Larsen, A.Skou, *Bisimulation through probabilistic testing*, Proc. of 16th ACM Symp. on Principles of Programming Languages, Austin, TX, 1989.
15. F. W. Vaandrager, *Two simple protocols*, In: *Applications of Process algebra*, Cambridge University Press, J.C.M. Baeten ed., pp. 23-44, 1990.
16. M.Y. Vardi, *Automatic verification of probabilistic concurrent finite state programs*, Proc. of 26th Symp. on Foundations of Com. Sc., IEEE Comp. Soc. Press, pp. 327-338, 1985.
17. C. Verhoef, *A general conservative extension theorem in process algebra*, Proc. of PROCOMET'94, IFIP 2 Working Conference, San Miniato, E.-R. Olderog ed., pp. 149-168, 1994.