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Periodic musical sequences and Lyndon words

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Abstract The study of periodic musical structures leads to the computation of sequences satisfying some specific property in such a way that two solutions which are cyclic shifts of one another are considered the same. There exists a powerful technique to do so developed in combinatorics on words under the name of Lyndon words. We illustrate this by two examples taken from African traditional music.

Keywords Periodic sequences, Lyndon words

1 Introduction

A formal system is a system of symbols together with rules to combine them into sequences of symbols which are considered as meaningful. One of the first assumptions underlying this approach is that one can recognize that two sequences of symbols are identical (for instance, among the three sequences *abc*, *abc*, *cba*, the first two are identical). A very particular type of sequence of symbols is the one obtained by combining several copies of the same sequence in such a way that this sequence is repeated endlessly, as in *abcabcabc*... Such a sequence is called "periodic".

Periodic structures are a fundamental feature of music. This fact is illustrated in classical music by well-known forms such as the chaconne. But this features appears to be much more important in traditional music, for instance in Central Africa, where most of the repertoires are based on cyclic forms.

From a musical point of view, one can generally consider that two periodic sequences which only differ by a finite number of elements at the beginning are basically the same periodic sequence. In fact, you cannot distinguish them if you do not hear their first notes. It is possible to formalize this idea by defining an equivalence relation on periodic sequences called "conjugacy relation".

The study of periodic musical phenomena may sometimes lead to the computation of periodic structures sharing some specific property. In this case, the problem is to compute solutions which are not conjugate one with the other. There is a concept in combinatorics on words, named Lyndon words, which can help in selecting one representative among conjugacy classes. Roughly speaking, a Lyndon word is defined as a finite word which is minimal for the lexicographic order in its conjugacy class. The properties of these words have been studied in details [10], and there exists an efficient linear algorithm to compute them [8].

In this article, we show how Lyndon words can be used in the study of periodic musical structures, and we illustrate this by two examples taken from African traditional music.

2 Periodic sequences

A *periodic* sequence is a function u from **N** to a given set A called the alphabet, such that there exists an integer m satisfying

$$u(n + m) = u(n)$$

for any integer n. We shall say in this case that u is m*-periodic*, and we define the *period* of u as the least integer m such that u is m-periodic. The set of periodic sequences over A is denoted by Per_A, and the set of periodic sequences with period m is denoted by $Per_A(m)$.

A *finite word* over the alphabet A is a finite sequence of elements of A. The set of all finite words over A is denoted by A^*, and we denote by $|w|$ the length of a finite word w, by $|w|_a$ the number of symbols equal to a in w, and by ε the empty word.

The cyclic shifts of a finite word are defined by introducing the permutation δ of A^* such that

$$\delta(ax) = xa, \quad a \in A, x \in A^*, \qquad \delta(\varepsilon) = \varepsilon.$$

The *cyclic shifts* of w are the words of the form $\delta^k(w)$ for any integer k. Finite words which are cyclic shifts of one another are called *conjugate*, and the conjugacy relation is an equivalence relation on A^*.

One can relate periodic sequences with finite words, since periodic sequences with period m are entirely defined by their finite prefix of length m. In fact, the letters occurring in a periodic sequence always repeat those occurring in its prefix. Thus there is a natural bijection between periodic sequences and finite words over A. For each finite word u of length m, we denote by u^ω the periodic sequence with period m, which has u as its finite prefix of length m. The function associating u with the periodic sequence u^ω maps bijectively A^* onto Per_A.

The notion of conjugacy may be extended to periodic sequences, since infinite sequences obtained from a given periodic one by deleting some of its first elements are also periodic, with the same period, and the only way to distinguish them from the original one is to look at a finite number of elements at the beginning of the sequence. The translated sequence Tu is defined by

$$Tu(n) = u(n + 1)$$

for any integer n. We shall say that two periodic sequences are *conjugate* iff one is the translated of the other, and this defines an equivalence relation on periodic sequences. The periodic sequences equivalent to a given one u are the translated $T^k u$ for any integer k.

There is an obvious relation between translated m-periodic sequences, and the cyclic shifts of their prefix of length m, which can be written for every u^ω in Per_A and for any integer k

$$T^k(u^\omega) = \delta^k(u)^\omega.$$

3 Lyndon words

When one wants to compute periodic musical structures satisfying some specific property, two conjugate sequences are considered as the same solution. It is thus necessary to limit the computation to only one representative for each conjugacy class. The natural way to do so is to use Lyndon words, as we shall see in this section.

Let us denote by the same symbol ~ the conjugacy relations on both Per_A and A^* (this convention is justified by the fact that $u^\omega \sim v^\omega$ in Per_A if and only if $u \sim v$ in A^*, as we have seen in the previous section).

For any subset R of A^*, we define a *cross-section* of R for the conjugacy relation as a set T satisfying the following conditions
(1) two elements of T cannot belong to the same conjugacy class,
(2) any element of R is conjugate to at least one element of T,
(3) any element of T is conjugate to at least one element of R.
A similar definition is given for any subset R of P_A.

A *Lyndon word* is a finite word which is minimal for the lexicographic order in its conjugacy class of A^*, and which is not a power of another word (for instance $ababab = (ab)^3$ is a power of ab). For instance, *aabab* is the Lyndon word of the conjugacy class which contains the words *aabab, ababa, babaa, abaab* and *baaba*. The minimal element being unique, the set of Lyndon words is a cross-section for the conjugacy relation on finite words. This can be extended to periodic sequences, by taking w^ω for all Lyndon words *w*. The resulting set of periodic sequences is a cross-section for the conjugacy relation on Per_A.

From a practical point of view, the computation of Lyndon words may be quite efficient. Jean-Pierre Duval has given an algorithm which gives the list in order of all Lyndon words of length less than an integer *n* [8]. The simplicity and efficiency of this algorithm are incredible. In Fig. 1, letters *a* and *M* are respectively the first and last letters of the alphabet, and for every letter *x* excepted *M*, *s*(*x*) is the letter that follows *x* in the alphabet. We denote by *w*[1...*n*] an array of letters of dimension *n*.

```
1    w[1] ← a
2    i ← 1
3    repeat
4        for j = 1 to n–i
5            do w[i+j] ← w[j]
6        /* at this point, w[1...i] is a Lyndon word */
7        i ← n
8        while i > 0 and w[i] = M
9            do i ← i–1
10       if i > 0 then w[i] ← s(w[i])
11   until i = 0
```

Fig. 1. Duval's algorithm for the computation of Lyndon words of length less than n

For *n* = 4, we give the successive values of *i*, the corresponding values of array *w* in line 6 of the algorithm, and the associated Lyndon words *w*[1...*i*] which appear in lexicographic order.

i	*w*	Lyndon words length ≤ 4
i = 1	*aaaa*	*a*
i = 4	*aaab*	*aaab*
i = 3	*aaba*	*aab*
i = 4	*aabb*	*aabb*
i = 2	*abab*	*ab*
i = 3	*abba*	*abb*
i = 4	*abbb*	*abbb*
i = 1	*bbbb*	*b*

Lyndon words have already been used in studies on circular musical structures, even when their use was not explicit. It is the case in the American theory of *pitch class sets*. A "normal order" for any pc set, e.g. any ordered set of pitches from the chromatic scale between 0 and 11 where 0 represents the note C and 11 the note B, is defined by Allen Forte as one of its cyclic shifts which has

> "the least difference determined by subtracting the first integer from the last" ([9] page 4).

But since two cyclic shifts may have the same value for the difference, Forte adds a "Requirement 2" when the previous condition gives many solutions

> "If the least difference of first and last integers is the same for any two permutations, select the permutation with the least difference between first and second integers. If this is the same, select the permutation with the least difference between the first and third integers, and so on" ([9] page 4).

We observe that this second condition corresponds to the definition of a Lyndon word among the cyclic shifts of the sequence of intervals. But in fact, it is not taken into account when the first condition gives a unique permutation. Thus it follows that Forte's normal order does not always correspond to the Lyndon word. For instance, the pc set denoted as 4-10 in Forte's table has the normal order [0, 2, 3, 5], corresponding to the notes C, D, D#, F. The sequence of intervals is 2 1 2 7 (the last interval is from 5 to 12, since 0 and 12 represent the same pc corresponding to the note C). The associated Lyndon word is 1 2 7 2, but it was not chosen as a normal order in this case, because the first condition gave 2 1 2 7 as a unique solution.

The general idea developed in the present article for the study of periodic musical structures is that when one wants to compute a cross-section of a given set R of periodic sequences over the alphabet A with regard to the conjugacy relation, one first computes the Lyndon words of A^*, and then one check whether their periodic counterparts in Per_A are conjugate to sequences of R or not.

4 Length decreasing functions

Sometimes one can improve the process by introducing a length decreasing function associated with an auxiliary alphabet B. We shall say that a function f from A^* to B^* is *length decreasing* if $f(x)$ is a word of B^* which is shorter than the corresponding word x of A^*

$$|f(x)| < |x|.$$

The idea is to replace the computation of Lyndon words of A^* by the computation of Lyndon words of B^*. This is possible when the function f is "compatible" with the conjugacy relation, in a sense that we shall make more precise. When f is also length decreasing, the computation becomes more efficient since $f(x)$ is shorter than the corresponding word x of A^*.

A mapping f from a subset R of A^* to B^* is said to be *compatible* with the conjugacy relation iff for any $x, y \in R$, $f(x) \sim f(y)$ is equivalent to $x \sim y$. In this case, Proposition 4.1 proves that in order to get a cross-section of R, it suffices to compute the Lyndon words w of B^*, and for those which are equivalent to words of $f(R)$, to choose a word in $f^{-1}(w)$.

Proposition 4.1 *If f is a mapping from a subset R of A^* to B^* which is compatible with the conjugacy relation, then it suffices to choose a word in $f^{-1}(w)$ for any Lyndon word w of B^* equivalent to a word of $f(R)$, to get a cross-section of R for the conjugacy relation.*

Proof: We first show that two words x and x' obtained in this way cannot belong to the same conjugacy class of R. If it were so for two words $x \in f^{-1}(w)$ and $x' \in f^{-1}(w')$, then $x \sim x'$ would imply $f(x) \sim f(x')$ according to the compatibility hypothesis, thus $w \sim w'$ and since w, w' are both Lyndon words, $w = w'$.

We now show that every class of R is represented once. In fact, for the class c_y of any element y of R, let w be the Lyndon word equivalent to $f(y)$ in B^*, and x the chosen word in $f^{-1}(w)$. One has $f(x) \sim f(y)$, which implies by hypothesis $x \sim y$, so that x is a representative for the class c_y.

A similar proposition may be stated for periodic sequences, the notion of "compatible mapping" being extended in a natural way to the conjugacy relation on Per_A and Per_B.

Proposition 4.2 *If f is a mapping from a subset R of Per_A to Per_B which is compatible with the conjugacy relation, then it suffices to choose a periodic sequence in $f^{-1}(w^\omega)$ for any Lyndon word w of B^* such that w^ω is equivalent to an element of $f(R)$, to get a cross-section of R for the conjugacy relation.*

When the mapping f is length decreasing, Proposition 4.1 shows that it is possible to replace Lyndon words of A^* by shorter Lyndon words over the alphabet B, the computation of which is made faster because of their smaller length. In the case of periodic sequences, we shall say that a mapping is length decreasing if the period of $f(w)$ is strictly less than the period of w. Then Proposition 4.2 adapts the same idea to the case of periodic sequences.

It seems that this method involving an *ad hoc* length decreasing mapping f is a general technique, which applies to different musical situations. We have encountered two examples of this type, in music formalization researches made on African traditional repertoires. The first one deals with harp melodic canons played by Nzakara people from Central African Republic. The second one deals with asymmetric rhythmic patterns which can be found in many cultures in Central Africa.

5 Two applications dealing with African traditional music

5.1 Computation of Nzakara melodic canons

The first musical example involving a mapping which is compatible with the conjugacy relation is related to ethnomusicological researches I have made on harp music from Nzakara people of Central African Republic. Each piece of poetry sung with the accompaniment of the harp relies on a short harp formula which is repeated endlessly as an *ostinato*. The traditional repertoire contains many such formula, which were strongly related to the political organization of the former Nzakara kingdom. Some of these formula have a quite regular structure, the strings being plucked by pairs, one with another. The formula can thus be considered as the superimposition of two melodic lines, one made with the upper note and the other one made with the lower note of each pair. Fig. 2 shows an example of such a formula (time is on the horizontal axis, and points indicate which strings are plucked). As one can see, the upper voice is reproduced, with just a few exceptions, in the lower voice, with a delay as indicated by the two broken lines. This formula has the structure of a two-voice melodic "canon" (see [2, 5, 7] for more details, and [3, 4] for audio samples of these harp canons).

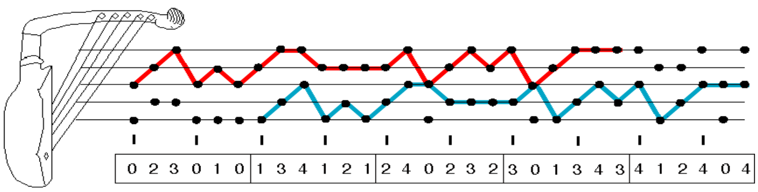

Fig. 2. A canon formula from Nzakara harp music

In order to describe more precisely the structure of these Nzakara harp canons, let E be the set of the five strings of the harp denoted in ascending order $E = \{c_1, c_2, c_3, c_4, c_5\}$. We consider as an alphabet a subset of the Cartesian product $E \times E$, restricted to the five combinations of strings plucked simultaneously, actually played by Nzakara musicians, as shown in Fig. 2. We denote these combinations as integers $0 = (c_1, c_3)$, $1 = (c_1, c_4)$, $2 = (c_2, c_4)$, $3 = (c_2, c_5)$ and $4 = (c_3, c_5)$, so that the alphabet is equal to the set $\{0, 1, 2, 3, 4\}$.

The canon of Fig. 2 can be factorized into five words $w = v_0v_1v_2v_3v_4$, each of them being obtained from the previous one by adding the same value to its elements, until we reach the initial word again.

v_0 = 0 2 3 0 1 0,
v_1 = 1 3 4 1 2 1,
v_2 = 2 4 0 2 3 2,
v_3 = 3 0 1 3 4 3,
v_4 = 4 1 2 4 0 4.

For instance, 1 added to each element of v_0 = 0 2 3 0 1 0 gives v_1 = 1 3 4 1 2 1, and so on. This appears to be a general construction in the Nzakara repertoire, since every harp canon is based on the same principle which consists in translating a given word several times, by adding the same value to its elements.

In this section, we shall consider a finite Abelian group G, and we study periodic sequences over G taken as an alphabet. We define the *difference sequence Du* of u by

$$Du(n) = u(n+1) - u(n)$$

for any integer n. In the case of Nzakara harp canons, the alphabet is identified with the finite group $G = \mathbf{Z}/5\mathbf{Z}$. Let us consider the formula given Fig. 2 as the finite prefix of length $m = 30$ of a periodic sequence u with period m. The preceding construction may be expressed by the fact that the period of Du divides the period of u. Actually, one can verify that Du has period $r = 6$. The following proposition is taken from [7]. It has a converse part which is restricted to a very special case, and the interesting fact is that Nzakara harp formulas precisely fall into this case.

Proposition 5.1 *If two periodic sequences u and v are conjugate, then Du and Dv are conjugate. Conversely, if Du and Dv are conjugate, and the period of Dv divides the period of v in such a way that the quotient of the two periods is equal to card(G), then u and v are conjugate.*

In the Nzakara case, the quotient of the periods of u and Du is always equal to 5, which is the cardinal of the group $G = \mathbf{Z}/5\mathbf{Z}$. Let R be the subset of Per_G containing periodic sequences such that their period is equal to the product of card(G) and the period of their difference sequence. We define a mapping f as the restriction of the operator D to the set R. Proposition 5.1 may be rewritten as

Proposition 5.2 *The mapping f from the subset R of Per_G to Per_G, associating to each periodic sequence its difference sequence, is a length decreasing mapping compatible with the conjugacy relation.*

Thus we want to compute a cross-section of the set $R_m = R \cap Per_G(m)$ of sequences u with period $m = \mathrm{card}(G) \times r$, where r is the period of Du. Then $f(R_m)$ is included in $Per_G(r)$, and $f(R_m) = S_r$, where S_r is the subset of sequences v of $Per_G(r)$ satisfying the following condition

(5.1) the sum of the elements occurring in the prefix of length r of v is equal to an element of G with its order equal to card(G)

(see [7] for details). Note that if this condition is true, then it is true for any conjugate sequence. In the Nzakara case, the sum can be any non-zero element of G, since card(G) = 5 is prime.

Proposition 5.2 shows that for each Lyndon word of length r such that the corresponding periodic sequence v satisfies condition (5.1), we can choose a word u in $f^{-1}(v)$. Then the set of corresponding periodic sequences with period m is a cross-section of R_m for the conjugacy relation.

Thus Proposition 5.2 gives an efficient algorithm to compute a cross-section of R_m. In fact, words in $f(R_m) = S_r$ have a much shorter period than those in R_m. For instance, in the example above, $m = 30$ and $r = 6$, so that the computation of Lyndon words of length 6 is much faster than the computation of those of length 30.

5.2 The rhythmic oddity property

Many musical traditions all over the world have asymmetric rhythmic patterns based on two different durations of two and three units (these patterns are sometimes called "aksak rhythm", such as the Turkish pattern 2 2 2 3). In central Africa, there is a particular type of asymmetric patterns, satisfying a property called "rhythmic oddity property" discovered by Simha Arom [1]. Let us consider the Aka pygmies pattern represented as a circle on Fig. 3. The property asserts that one cannot break the circle into two parts of equal length whatever the chosen breaking point. There is always one unit lacking on one side.

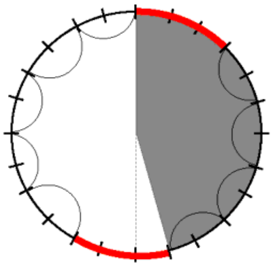

Fig. 3. Aka pygmies pattern 3 2 2 2 2 3 2 2 2 2 2
with no breaking point giving two parts of equal length

The asymmetry of the pattern is to some extent intrinsic, in the sense that there exists no breaking point giving two parts of equal length. Note that the oddity property requires that the circle is divided into an even number of units. We have described in [6] an algorithm for enumerating all the patterns satisfying the rhythmic oddity property.

In this section, the alphabet is $A = \{2, 3\}$. For any finite word u in A^*, we define the *height* of u, denoted by $h(u)$, as the sum of its symbols.

A finite word w of A^* satisfies the *rhythmic oddity property* if and only if
(1) $h(w)$ is even, and
(2) no cyclic shift of w can be factorized into words uv such that $h(u) = h(v)$.

Our construction for the computation of words satisfying the rhythmic oddity property (see [6]) relies on two functions a and b from $A^* \times A^*$ into itself defined by

$$a(u, v) = (3u, 3v), \qquad b(u, v) = (v, 2u).$$

Considering the free monoid B^* on the alphabet $B = \{a, b\}$, we identify the concatenation with the composition of functions. Thus any word α of B^* is identified with a function from $A^* \times A^*$ into itself. We proved the following characterization.

Proposition 5.1 *A word w satisfies the rhythmic oddity property if and only if there exists a unique word $\alpha \in B^*$ with $|\alpha|_b$ being odd, such that $w = uv$ or $w = vu$ with $(u, v) = \alpha(\varepsilon, \varepsilon)$.*

Let S, S' be subsets of A^* defined by

$$S = \{uv, \exists\alpha \text{ unique} \in B^*, |\alpha|_b \text{ odd}, (u,v) = \alpha(\varepsilon,\varepsilon)\},$$
$$S' = \{vu, \exists\alpha \text{ unique} \in B^*, |\alpha|_b \text{ odd}, (u,v) = \alpha(\varepsilon,\varepsilon)\}.$$

Proposition 5.1 implies that the set R of words of A^* satisfying the rhythmic oddity property is equal to

$$R = S \cup S'.$$

Note that words of S' are cyclic shifts of words of S.

We define a mapping f from S to B^* associating to each word $w = uv$ the corresponding word $\alpha = f(w)$ of B^*. This is possible since α is unique by definition of

S. It can be established that for any $w, w' \in S$, $f(w')$ is a cyclic shift of $f(w)$ if and only if w' is a cyclic shift of w. Furthermore, f is length decreasing because the number of 3 in w is equal to twice the number of a in α, and the number of 2 is equal to the number of b. Then one has the following result.

Proposition 5.2 *The mapping f from the subset S of A* to B*, associating to each word* $w = uv$ *of S the corresponding word* α *such that* $(u, v) = \alpha(\varepsilon, \varepsilon)$, *is a length decreasing mapping compatible with the conjugacy relation.*

Thus we want to compute a cross-section of the set $R = S \cup S'$ of words of A^* satisfying the rhythmic oddity property. One can notice that $\alpha(\varepsilon, \varepsilon) = \beta(\varepsilon, \varepsilon)$ implies $\alpha = \beta$, which means that as soon as α exists satisfying Proposition 5.1, then α is unique. This implies that f is surjective from S onto the subset of all words of B^* with an odd number of b. Furthermore, if $f(w) = f(w')$, then $(u, v) = (u', v')$ where $w = uv$ and $w' = u'v'$, thus $w = w'$. This means that f is injective, so that f is a bijection from S to the set of words of B^* with an odd number of b.

Proposition 5.2 shows that for each Lyndon word α of B^* with an odd number of b, we can take the word $f^{-1}(\alpha) = uv$ such that $(u, v) = \alpha(\varepsilon, \varepsilon)$ to get a cross-section of S for the conjugacy relation. But since every word of S' is a cyclic shift of a word of S, it follows that the resulting subset is also a cross-section of R for the conjugacy relation.

As in the previous musical example, f is length decreasing. It follows that we obtain an efficient algorithm to compute a cross-section of R. In fact, words in $f(S)$ have a shorter length than those in S. For instance, considering

$$w = 3333332, \qquad f(w) = aaab,$$

one has $|w| = 7$, whereas $|f(w)| = 4$.

In [6], we published a table giving the numbers of patterns satisfying the rhythmic oddity property, depending on the numbers n_2, n_3 of durations of two and three units respectively. We used a model of the problem as a constraint satisfaction problem designed by Charlotte Truchet, and Louis-Martin Rousseau implemented this model in an ILOG solver. The maximal value that we obtained was 4389, corresponding to $n_2 = 17$, $n_3 = 12$. Using Duval's algorithm implemented in Common Lisp, the computation of a bigger table up to $n_2 = 21$, $n_3 = 16$ took only about 15 minutes, and it gave the following new values.

	17	19	21
12	4389	7084	10966
14	14421	25300	42288
16	43263	82225	148005

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