NEW RESULTS ON REGULAR AND IRREGULAR SAMPLING BASED ON WIENER AMALGAMS

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1 INTRODUCTION

This paper should be seen as a companion to the article [F7] "WIENER AMALGAMS OVER EUCLIDEAN SPACES AND SOME OF THEIR APPLICATIONS" published in this issue. We are going to show how Wiener amalgam spaces (and not just ordinary amalgam spaces) can be used to derive results of interest in sampling theory. In this sense it represents a continuation of a series of papers [F8], [FG4–6] on the theoretical background of the irregular sampling problem for band-limited functions.

In view of the close connection between the two articles we will keep the same notations and have decided to avoid duplication in the list of references. Thus references which are *not* listed in the bibliography of the present note will be found in the previous note.

2 VARIATIONS ON THE SAMPLING THEOREM, L^p-ERROR ANALYSIS

One of the corner-stones of digital signal analysis is the so-called sampling theorem, according to which a band-limited signal can be completely reconstructed from the sampling values taken at any sufficient fine lattice. In fact, the critical rate, also known as the Nyquist rate, is inversely proportional to the bandwidth of the signal ((2·maximal frequency)$^{-1}$ in the usual engineering terminology). Usually this result is presented as a Hilbert space result. Using Plancherel's theorem and Poisson's formula it can be verified that the classical Shannon sampling theorem is indeed equivalent to the Fourier series expansion of periodic functions, a result which is the prototype of the concept of general orthogonal expansion in a Hilbert space (cf. [Br], [Pa2], [LO], and [Bu]; [Je] or [Ma1] for surveys).

We will describe the setting in a function space terminology and show how statements about the sampling series (and later results on irregular sampling) can be derived by arguments based on the use of Wiener amalgam spaces. Most of the results in this section are new and cover limiting cases or variants of results given in a series of papers on the irregular sampling problem for band-limited functions [FG4-6].

Definition 1. A tempered distribution $\sigma \in \mathcal{S}(I\!R^m)$ is called *band-limited* with *spectrum* Ω if the (generalized) Fourier transform $\hat{\sigma}$ vanishes on the complement of the closed, bounded set $\Omega \subseteq I\!R^m$. For integrable functions f the statement $\mathrm{spec}(f) \subseteq \Omega$ simply means that $\hat{f}(s) = 0$ for $s \notin \Omega$.

It is a standard result due to Paley and Wiener, and, in its most general version, to Schwartz, that band-limited tempered distributions are represented by analytic, hence continuous and differentiable (ordinary) functions. Therefore single function values are well defined for band-limited functions in L^p-spaces, for $1 \le p \le \infty$. However, we shall not base our arguments on this fact, but use the following result instead (this is a special case of Thm.5 of [F1] and holds for lca. groups).

Lemma 1. Let a compact subset Ω of $I\!R^m$ and $\alpha \ge 0$ be given. Then there exists some constant C_Ω (only depending on Ω and α) such that for all $p \ge 1$

$$\|f\,|W(C^0, L^p_w)\| \le C_\Omega \cdot \|f\|_{L^p_w} \tag{1}$$

for $f \in L^p_w(I\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$, and all weights with $L^1_\alpha * L^p_w \subseteq L^p_w$.

Proof. We choose an arbitrary $h \in \mathcal{S}(I\!R^m) \subseteq W(C^0, L^1_\alpha)$ satisfying $\hat{h}(\omega) \equiv 1$ on Ω. Then $f = h * f$, and by the convolution relations for amalgams

$$\|f\,|W(C^0, L^p_w)\| \le C_1 \|h\,|W(C^0, L^1_\alpha)\| \cdot \|f\,|W(L^1, L^p_w)\| \le C_\Omega \cdot \|f\|_{L^p_w}\ .$$

□

Although the following result is true (by almost the same argument) for lca. groups, we present it for simplicity in the setting of $I\!R^m$. The product of multi-indices such as $\boldsymbol{a}n$ is to be understood as $(a_1 n_1, \dots, a_m n_m)$.

Theorem 2 (*(Weighted L^p-version of the Classical Sampling Theorem)*. Given a compact set $\Omega \subseteq I\!R^m$ and a band-limited function $h \in L^1_s(I\!R)$ satisfying $\hat{h}(t) = 1$ on Ω there exists $\boldsymbol{c} = (c_1, \dots c_m),\ c_i > 0$ (depending only on h) such that for any $\boldsymbol{a} \le \boldsymbol{c}$ (coordinatewise) one has:

Any band-limited function $f \in L^p_w(I\!R^m)$ with $\mathrm{spec}(f) = \mathrm{supp}(\hat{f}) \subseteq \Omega$ can be reconstructed from the sampling values over the lattice $(\boldsymbol{a}n)_{n \in \boldsymbol{Z}^m}$ by means of the *cardinal series*

$$f = \sum_{n \in \boldsymbol{Z}^m} \boldsymbol{a}^{-1} f(\boldsymbol{a}n) T_{\boldsymbol{a}n} h \tag{2}$$

Unconditional convergence of the series takes place in the $W(C^0, L^p_w)$-norm, hence in L^p_w as well as uniformly over compact subsets of $I\!R^m$ for $1 \le p < \infty$.

Proof. We shall use the symbol III for the so-called 'shah-distribution' $\mathrm{III} = \mathrm{III}_1$, given by

$$\mathrm{III}_{\boldsymbol{a}} := \sum_{n \in \boldsymbol{Z}^m} \delta_{\boldsymbol{a}n}\quad .$$

It is clear that $\mathrm{III}_{\boldsymbol{a}} \in W(M, L^\infty)$ for each $\boldsymbol{a}$, actually $\boldsymbol{a} \cdot \mathrm{III}_{\boldsymbol{a}} := \prod_{j=1}^m a_j \cdot \mathrm{III}_{\boldsymbol{a}}$ is uniformly bounded in $W(M, L^\infty)$. Since f is continuous by Lemma 1, $f \cdot \mathrm{III}_{\boldsymbol{a}}$ is well defined as a discrete Radon measure, but the pointwise multiplier result for amalgams gives more: $f \cdot \boldsymbol{a}\mathrm{III}_{\boldsymbol{a}} \in W(C^0, L^p_w) \cdot W(M, L^\infty) \subseteq W(M, L^p_w)$, and even uniform bounded in $W(M, L^p_w)$ with respect to $\boldsymbol{a}$.

Interpretation of (2) in the distribution theoretic sense shows that we have to verify

$$(\boldsymbol{c}f \cdot \text{III}_{\boldsymbol{c}}) * h = f \ . \tag{3}$$

Given Poisson's formula in the form $(\boldsymbol{a}\text{III}_{\boldsymbol{a}})^\wedge = \text{III}_{\boldsymbol{b}}$ for $\boldsymbol{b} = \boldsymbol{a}^{-1}$ we may rewrite (applying the usual rules for the Fourier transform on $\mathcal{S}'(\mathbb{R}^m)$) that this conditions is equivalent to $(\hat{f} * \text{III}_{\boldsymbol{b}}) \cdot \hat{h} = \hat{f}$. Drawing a picture of the compactly supported function $\hat{f}$ and its β-periodic extension $\hat{f} * \text{III}_{\boldsymbol{b}}$ the reader will immediately verify that the given plateau-condition allows to find $\boldsymbol{c}$ such that $\boldsymbol{b} = \boldsymbol{c}^{-1}$ is large enough for the formula to hold (Ω has to fit into a rectangle of the form $[c_1^{-1}, \dots, c_m^{-1}]$).

In order to check convergence let us observe that Lemma 1 implies convergence of $\boldsymbol{a} \cdot \sum_{|n|\le k} f(\boldsymbol{a}n)\delta_{\boldsymbol{a}n}$ to $\boldsymbol{a}f \cdot \text{III}_{\boldsymbol{a}}$ for $k \to \infty$ in the norm of $W(M, L^p_w)$ for any $p < \infty$.
Applying $W(M, L^p_w) * W(C^0, L^1_s) \subseteq W(C^0, L^p_w)$ we derive that

$$\boldsymbol{a} \cdot \sum_{|n|\le k} f(\boldsymbol{a}n)\, \delta_{\boldsymbol{a}n} * h = \boldsymbol{a} \cdot \sum_{n=-k}^{k} f(\boldsymbol{a}n)\, T_{\boldsymbol{a}n} h$$

is convergent in $W(C^0, L^p_w)$, and therefore in L^p_w as well as locally uniform. □

Remark 1. Using invertible linear transformations of $\mathbb{R}^m$ the above results can be easily transformed into a result on more general lattices and no significant change in the arguments is required. That this is the most general approach to the sampling problem based on a Poisson-type formula can be seen from a recent result of Cordoba [Co].

We want to show next how amalgams can be used to describe the aliasing error in L^p-norms, i.e. the consequences of applying the above formula to $f \in L^p$ which are not band-limited. Obviously the part of $\hat{f}$ exceeding Ω will be responsible for the error, but a simple L^p-estimate of that part is certainly not sufficient. After all, the sampling points are just a set of measure zero in $\mathbb{R}^m$. A sufficient extra condition on $\hat{f}$ is integrability, which allows one to obtain *uniform* estimates for the aliasing error (cf. [BSS], Theorem 3.8). We show that $W(C^0, L^{p'})$-estimates can be obtained under a slightly stronger $W(L^p, L^1)$ assumption on $\hat{f}$.

Lemma 3. (*aliasing error estimate using amalgams*)

Assume $\hat{f} \in W(L^p, L^1)$ for some $p \in [1, 2]$. Then $f \in W(C^0, L^{p'})$, and the aliasing error can be estimated as follows. For any $\hat{h}(t) \equiv 1$ on Ω and $0 \le \hat{h}(t) \le 1$ on $\mathbb{R}^m$ there exists some $C_2 > 0$ such that

$$\|f - (\boldsymbol{a}f \cdot \text{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \le C_2 \,\|\hat{f} - \hat{f} \cdot \mathbf{1}_\Omega \,|W(L^p, L^1)\| \ . \tag{4}$$

In particular, the aliasing error tends to zero for $\boldsymbol{a} \to (0, \dots, 0)$.

Proof. The first statement is a simple consequence of the generalized HY-theorem, by which $\mathcal{F}$ maps $W(L^p, L^1) \subseteq W(\mathcal{F}L^{p'}, L^1)$ to $W(\mathcal{F}L^1, L^{p'}) \subseteq W(C^0, L^{p'})$ (cf. [F2], see [F3] for weighted versions). In order to estimate the aliasing error we split f into a good and a bad part by setting $f_\Omega := \mathcal{F}^{-1}(f \cdot \mathbf{1}_\Omega)$ and $f_r := f - f_\Omega$. Then of course $f_\Omega = (\boldsymbol{a}f_\Omega \text{III}_{\boldsymbol{a}}) * h$, and therefore the aliasing error can be estimated by

$$\begin{aligned}
&\|f - (\boldsymbol{a}f \cdot \text{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| = \|f_r - (\boldsymbol{a}f_r \text{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \le \\
&\quad \le \ \|f_r \,|W(C^0, L^{p'})\| + \|(\boldsymbol{a}f_r \text{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \le \\
&\quad \le \ \|f_r|W(C^0, L^{p'})\| + C \cdot \|\boldsymbol{a}\text{III}_{\boldsymbol{a}} \,|W(M, L^\infty)\| \,\|h \,|W(C^0, L^1)\| \cdot \|f_r \,|W(C^0, L^{p'})\| \ .
\end{aligned}$$

Since the family $\boldsymbol{a}\mathrm{III}_{\boldsymbol{a}}$ is uniformly bounded in $W(M, L^\infty)$ we obtain

$$\|f - (\boldsymbol{a} f \cdot \mathrm{III}_{\boldsymbol{a}}) * h \,|W(C^0, L^{p'})\| \le C_1 \cdot \|f_r \,|W(C^0, L^{p'})\| \le C_2 \cdot \|\hat{f} - \hat{f} \cdot \mathbf{1}_\Omega \,|W(L^p, L^1)\| \ .$$

From this it is clear that the aliasing error tends to zero as Ω increases to $I\!\!R^m$, and even the speed of convergence can be controlled by the decay of the norm of the high frequency tails, measured in the norm of $W(L^p, L^1)$. □

In Theorem 2 we have restricted ourselves to the use of band-limited functions $h \in L^1_s$ in the reconstruction process, because we wanted to have the result for the full range of values $p \ge 1$ and weights. In fact, the use of the traditional sinc-function, the inverse Fourier transform of the rectangular function, which is *not* in L^1, has to be excluded from the discussion for this reason. On the other hand, the sinc-function (or its multidimensional analog, obtained by pointwise products), belongs to L^p, for any $p > 1$. Thus there is some hope that estimates involving the sinc-function can be obtained in this case. It turns out, however, that the convolution estimate based on the fact that $\sum_{n \in \boldsymbol{Z}^m} f(\boldsymbol{a}n)\, \delta_{\boldsymbol{a}n} \in W(M, L^p_w)$, and therefore $\in W(C^0, L^r)$ for any $r > 1$, is too weak to ensure L^p_w (or even $W(C^0, L^p_w)$ convergence) of the sampling series, even for the trivial weight w. The problem even gets worse if one wants to study jitter errors, because then the usual argument for the L^2-case (it is based on orthogonal series expansions) breaks down too. We will show how the generalized HY-theorem can be used to establish appropriate estimates. We write $\mathbf{sinc}(\boldsymbol{x}) := \mathrm{sinc}(x_1) \cdots \mathrm{sinc}(x_m),\ \boldsymbol{x} \in I\!\!R^m$.

But first a very useful corollary to the generalized HY inequality.

Lemma 4. The *ideal low pass filter*, i.e. convolution by **sinc**, defines a bounded multiplier from $W(M, L^p)$ into $W(C^0, L^p)$. In particular, for any convergent sequence $(\mu_n)_{n \ge 1}$ in $W(M, L^p)$, with limit μ_0, the sequence $\mu_n * \mathbf{sinc}$ is convergent in $W(C^0, L^p)$.

Proof. We will apply the generalized HY inequality twice. First we observe that $\mu \in W(M, L^p) \subseteq W(\mathcal{F}L^\infty, L^p)$ implies that $\hat{\mu} \in W(\mathcal{F}L^p, L^\infty)$. Since $\mathrm{rect} = \mathcal{F}(\mathbf{sinc})$ is known to be a bounded pointwise multiplier for $\mathcal{F}L^p$ for $1 < p < \infty$ (cf. [Pe], Chap.7 or [St], Chap.4), hence $\hat{\mu} \cdot \mathrm{rect} \subseteq \mathrm{W}(\mathcal{F}\mathrm{L}^p, \mathrm{L}^1)$, thus $\mu * \mathbf{sinc} \in \mathcal{F}^{-1}(W(\mathcal{F}L^p, L^1)) \subseteq W(\mathcal{F}L^1, L^p) \subseteq W(C^0, L^p)$, and the required norm estimates hold as well. □

The following result is a partial extensions of Theorem 5 in [Go] to the *irregular* case (cf. Thm.14 below).

Corollary 5. Let $X = ((x_i)_{i \in I})$ be a relatively separated family in $I\!\!R^m$ and $\Omega \subseteq I\!\!R^m$ be compact.
Then

$$S_X f := \sum_{i \in I} f(x_i) T_{x_i} \mathbf{sinc}$$

is unconditionally convergent in $W(C^0, L^p)$, and there exists $C_2 = C(X, \Omega) > 0$ such that

$$\|S_X f \,|W(C^0, L^p)\| \le C_2 \cdot \|f\|_p \qquad (5)$$

for any $p \in (1, \infty)$ and any $f \in L^p(I\!\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$.

Proof. If X is relatively separated then $\delta_X := \sum_{i \in I} \delta_{x_i} \in W(M, L^\infty)$, and by Lemma 1

$$\sum_{i \in I} f(x_i) \delta_{x_i} = f \cdot \delta_X \in W(C^0, L^p) \cdot W(M, L^\infty) \subseteq W(M, L^p), \qquad (6)$$

for any band-limited $f \in L^p(I\!\!R^m)$ and the previous Lemma applies. □

In various situations, especially in the discussion of the irregular sampling problem with highly irregular sampling sets (which might have arbitrary high density at some places), the following modification is of interest.

Lemma 6. Let $\Psi = (\psi_i)_{i\in I}$ be a family of measurable functions which satisfy $0 \le \psi_i(x) \le 1$, $\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i)$ for all $i \in I$ and $\sum_{i\in I} \psi_i(x) \le C_\Psi < \infty$ for $x \in I\!R^m$. If $c_i := \|\psi_i\|_1$ then the discrete measure $\mu_\Psi := \sum_{i\in I} c_i \delta_{x_i}$ belongs to $W(M, L^\infty)$, and $\|\mu \,| W(M, L^\infty)\| \le C_\delta \cdot C_\Psi$ for some $C_\delta > 0$.

Proof. We shall make use of the duality $W(M, L^\infty) = W(C^0, L^1)'$. Thus we only have to obtain an estimate for $f \in \mathcal{K}(I\!R^m)$ (using the positivity of c_i)

$$|\mu_\Psi(f)| \le \sum_{i\in I} c_i |f(x_i)| = \| \sum_{i\in I} f(x_i)\psi_i \|_1 \tag{7}$$

Fixing $h \in \mathcal{K}(I\!R^m)$ with $h(x) \equiv 1$ on $B_\delta(0)$ we check that $\sum_{i\in I} f(x_i)\psi_i(x) \le \|T_x h \cdot f\|_\infty$, and consequently we complete the proof by

$$|\mu_\Psi(f)| \le \|T_x h \cdot f\|_1 = \|F_h\|_1 = \|f \,| W(C^0, L^1)\| \quad .$$

□

Corollary 7. For Ψ, δ as above and any compact set $\Omega \subseteq I\!R^m$ there exists $C = C(X, \delta, \Omega) > 0$ such that $S_X f := \sum_{i\in I} f(x_i)\, c_i \cdot T_{x_i}\mathbf{sinc}$ is unconditionally convergent in $W(C^0, L^p)$, and satisfies

$$\|S_X f \,| W(C^0, L^p)\| \le C \cdot \|f\|_p \tag{8}$$

for any $p \in (1, \infty)$ and any $f \in L^p(I\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$.

Proof. This corollary follows from Lemma 6 by means of Lemma 4.

In the discussion of the sampling theorem various kinds of error analysis are of interest. In some of the classical papers (cf. [Pa1,2]) uniform error estimates for L^2-data were considered as sufficient for practical purposes). As we have seen, one may expect $W(C^0, L^p)$ estimates in many cases. That these can be obtained has been shown for a variety of situations in [FG3]. Some limiting cases and situations mostly not covered in [FG3] are discussed in the sequel.

Theorem 8 (*Jitter error estimate for p-norms*). Let X be relatively separated , Ω bounded in $I\!R^m$ and $p, 1 < p < \infty$ be given. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any family $\tilde{X} = (\tilde{x}_i)_{i\in I}$ satisfying $|x_i - \tilde{x}_i| \le \delta$ the jitter error is small in the $W(C^0, L^p)$-sense, i.e.

$$\| \sum_{i\in I} (f(x_i) - f(\tilde{x}_i))\, T_{x_i}\mathbf{sinc} \,| W(C^0, L^p)\| \le \varepsilon \cdot \|f\|_p \tag{9}$$

Proof. Without loss of generality we may assume that $f \in \{f \in L^p(I\!R^m),\ \|f\|_p \le 1 \text{ and } \mathrm{spec}(f) \subseteq \Omega\}$, a set which is known to be equicontinuous in $L^p(I\!R^m)$. Next we observe that

$$|f(x_i) - f(\tilde{x}_i)| \le \mathrm{osc}_\delta f(x_i) \text{ for } i \in I \ . \tag{10}$$

By Theorem 3 of [F7] $\mathrm{osc}_\delta f \in W(C^0, L^p)$ (with arbitrary small norm for sufficiently small δ), thus it is possible to find to any given $\eta > 0$ some $\delta > 0$ such that

$$\| \sum_{i\in I} (f(x_i) - f(\tilde{x}_i))\delta_{x_i} \,| W(M, L^p_w)\| \le \eta \quad .$$

An application of Lemma 4 of [F7] concludes the proof. □

Let et us look at a delicate point in the above argument: We did *not* go to absolute values of the sinc-function in the proof. However, we used the fact that smaller (by the absolute value) complex coefficients in the series allow better norm estimates of the corresponding discrete measures in $W(M, L^p)$.

The above result has an important corollary.

Corollary 9. Let Ω be a bounded subset of $\mathbb{R}^m$ and $1 < p < \infty$ be given. Then for any sufficiently small $\boldsymbol{a} > 0$ there exists some $\delta = \delta(\boldsymbol{a}, \Omega, p)$ such that the operator $f \mapsto S_X f$ is invertible over $L^{p,\Omega}(\mathbb{R}^m)$ if $|x_n - \boldsymbol{a}n| \leq \delta$ for all $n \in \boldsymbol{Z}^m$. In particular, it is possible to recover f from the irregular sampling values $(f(x_n))_{n\in\boldsymbol{Z}^m}$ by applying the inverse operator S_X^{-1} to $S_X f$ (it can be obtained by Neumann's series).

Remark 2. The above result can also be described alternatively as an iterative algorithm (involving iterative application of $Id - S_X$), which is convergent in the sense of the $W(C^0, L^p)$-norm. Actually, we have just shown that the family $\{T_{x_i}\mathbf{sinc}\}$ is a *Banach frame* in the sense of [Gr] in the Banach space

$$L^{p,\Omega}(\mathbb{R}^m) := \{\, f \in L^p(\mathbb{R}^m),\ \mathrm{spec}(f) \subseteq \Omega \,\} \quad .$$

This means that the mapping $f \mapsto \langle f, T_{x_i}\mathbf{sinc}\rangle = f(x_i)$ is a mapping from $L^{p,\Omega}$ to the sequence space ℓ^p, that $(\sum_{i\in I} |f(x_i)|^p)^{1/p}$ defines an equivalent norm on $L^{p,\Omega}$, and that there is a bounded operator $U : \ell^p \to L^{p,\Omega}$, with $U \circ S_X = Id$ on $L^{p,\Omega}$.

The jitter error discussed above is the traditional one. Plotkin (cf. [PRS1,2]) also mentioned a *jitter error of second kind*, arising in the synthesis process, i.e. from the considerations of sums $\sum_{i\in I} \lambda_i T\tilde{x}_i h$ instead of sums $\sum_{i\in I} \lambda_i T_{x_i} h$. It is an open question whether it is possible to give a satisfactory estimate for this jitter error in the L^p-norm for the case of the **sinc**-function (another argument for the use of better decaying kernels), but we can at least prove the following (which at least guarantees uniform estimates).

Proposition 10 (*Jitter error of the second kind*). Let $h \in W(C^0, L^1)$, $p \in [1, \infty)$, and a relatively separated family X be given. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\sum_{i\in I} \lambda_i T\tilde{x}_i h - \sum_{i\in I} \lambda_i T_{x_i} h \,|W(C^0, L^p)\| \leq \varepsilon \cdot \|\sum_{i\in I} \lambda_i \delta_{x_i} \,|W(M, L^p)\| \qquad (11)$$

$$\text{whenever} \quad |x_i - \tilde{x}_i| \leq \delta \text{ for } i \in I \ .$$

In the limiting case of $h = \mathbf{sinc}$ we can obtain an estimate in $W(C^0, L^r)$ for any r, $p < r < \infty$, and in particular the uniform jitter error will be small.

Note. The typical application of this result involves $h \in L^1(\mathbb{R}^m)$, which is band-limited, e.g. a classical de la Vallée Poussin kernel (cf. Lemma4).

Proof. In this case we are forced to use absolute values involving h.

The pointwise estimate $|T\tilde{x}_i h - T_{x_i} h| \leq T_{x_i}(\mathrm{osc}_\delta h)$, gives via Lemma 4

$$\|\sum_{i\in I} \lambda_i T\tilde{x}_i h - \sum_{i\in I} \lambda_i T_{x_i} h \,|W(C^0, L^p)\| \leq C \cdot \|\sum_{i\in I} \lambda_i \delta_{x_i} \,|W(M, L^p)\| \cdot \|\mathrm{osc}_\delta h \,|W(C^0, L^1)\| \quad .$$

By Theorem 3 $\mathrm{osc}_\delta h$ will be small in $W(C^0, L^1)$ for sufficiently small δ. The choice $h = \mathbf{sinc}$ is not covered by this statement, but since the **sinc**-function belongs only to $L^s(\mathbb{R}^m)$ for any $s > 1$, hence to $W(C^0, L^s)$ by Lemma 4 (actually the norms tend to infinity for $s \to 1$). Therefore the following estimate is possible (but requires smaller and smaller δ for $\varepsilon > 0$, as s goes to 1):

$$\|\sum_{i\in I} \lambda_i T_{\tilde{x}_i}\mathbf{sinc} - \sum_{i\in I} \lambda_i T_{x_i}\mathbf{sinc} \,|W(C^0, L^r)\| \leq$$

$$\leq C \cdot \| \sum_{i \in I} \lambda_i \delta_{x_i} \, | W(M, L^p) \| \cdot \| \mathrm{osc}_\delta \mathbf{sinc} \, | W(C^0, L^s) \| \ \text{ for } 1/r = 1 - (1/p + 1/s) \quad .$$

Uniform convergence follows in each case, or by the choice $s = p'$. □

Corollary 11 (*uniform jitter estimate for sinc-functions*). For any compact Ω, relatively separated X, and p, with $1 < p < \infty$, the uniform total jitter is small if $\tilde{X}$ is close to X, i.e. for any $f \in L^p(I\!R^m)$ with $\mathrm{spec}(f) \subseteq \Omega$

$$\| \sum_{i \in I} f(x_i) T_{x_i} \mathbf{sinc} - f(\tilde{x}_i) T_{y_i} \mathbf{sinc} \|_\infty \leq \varepsilon \cdot \|f\|_p \tag{12}$$

as long as $|x_i - y_i| \leq \gamma$ and $|\tilde{x}_i - x_i| \leq \gamma$ for some $\gamma \leq \gamma_0 = \gamma_0(\varepsilon)$. If **sinc** is replaced above by some function $h \in W(C^0, L^1)$ the estimate even holds true in the sense of $W(C^0, L^p)$ and for $p \geq 1$.

There is, however, no hope to extend the jitter error estimate for the sinc-function to the case $p = \infty$, as can be shown by the following one-dimensional counterexample. It also gives an answer to the following problem: Given an irregular sampling family $X = (x_n)_{n=1}^\infty$, is it possible that the series $\sum_{n=1}^\infty T_{x_n} \,\mathrm{sinc}$ can have a zero, in other words, is it possible that for some choice of x_n one has $\sum_{n=1}^\infty \mathrm{sinc}(x_n - x_0) = 0$ for some x_0? This series may be considered as a low pass filtered version of the discrete measure $\delta_X := \sum_{n=1}^\infty \delta_{x_n}$ (using the rectangular filter), and is proposed as a correction term in the reconstruction procedure suggested by Plotkin (cf.[PRS1] and [PRS2]).

Proposition 12. Given any lattice constant $\boldsymbol{a} > 0$ and any $\delta > 0$ (the allowed jitter constant). Then for any $r \in I\!R$ (or $r = \infty$) it is possible to find a jitter-sequence j_n with $|j_n| \leq \delta$, such that

$$\sum_{n=1}^\infty \mathrm{sinc}(\boldsymbol{a} n - j_n) = r \quad ,$$

i.e. the 'jittered series' can take arbitrary values at zero, or may be divergent, even if the jitter error is uniformly small (it would be easy to use summation over $\boldsymbol{Z}$ as well).

Proof. The argument is based on the fact that the sinc-function, defined by $\mathrm{sinc}(x) := \sin(\pi x)/(\pi x)$ for $x \neq 0$, decays only like $1/x$, and that the harmonic series $\sum_{n=1}^\infty 1/n$ is known to be divergent. Without loss of generality we assume that $\boldsymbol{a} = (1, \ldots, 1)$, and that $r > 0$. By means of dilations and choosing some j_n's negative, our argument covers the remaining cases.

Setting $M := \{1 + 2k, \ k \in I\!N\}$ we plan to keep $j_n = 0$ for $n \notin M$, and to choose j_n in a suitable way for $n \in M$. We will assume for simplicity that $\delta \leq 1/8$.

The main estimate concerns an estimate for the derivative of the sinc-function over the intervals I_k, defined by $I_k := [2k + 3/8, 2k + 5/8]$, $k \geq 2$. One obtains (since $y > 4$, hence $\pi y > 6$ for $y \in I_k, k \geq 2$):

$$\mathrm{sinc}'(y) = \frac{\cos(\pi y)}{y} - \frac{\sin(\pi y)}{\pi y^2} \leq \frac{(\cos(\pi y) + 1/6)}{y} \quad . \tag{13}$$

Noting furthermore that $\cos(z) \leq -\frac{1}{2y} < -\frac{2}{3}$ if $|(2k+1)\pi - z| < \frac{\pi}{4}$ we conclude that

$$\mathrm{sinc}'(y) < \frac{-2/3 + 1/6}{y} = -\frac{1}{2y} \leq -\frac{1}{4k} \ \text{ for } y \in I_k \quad .$$

The mean value theorem implies that for any $u \in I\!R$

$$\mathrm{sinc}(u - j_n) = \mathrm{sinc}(u) - j_n \mathrm{sinc}'(\xi_n) \ \text{ for some } \xi_n \in (n - j_n, n) \quad , \tag{14}$$

hence

$$\mathrm{sinc}(2k+1-\delta) \geq \frac{\delta}{4k} \quad \text{for } k \in \mathbb{N} \quad . \tag{15}$$

Since the partial sums of the harmonic series are unbounded there exists $k_1 \in \mathbb{N}$ such that

$$s := \sum_{k=0}^{k_1-1} \mathrm{sinc}(2k+1-\delta) \leq r \ , \quad \text{but} \quad s + \mathrm{sinc}(2k_1+1-\delta) > r \ .$$

Using the fact that sinc is strictly decreasing and continuous on $[2k_1+1-\delta, 2k_1+1]$ we can find some δ_1 with $0 \leq \delta_1 \leq \delta$ such that $\mathrm{sinc}(2k_1+1-\delta_1) = r - s$. Setting $j_n = \delta$ for $n = 2k+1$, $2 \leq k \leq k_1$, $j_{k_1} := \delta_1$ and $j_n = 0$ otherwise, we find that $\sum_{n=1}^{\infty} \mathrm{sinc}(n - j_n) = r$, i.e. the jittered sampling series can take any prescribed value. □

As an immediate corollary we obtain the counterexample, showing that the low-pass filtered version of an almost regular lattice may have zeros.

Since $\mathrm{sinc}(0) = 1$ it is sufficient to choose j_n in such a way that $\sum_{n=1}^{\infty} \mathrm{sinc}(n - j_n) = -1$ (note that $\mathrm{sinc}(k) = 0$ for any $k \in \mathbb{Z}$, $k \neq 0$). Note that it would be possible to ask for *no* jitter error for any term with $n \leq n_0$ (some given number), and still having this disastrous phenomenon. It is also possible to show that there is a uniform error estimate if the sequence of jitter errors is itself square summable, i.e. if $\sum_{n=1}^{\infty} j_n^2 < \infty$. Results in this direction will be discussed elsewhere.

Remark 2. The above arguments also can be used to obtain an L^p-error estimate for the jitter error for reconstruction from sampling values together with derivates as described in [BDo]. One has only to observe that the derivative of a band-limited function in L^p is itself a band-limited function in L^p.

In a discussion of band-limited functions on $\mathbb{R}$ ([Cl],[CC]) the following question came up: Given a band-limited, square integrable function and a smooth deformation of the real line $\varphi : \mathbb{R} \to \mathbb{R}$, what can be said about $f \circ \varphi$? Since band-limited functions are differentiable it makes sense to discuss only differentiable functions φ which are strictly increasing, i.e. satisfying $0 < \varphi'(t) < \infty$ for all $t \in \mathbb{R}$. According to the conjecture stated in [Cl] the composition mapping $f \mapsto f \circ \varphi$ should always produce some non-band-limited functions from band-limited ones, except the special case where φ is an affine mapping of the form $\varphi(z) = \alpha z + \beta$ for $\alpha > 0$ and $\beta \in \mathbb{R}$. Along with this question, however, it has to be checked under which circumstances the composition mapping preserves square integrability. Using Wiener amalgam spaces we can give a partial answer.

Proposition 13 (*Preservation of p-integrability under composition*). Assume that for some positive $a > 0$ and $K \geq 0$ we have $\varphi(t) \in [a(t-K), a(t+K)]$ for all $t \in \mathbb{R}$ (i.e. that the graph of φ is contained in a strip in $\mathbb{R}^2$), then for some $C = C(\Omega, K, a) > 0$

$$\|f \circ \varphi\|_p \leq C \cdot \|f\|_p \quad \forall f \in L^p \text{ with } \mathrm{spec}(f) \subseteq \Omega \quad . \tag{16}$$

Note that this statement is *not* valid without the hypothesis of band-limitedness on f. Let us demonstrate this by an L^2-example. Assume that for some point $t_n \in \mathbb{R}$ with $|\varphi'(t_n)| \leq \frac{1}{2n^2}$, then $|\varphi'(t)| \leq 1/n^2$ for some interval $[a_n, b_n]$. Setting $\alpha_n := \varphi^{-1}(a_n)$ and $\beta_n := \varphi^{-1}(n_n)$ it follows from the mean value theorem, that $|a_n - b_n| \geq n^2 |\alpha_n - \beta_n|$, and that the indicator function $f := \mathbf{1}_{[\alpha_n, \beta_n]}$ satisfies $\|f \circ \varphi\|_2 \geq n \cdot \|f\|_2$.

Proof. Choosing some function $k \in \mathcal{K}(\mathbb{R}^m)$ such that $k(x) \equiv 1$ on $[-aK, aK]$ we have

$$|f \circ \varphi(t)| \leq \sup_{z \in [a(t-K), a(t+K)]} |f(z)| \leq \|T_{at} k \cdot f\|_\infty = F_k(at) \ .$$

Since we know that band-limited functions belong to $W(C^0, L^p)$, with $\|F_k\|_p \le C_\Omega \|f\|_p$, we end up with the estimate (using the dilation invariance of L^p)

$$\|f \circ \varphi\|_p \le a^{m(1-1/p)} \cdot \|F_k\|_p \le C \cdot \|f\|_p \quad . \tag{17}$$

□

The above proposition gives a sufficient condition to preserve square integrability of band-limited L^p-functions. Improving on the necessary conditions on φ (in order to preserve band-limitedness) as given in [Cl] we mention the following result: Any function φ, which has an unbounded derivative will produce functions which are *not* band-limited!

In fact, band-limited functions satisfy (what is usually called Bernstein's inequality, cf. [FG2], Prop.3.4 for an L^p-version)

$$\sup_{t \in I\!R} |f'(t)| \le C \sup_{t \in I\!R} |f(t)| \quad .$$

Functions φ with unbounded $\gamma'(t)$ (arbitrary steep parts in the graph of φ) apparently destroy this property, i.e. $f \circ \varphi$ will not satisfy the last estimate for such functions φ and cannot be band-limited for that reason.

3 GEOMETRIC CONDITIONS ON THE SAMPLING SETS

One of the basic estimates in irregular sampling theory is the following (we state the L^p-version here), taking a discrete sampling family $X = (x_i)_{i \in I}$.

For any bounded set Ω and p, $1 \le p < \infty$, there is some $C_\Omega > 0$ such that

$$\left(\sum_{i \in I} |f(t_i)|^p \right)^{1/p} \le C_\Omega \|f\|_p \; \forall f \in L^p(I\!R^m) \text{ with } \mathrm{spec}(f) \subseteq \Omega \quad . \tag{18}$$

This estimate has been first shown for $f \in L^2(I\!R^m)$ and regular lattices by S. Nikolskij, and is known as Nikolskij's inequality. In a more classical setting such estimates have been proved by Plancherel and Polya (cf. [PP]). Recently an irregular version (for sets in the plane which arise as products of irregular sets in each coordinate) has been proposed by Butzer and Hinsen (cf. [BH1,2]). For certain irregular sets these results were proved by Duffin and Shaeffer in [DS] in the one-dimensional case (cf. [DS], or [Y]).

It is clear, that X must not be too dense that such an inequality holds, even if we have very smooth and nice functions f. Certainly there must be no accumulation points to the family X. Since the L^p-spaces are translation invariant the correct necessary condition is the following one.

Definition 2. A discrete set $X = (x_i)_{i \in I}$ is called *relatively separated* if there exists an upper bound on the local density of X in $I\!R^m$ in the following sense: For some $r_0 > 0$ there is a uniform bound on

$$d(y) := \#\{i \,|\, x_i \in B_{r_0}(y)\}, \text{ i.e. } d_r(X) := \sup_{y \in I\!R^m} d(y) < \infty \; . \tag{19}$$

It is then clear (any ball can be covered by a finite family of balls of any given size) that for any ball $B \subseteq I\!R^m$ the following is true.

The number of elements in any translate $x + B$ is uniformly bounded.

(I) For some $C_B > 0$ we have $\#\{i \,|\, t_i \in (x+B)\} \leq C_B \;\; \forall x \in I\!R^m$.

(II) For any $r > 0$ the family $(B_r(x_i)_{i\in I})$ of balls of radius r is of bounded height $h(r)$, i.e. there is a maximal number $h(r)$ of balls $B_r(x_i)$ covering any given point. Actually, somewhat more holds.

(II′) Given any compact subset $K \subseteq I\!R^m$ there is a uniform bound on the balls intersecting $y+K$, independently of y , i.e.

$$\sup_{y\in I\!R^m} \#\{i \,|\, (y+K)\cap B_r(x_i) \neq \emptyset\} < \infty \; . \tag{20}$$

Theorem 14. A discrete family $X = ((x_i)_{i\in I})$ in $I\!R^m$ is relatively separated if one (hence all) of the following conditions are satisfied: (I), (II) or

(III) The measure $\sum_{i\in I}\delta_{x_i}$ belongs to $W(M, L^\infty)$, i.e. is *translation bounded* in the sense of [AL].

(IV) For some (any) $1 \leq p < \infty$ there is a constant $C > 0$ such that

$$(\sum_{i\in I} |f(x_i)|^p)^{1/p} \leq C \cdot \|f\,|W(C^0, L^p)\| \;\; \forall f \in W(C^0, L^p) \tag{21}$$

(or only for all functions $f = T_x k$, for some non-zero $k \in \mathcal{K}(I\!R^m)$).

(V) For any $\delta > 0$ the family is a finite union of *separated* families $X^k = (x_i^k)_{i\in I}$, each satisfying $|x_i^k - x_j^k| \geq \delta > 0$ for $i \neq j$.

(VI) X is a finite union of sets which are subsets of sequences each of which is *uniformly dense* in the sense of Duffin and Shaeffer:

$$|x_n - \alpha n| \leq L \;\; \forall n \in \boldsymbol{Z}^m (\text{ for some } \alpha > 0 \text{ and } L > 0) \; . \tag{22}$$

Proof. Taking for granted that the concept of relative separation does not depend on the choice of the radius r_0 we obtain the stronger version of (II) by choosing some r_1 such that $K \subseteq B_{r_1}(x_0)$, observing then that $(y+K)\cap B_r(x_i) \neq \emptyset$ only if $x_i \in B_{r_2}(y)$, for $r_2 = r_1 + r$, showing thus that (20) is equivalent with (19). In order to show the equivalence with (III) choose some $k \in \mathcal{K}(I\!R^m)$ such that $0 \leq k(y) \leq 1$, $k(x) \equiv 1$ on $B_s(0)$ and $\text{supp}(k) \subseteq B_r(0)$. Then we have for $\mu := \sum_{i\in I}\delta_{x_i}$

$$M_k(x) := \|(T_x k)\cdot \mu\|_M = \sum_{i\in I} |k(x_i - x)| \leq \#\{i | x_i \in B_r(x)\} \;\; , \tag{23}$$

but on the other hand $M_k(x) \geq \#\{i | x_i \in B_s(x)\}$.

To check that a relatively separated family X satisfies (IV) we use

$$\mu \cdot f \in W(M, L^\infty)\cdot W(C^0, L^p) \subseteq W(M, L^p) \; .$$

Assume conversely, that X is *not* relatively separated. Then for each $r > 0$ there are points x_n in $I\!R^m$ such that $\#\{i | x_i \in B_r(x)\} \geq n$. Let now $k \in \mathcal{K}(I\!R^m)$ be any function with $\min_{y\in B_r(0)}(k(y)) \geq \eta > 0$. Then the sequence $T_{x_n}k$ is bounded in $W(C^0, L^p)$ for any $p \geq 1$, but $(\sum_{i\in I} |T_{x_n}k(x_i)|^p)^{1/p} \geq \eta \cdot n^{1/p}$ for each $n \geq 1$, in contradiction to (IV).

(V): It is left as an exercise to the reader to verify that a relatively separated set satisfies (V). Conversely, let X be relatively separated. We cover $I\!R^m$ by (almost) disjoint cubes of side length $\geq \delta > 0$. Then in each of these cubes together with all its 2^m neighbors there are a maximal number n_1 of points available. This can be easily used to split X into at most n_1 δ-separated subsequences (cf. [FGr]). Conversely any separated set is obviously relatively separated, and the same is true for finite unions.

(VI): We observe first that property (18) has the following features: If this property holds for any subset Y of set X satisfying (18), and it also holds for finite unions of sets X_i, each of them satisfying (18). Since a set satisfying (22) is relatively separated, the sets described in (VI) are relatively separated by the argument just given. The converse requires only slight modifications of arguments used for (V). □

The above result also sheds light on the so-called Parseval relationship for non-uniform sampling appearing in a note by Marvasti and Chuande [MC] which has just appeared. Their proof makes implicit use of the definition of a 'sampling set' in the sense of Duffin and Schaeffer [DS], which implies that we have an estimate of the form (21) for $p = 2$, which means that X is relatively separated by Theorem 6. Actually, the argument used in [MC] is not valid without relative separation. In fact, for general frequencies $(\lambda_n)_{n=1}^\infty$ the convergence of $\sum_{n=1}^\infty a_n e^{-i\pi\lambda_n}$ for sequences $(a_n)_{n=1}^\infty$ in ℓ^2 is only guaranteed in the sense of some mean (cf. [Be] for details), and *not* locally in L^2 (it is not difficult to set up simple counter-examples which are divergent over some interval). Given this restriction we can give a proof of Parseval's relationship for non-uniform sampling in several variables.

Theorem 15. Let $(x_n)_{n=1}^\infty$ be a relatively separated sampling sequence in $I\!R^m$, and M_{lp} be the Fourier transform of $S_X f := \sum_{n=1}^\infty f(x_n) \cdot T_{x_n} g$, where $g \in L^2$ satisfies $\hat{g}(t) \equiv 1$ on Ω. Then M_{lp} belongs locally to L^2, and the following relation holds for any $f \in L^2$ with $\mathrm{spec}(f) \subseteq \Omega$

$$\sum_{n=1}^{\infty} |f(x_n)|^2 = \int_{I\!R^m} \hat{f}(s) \cdot \overline{M}_{lp}(s)\, ds \quad . \tag{24}$$

Proof. We have already discussed convergence of the series on the left side. On the other hand Corollary 5 shows that $S_X f \in W(M, L^2) \subseteq W(\mathcal{F}L^\infty, L^2)$ and thus by the generalized HY inequality $M_{lp} \in W(L^2, L^\infty) \subseteq L^2_{loc}(I\!R^m)$. Since $\hat{f}$ is a compactly supported L^2-function converence of the right hand integral follows. These observations allows us to use the duality pairing $\langle \cdot, \cdot \rangle$ (this time considered as the natural extension of the Hilbert space duality in the argument below) in varying pairs in order to obtain.

$$\sum_{n=1}^{\infty} |f(x_n)|^2 = \langle |f|^2, \delta_X \rangle = \langle f, \delta_X \cdot f \rangle = \langle f * g^*, \delta_X \cdot f \rangle = \tag{25}$$

$$= \langle f, (\delta_X \cdot f) * g \rangle = \langle f, S_X f \rangle = \langle \hat{f}, M_{lp} \rangle = \int_{I\!R^m} \hat{f}(s) \overline{M}_{lp}(s)\, ds$$

the third step following from the fact that f has spectrum in Ω and that convolution by $g^* := \mathcal{F}^{-1}(\hat{g}^-)$ acts therefore trivial. The last step uses Plancherel's theorem. □

Remark 4. Observe that this result is not only true in several dimensions but is *not* restricted to particular sampling schemes arising as product sets of one-dimensional irregular sampling sets, as in the two-dimensional result given in [MC].

4 PRODUCT CONVOLUTION OPERATORS AND SIGNAL RECOVERY

Estimates for certain product-convolution operators are at the heart of a reconstruction method suggested by Donoho and Stark [DS1,2]. The situation discussed there is the following one: A function f is given with several parts being missing. This missing information is complemented by some a priori information on its Fourier transform. The well known Papoulis–Gerchberg algorithm covers the case where the spectrum is known to be contained in some bounded spectral set, i.e. covers the case of band-limited functions. Given only a small part of the function it is then possible to recover the full function by an iterative procedure. However, the method may be very instable and sensitive to noise. It also does not cover the case of a possibly unbounded spectrum. The results of Donoho and Stark [DS1,2] show that it is possible to solve the problem in certain cases, e.g. if the spectrum is unbounded, but has finite measure. Of course, there has to be a trade-off between the size of the set T of missing values, and the set Ω on which the Fourier transform $\hat{f}$ of f is concentrated (or unknown). It turns out that one has stable reconstruction by means of an iterative algorithm if $|T|\,|\Omega| < 1$, i.e. if the product of the (Lebesgue) measures of these sets is small enough. Although only the one-dimensional case is treated explicitly in [DS1,2] their arguments extend to m dimensions, and even to locally compact abelian groups, as pointed out by Smith [Sm].

The key estimate for the recovery (cf. [DS1]), concerns the operator

$$f \mapsto PQf \quad \text{with} \quad Q : f \mapsto \mathcal{F}^{-1}(\mathbf{1}_\Omega \hat{f}), \quad P : f \mapsto \mathbf{1}_T f \quad , \tag{26}$$

where $\mathbf{1}_\Omega$ denotes the indicator function of the set $\Omega \subseteq \mathbb{R}^m$ and $T \subseteq \mathbb{R}^m$ is some subset of $\mathbb{R}^m$. Clearly, another way of considering this operator is to see it as a convolution product, with convolution by $\mathcal{F}^{-1}(\mathbf{1}_\Omega)$, followed by pointwise multiplication with $\mathbf{1}_T$. It is shown that $\|P \circ Q\| \leq |W|\,|T|$, the operator norm being for L^p, $1 \leq p \leq 2$. It is easy to verify this result (due to K. Smith [Sm]). By Hausdorff-Young $\|h_\Omega\|_{p'} \leq \|\mathbf{1}_\Omega\|_p = |\Omega|^{1/p}$, and by Hölder's inequality $L^p * L^{p'} \subseteq C^0$, thus

$$\|PQf\|_p \leq \|P(Qf)\|_p \leq \|\mathbf{1}_T\|_p \|Qf\|_\infty \leq (|T|\,|\Omega|)^{1/p}\,\|f\|_p \quad \forall f \in L^p(\mathbb{R}^m) \quad . \tag{27}$$

It is clear that under the given circumstances $Q \circ (Id - P) \circ Q$ is invertible as an operator on $L^{p,\Omega}$. However, if $f \in L^{p,\Omega}$ is given over $\mathbb{R}^m \backslash T$, we exactly know $(Id - P) \circ Q$, and therefore, applying the inverse operator, we are able to recover f. Of course, the inversion is carried out by means of Neumann's series and can thus be formulated as an iterative procedure (cf. [DS1], section 4).

The other way of looking at their result was to decompose the mapping $f \mapsto PQf$ into 4 different mappings, which in principle could go through arbitrary Banach spaces of functions or distributions (not only through L^p-spaces).

With $B = L^p$ (on any lca. group) the following is natural (and gives the same result as mentioned above). Consider the sequence of mappings

$$f \mapsto \mathcal{F}f \mapsto \mathbf{1}_\Omega \cdot \mathcal{F}f \mapsto \mathcal{F}^{-1}(\mathbf{1}_\Omega \cdot \mathcal{F}f) \mapsto \mathbf{1}_T \cdot \mathcal{F}^{-1}(\mathbf{1}_\Omega \mathcal{F}f) \tag{28}$$

The composed operator is treated as a composition of operators each of which is either a (inverse) Fourier transform or a pointwise multiplier of some indicator function. Using the fact that L^p is (contractively) embedded into the pointwise multiplier algebra from $L^{p'}$ into L^1, we see that the optimal way of looking at the above sequence as operators between the spaces L^p, $L^{p'}$, L^1, L^∞ and L^p (in this order), and in each case the norm of the multiplication operator is just the $|\cdot|^{1/p}$ (a power of the volume of the underlying set).

It is now evident, that the above chain (28) can run through various other spaces. The general idea behind such an approach is of course to describe situations, which are not covered by the above estimates, but still allow (maybe under some extra conditions on f) to apply the signal recovery algorithm. As a typical result in this direction obtained by using amalgam spaces we discuss a theorem concerning L^2-functions. Furthermore we fix some r such that $r \geq 2$. Then for $1/p := 1/2 + 1/r$

$$\|f\mathbf{1}_\Omega \,|W(L^2,\ell^p)\| \leq \|\hat{f}\,|W(L^2,\ell^2)\| \cdot \|\mathbf{1}_\Omega \,|W(L^\infty,\ell^r)\| \quad , \tag{29}$$

and by the Hausdorff-Young theorem for Wiener amalgams an estimate for Qf:

$$\|Qf\,|W(L^{p'},\ell^2)\| \leq C \cdot \|\hat{f}\mathbf{1}_\Omega \,|W(L^2,\ell^p)\| \leq C \cdot \|f\|_2 \,\|\mathbf{1}_\Omega \,|W(L^\infty,\ell^r)\| \quad . \tag{30}$$

Applying now the pointwise multiplier rule for amalgams one has

$$\|PQf\|_2 = \|PQf\,|W(L^2,\ell^2)\| \leq \|Qf\,|W(L^{p'},\ell^2)\| \cdot \|\mathbf{1}_T \,|W(L^r,\ell^\infty)\| \leq$$

$$\leq C \cdot \|f\|_2 \,\|\mathbf{1}_\Omega \,|W(L^\infty,\ell^r)\| \cdot \|\mathbf{1}_T \,|W(L^r,\ell^\infty)\| \quad .$$

We have shown that at the expense of a more sensitive measurement of W ($\|\mathbf{1}_\Omega \,|W(L^\infty,\ell^2)\|$ instead of $\|\mathbf{1}_\Omega\|_2 = |W|^{1/2}$) we can replace $\|\mathbf{1}_T\|_2 = |T|^{1/2}$ by the much less sensitive measure $\|\mathbf{1}_T \,|W(L^2,\ell^\infty)\| = \sup_{x\in\mathbb{R}^m} |T \cap (x+Q)|$, where Q is the unit cube in $\mathbb{R}^m$ (which may be considered as a *local density measure*). This result can be used as follows.

Assume we know that the set W consists of few disjoint intervals (or cubes), far apart from each other (so that the band-width or even the diameter of the spectrum is large). Then, roughly speaking, the norm of $\mathbf{1}_\Omega \in W(L^\infty,\ell^r)$ corresponds to $k^{1/r}$ if k is the number of intervals of unit length needed to cover W.

Theorem 16. For $r > 0$ and Q open in $\mathbb{R}^m$, with compact closure, there is some $\gamma > 0$ such that any $f \in L^2(\mathbb{R}^m)$, with spec(f) contained in at most n balls of radius r, can be completely recovered from $f\mathbf{1}_M$, if only $\inf_{x\in\mathbb{R}^m} |M \cap (x+Q)| \geq |Q| - \gamma$, i.e. if the local density of the set of missing values is not too large.

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