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Reflectionless Analytic Difference Operators (AΔOs): Examples, Open Questions and Conjectures

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Abstract

We present a scenario concerning the existence of a large class of reflectionless self-adjoint analytic difference operators. In order to exemplify this scenario, we summarize our results on reflectionless self-adjoint difference operators of relativistic Calogero-Moser type.

1 Introduction

The following serves a twofold purpose. On the one hand, we present a short sketch of some recent results on explicit eigenfunctions for AΔOs of relativistic Calogero–Moser type, restricting attention to couplings for which the eigenfunctions are reflectionless. This is a rather small family, but it contains various instructive examples that suggest a more general scenario.

On the other hand, then, we would like to draw attention to some natural questions and speculations concerning the existence and properties of a far larger family of AΔOs with reflectionless eigenfunctions. The class of reflectionless AΔOs whose eventual existence is at issue can be delineated as follows: It should not only give rise to the class of reflectionless Schrödinger operators associated to soliton solutions of the KdV equation (by taking the step size to 0), but also to the class of reflectionless Jacobi operators associated to soliton solutions of the infinite Toda lattice (by analytic continuation and discretization). We begin by recalling some salient features of the latter two families [1, 2, 3, 4].

The relevant Schrödinger operators are of the form

$$H = -\frac{d^2}{dx^2} + V(x), \tag{1}$$

with $V(x)$ a real-valued Schwartz space function. They are reflectionless when there exist eigenfunctions for all eigenvalues $p^2, p \geq 0$, satisfying

$$\Psi_H(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ a_H(p)\exp(ixp), & x \to -\infty. \end{cases} \tag{2}$$

It has been known for half a century that such potentials $V(x)$ exist [5]. The function $a_H(p)$ (essentially the S-matrix or transmission coefficient) must be of the form

$$a_H(p) = \prod_{n=1}^{N} \frac{p - i\kappa_n}{p + i\kappa_n}, \qquad 0 < \kappa_N < \cdots < \kappa_1, \tag{3}$$

and for each such S-matrix there exists an N-dimensional family of potentials parametrized by normalization coefficients $\nu_1, \ldots, \nu_N \in (0, \infty)$. The Jost function $\Psi_H(x,p)$ defined by (2) has an analytic continuation to $p \in \mathbb{C}$, and yields bound states

$$\phi_n(\cdot) \equiv \Psi_H(\cdot, i\kappa_n), \qquad n = 1, \ldots, N, \tag{4}$$

satisfying

$$\int_{-\infty}^{\infty} |\phi_n(x)|^2 dx = 1/\nu_n, \qquad n = 1, \ldots, N. \tag{5}$$

Thus H gives rise to un unbounded self-adjoint operator on $L^2(\mathbb{R}, dx)$ with continuous spectrum $[0, \infty)$ and discrete spectrum $-\kappa_1^2, \ldots, -\kappa_N^2$.

The pertinent Jacobi operators are infinite matrices of the form

$$J_{jk} = b_j \delta_{jk} + a_j \delta_{j,k-1} + a_k \delta_{j,k+1}, \qquad j, k \in \mathbb{Z}, \tag{6}$$

with a_j and b_j sequences of real numbers rapidly converging to 1 and 0, resp., for $j \to \pm\infty$. They are reflectionless when there exist eigenfunctions for all eigenvalues $2\cos(\alpha p)$, $p \in [0, \pi/\alpha]$, $\alpha > 0$, satisfying

$$\Psi_J(j,p) \sim \begin{cases} \exp(ij\alpha p), & j \to \infty, \\ a_J(p)\exp(ij\alpha p), & j \to -\infty. \end{cases} \tag{7}$$

In this case $a_J(p)$ must be of the form

$$a_J(p) = \prod_{n=1}^{N} \frac{\sin[\alpha(p - i\kappa_n^+)/2]}{\sin[\alpha(p + i\kappa_n^+)/2]} \cdot \prod_{m=1}^{M} \frac{\cos[\alpha(p - i\kappa_m^-)/2]}{\cos[\alpha(p + i\kappa_m^-)/2]}, \tag{8}$$

where $0 < \kappa_N^+ < \cdots < \kappa_1^+$, $0 < \kappa_M^- < \cdots < \kappa_1^-$, and for each such S-matrix there exists an $(N+M)$-dimensional family of Jacobi operators labeled by normalization coefficients $\nu_1^+, \ldots, \nu_N^+, \nu_1^-, \ldots, \nu_M^- \in (0, \infty)$. The above Jost function yields bound states

$$\begin{aligned} \phi_n^+(\cdot) &\equiv \Psi_J\left(\cdot, i\kappa_n^+\right), \qquad n = 1, \ldots, N, \\ \phi_m^-(\cdot) &\equiv \Psi_J\left(\cdot, \frac{\pi}{\alpha} + i\kappa_m^-\right), \qquad m = 1, \ldots, M, \end{aligned} \tag{9}$$

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |\phi_n^+(j)|^2 &= 1/\nu_n^+, \qquad n = 1, \ldots, N, \\ \sum_{j \in \mathbb{Z}} |\phi_m^-(j)|^2 &= 1/\nu_m^-, \qquad m = 1, \ldots, M. \end{aligned} \tag{10}$$

Thus the matrix $\{J_{kl}\}_{k,l \in \mathbb{Z}}$ gives rise to a bounded self-adjoint operator J on $l^2(\mathbb{Z})$ that has continuous spectrum $[-2, 2]$ and discrete spectrum $2\cosh(\alpha\kappa_n^+)$, $-2\cosh(\alpha\kappa_m^-)$, with $n = 1, \ldots, N$, $m = 1, \ldots, M$.

As is well known, the reflectionless Schrödinger operators give rise to the N-soliton solutions of the KdV equation via the Inverse Spectral Transform (IST). More specifically, the parameters $\kappa_1, \ldots, \kappa_N$ determine the speeds of the N solitons, all of which move from left to right, whereas the coefficients $\nu_1, \ldots, \nu_N$ determine their positions for asymptotic times.

Similarly, the IST for the infinite Toda lattice ties in the Toda soliton solutions with the above reflectionless Jacobi operators. In this case the parameters κ_n^+ and κ_m^- determine the speeds of the N right-moving and the M left-moving solitons in a general soliton solution, with the normalization coefficients yielding once again the asymptotic positions.

The AΔOs at issue here may be viewed as generalizations of the above ordinary second-order differential and *discrete* difference operators H (1) and J (6). They are of the form

$$A = V_+(x)^{1/2}T_{i\beta}+V_-(x)^{1/2}T_{-i\beta}+V_0(x), \qquad \beta > 0, \tag{11}$$

where the translations $T_{\pm i\beta}$ are defined by

$$(T_z f)(x) = f(x-z), \qquad z \in \mathbb{C}. \tag{12}$$

We require that the functions V_+, V_- and V_0 be meromorphic and satisfy

$$V_\pm(x) \to 1, \qquad V_0(x) \to 0, \qquad x \to \pm\infty. \tag{13}$$

Of course, the simplest case is $V_\pm = 1$, $V_0 = 0$. Then one can solve the eigenvalue equation

$$A\Psi = 2\cosh(\beta p)\Psi, \qquad p \geq 0, \tag{14}$$

by taking $\Psi(x,p) = \exp(ixp)$. But when one multiplies this solution by an arbitrary $i\beta$-periodic meromorphic function (which may also depend on p), then one obtains a function $\hat{\Psi}$ that also solves (14).

The importance of this infinite-dimensional ambiguity for eigenfunctions of AΔOs (as opposed to the two-dimensional one for H and J) cannot be overestimated. It renders various existence and uniqueness methods useless, and is responsible for the absence (to date) of a well-developed Hilbert space theory. As a starting point for coping with this, we require in addition that the meromorphic functions $V_\pm$ and V_0 be such that A is at least *formally* self-adjoint on $L^2(\mathbb{R}, dx)$. (In particular, $V_0(x)$ must be real-valued for real x.)

With the above formal requirements in place, we can now introduce a notion of "reflectionless self-adjoint AΔO A": The eigenvalue equation (14) should admit a solution $\Psi_A(x,p)$ satisfying

$$\Psi_A(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ a_A(p)\exp(ixp), & x \to -\infty, \end{cases} \tag{15}$$

which can be used to *define* a self-adjoint operator (denoted by $\hat{A}$) on $L^2(\mathbb{R}, dx)$. We postpone an explanation of this procedure to the end of Section 4, in which we consider Hilbert space aspects. In Section 2 we collect algebraic features of the simplest special cases, and in Section 3 we discuss more general examples arising from relativistic Calogero–Moser systems. (For background information on the latter integrable systems we refer to Ref. [6].) In Section 5 we sketch further pertinent information on reflectionless AΔO-eigenfunctions, and formulate some conjectures.

2 The one-soliton operators

Consider the following two functions:

$$\Psi^+(x,p) \equiv e^{ixp}\frac{e^{i\beta\kappa/2}\sinh(\kappa x+\beta p/2) - e^{-i\beta\kappa/2}\sinh(\kappa x-\beta p/2)}{2(\cosh[\kappa(x+i\beta/2)]\cosh[\kappa(x-i\beta/2)])^{1/2}\sinh[\beta(p+i\kappa)/2]}, \tag{16}$$

$$\Psi^-(x,p) \equiv e^{ixp} \frac{e^{i\beta\kappa/2}\cosh(\kappa x + \beta p/2) + e^{-i\beta\kappa/2}\cosh(\kappa x - \beta p/2)}{2(\cosh[\kappa(x+i\beta/2)]\cosh[\kappa(x-i\beta/2)])^{1/2}\cosh[\beta(p+i\kappa)/2]}. \tag{17}$$

Clearly, we have

$$\Psi^+(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ \dfrac{\sinh[\beta(p-i\kappa)/2]}{\sinh[\beta(p+i\kappa)/2]}\exp(ixp), & x \to -\infty, \end{cases} \tag{18}$$

$$\Psi^-(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ \dfrac{\cosh[\beta(p-i\kappa)/2]}{\cosh[\beta(p+i\kappa)/2]}\exp(ixp), & x \to -\infty, \end{cases} \tag{19}$$

so these functions have a reflectionless asymptotics. Moreover, they satisfy

$$A^\delta \Psi^\delta = 2\cosh(\beta p)\Psi^\delta, \qquad \delta = +,-, \tag{20}$$

where A^+ and A^- are the formally self-adjoint AΔOs

$$\begin{aligned} A^\pm \equiv {} & \left(\frac{\cosh[\kappa(x-3i\beta/2)]\cosh[\kappa(x+i\beta/2)]}{\cosh^2[\kappa(x-i\beta/2)]}\right)^{1/2} T_{i\beta} + (i \to -i) \\ & \mp \frac{\sin^2(\kappa\beta)}{\cosh[\kappa(x-i\beta/2)]\cosh[\kappa(x+i\beta/2)]}. \end{aligned} \tag{21}$$

(This can be checked by a straightforward calculation, cf. also (29)–(32) below.)

Before explaining the context from which these AΔOs arise and entering into Hilbert space aspects, let us add a few more features that can be directly verified. First, the function

$$\phi^+(x) \equiv \Psi^+(x,i\kappa) = (4\cosh[\kappa(x-i\beta/2)]\cosh[\kappa(x+i\beta/2)])^{-1/2}, \tag{22}$$

with A^+-eigenvalue $2\cos(\beta\kappa)$, is in $L^2(\mathbb{R},dx)$, provided

$$\kappa\beta \neq (2k+1)\pi, \qquad k \in \mathbb{N}, \tag{23}$$

and the function

$$\phi^-(x) \equiv \Psi^-\left(x, \frac{i\pi}{\beta} + i\kappa\right) = e^{-\pi x/\beta}(4\cosh[\kappa(x-i\beta/2)]\cosh[\kappa(x+i\beta/2)])^{-1/2}, \tag{24}$$

with A^--eigenvalue $-2\cos(\beta\kappa)$ is in $L^2(\mathbb{R},dx)$, provided

$$\kappa\beta \neq (2k+1)\pi, \qquad k \in \mathbb{N}, \qquad \kappa\beta > \pi. \tag{25}$$

Second, one easily checks

$$\lim_{\beta\to 0}[A^+ - 2]/\beta^2 = -\frac{d^2}{dx^2} - \frac{2\kappa^2}{\cosh^2(\kappa x)}, \tag{26}$$

$$\lim_{\beta\to 0}\Psi^+(x,p) = e^{ixp}\frac{e^{\kappa x}(p+i\kappa) + e^{-\kappa x}(p-i\kappa)}{2\cosh(\kappa x)(p+i\kappa)}, \qquad \lim_{\beta\to 0}\phi^+(x) = \frac{1}{2\cosh(\kappa x)}. \tag{27}$$

Thus, this $\beta \to 0$ limit gives rise to a Schrödinger operator associated with a 1-soliton solution to the KdV equation.

Third, substituting

$$\beta \to i\alpha, \qquad x \to j\alpha, \qquad j \in \mathbb{Z}, \tag{28}$$

in A^δ, $\Psi^\delta(x,p)$ and $\phi^\delta(x)$, yields Jacobi operators and eigenfunctions associated with 1-soliton solutions to the Toda lattice, moving to the right for $\delta = +$ and to the left for $\delta = -$.

Fourth, one can obtain *all* 1-soliton Schrödinger and Jacobi operators in the way just sketched by taking the functions $\exp(ix_0p)\Psi^\delta(x - x_0, p)$, $\delta = +, -$, and the AΔOs A^δ (21) with $x \to x - x_0$ as a starting point. Indeed, letting x_0 vary over $\mathbb{R}$, one obtains all positive normalization coefficients (as is easily seen).

3 Relativistic Calogero–Moser AΔOs

The above AΔOs A^δ are derived from AΔOs H^δ given by

$$H^\pm \equiv \left(\frac{\cosh[\kappa(x - i\beta)] \cosh[\kappa(x + i\beta/2)]}{\cosh[\kappa x] \cosh[\kappa(x - i\beta/2)]} \right)^{1/2} T_{i\beta/2} \pm (i \to -i). \tag{29}$$

Specifically, one readily checks that

$$A^\pm = (H^\pm)^2 \mp 2, \tag{30}$$

and that the functions $\Psi^\delta(x,p)$ are H^δ-eigenfunctions as well:

$$H^+\Psi^+ = 2\cosh(\beta p/2)\Psi^+, \tag{31}$$

$$H^-\Psi^- = 2\sinh(\beta p/2)\Psi^-. \tag{32}$$

Thus, the eigenvalue property (20) may be viewed as a corollary of (30)–(32).

We proceed by relating $H^\pm$ to the three AΔO families

$$H_r(g) \equiv \left(\frac{\sinh[\kappa(x - ig\beta)]}{\sinh[\kappa x]} \right)^{1/2} T_{i\beta} \left(\frac{\sinh[\kappa(x + ig\beta)]}{\sinh[\kappa x]} \right)^{1/2} + (i \to -i), \tag{33}$$

$$H_a(g) \equiv \left(\frac{\cosh[\kappa(x - ig\beta)]}{\cosh[\kappa x]} \right)^{1/2} T_{i\beta} \left(\frac{\cosh[\kappa(x + ig\beta)]}{\cosh[\kappa x]} \right)^{1/2} + (i \to -i), \tag{34}$$

$$H_e(g) \equiv \left(\frac{\cosh[\kappa(x - ig\beta)]}{\cosh[\kappa x]} \right)^{1/2} T_{i\beta} \left(\frac{\cosh[\kappa(x + ig\beta)]}{\cosh[\kappa x]} \right)^{1/2} - (i \to -i), \tag{35}$$

which we studied in Ref. [7]. The subscripts stand for "repulsive", "attractive" and "extra". They denote three regimes that can be associated with (reduced, two-particle) relativistic Calogero–Moser systems. The choice $g \in \mathbb{N}^*$ we made in Ref. [7] ensures that the AΔOs (33)–(35) admit reflectionless eigenfunctions.

Clearly, when we take $\beta \to 2\beta$ in H^+ and H^-, we obtain $H_a(2)$ and $H_e(2)$, resp. Now $H_a(2)$ is of the form envisaged above (recall (11)–(15)), its reflectionless eigenfunction $\Psi_a(x,p)$ being given by $\Psi^+(x,p)$ (16) with $\beta \to 2\beta$. The substitution (28) in $H_a(2)$ and $\Psi_a(x,p)$ yields a Jacobi operator associated to a 2-soliton solution of the Toda lattice, consisting of solitons moving with equal speeds in opposite directions. Readers familiar with the (Kac/van Moerbeke) $\leftrightarrow$ (Toda) correspondence (see, e.g., pp. 71–75 in Toda's monograph [2]) will appreciate what is going on here.

By contrast, $H_e(2)$ is not of the form (11)–(14), since the coefficient of $T_{-i\beta}$ goes to -1 for $x \to \pm\infty$; accordingly, one obtains eigenvalue $2\sinh(\beta p)$ for its reflectionless eigenfunction $\Psi_e(x,p)$ (given by $\Psi^-(x,p)$ (17) with $\beta \to 2\beta$). However, $H_r(2)$ does give rise to the properties (11)–(15), its reflectionless eigenfunction being given by

$$\Psi_r(x,p) \equiv e^{ixp} \frac{e^{i\beta\kappa}\cosh(\kappa x + \beta p) - e^{-i\beta\kappa}\cosh(\kappa x - \beta p)}{2(\sinh[\kappa(x+i\beta)]\sinh[\kappa(x-i\beta)])^{1/2}\sinh[\beta(p+i\kappa)]}. \tag{36}$$

On the other hand, the substitution (28) in $H_r(2)$ does not yield a self-adjoint Jacobi operator. Similarly, its limit

$$\lim_{\beta\to 0}(H_r(2)-2)/\beta^2 = -\frac{d^2}{dx^2} - \frac{2\kappa^2}{\sinh^2(\kappa x)}, \tag{37}$$

yields a singular potential, which is not allowed for KdV solitons.

More generally, the AΔOs $H_s(N+1)$, with $s = r, a, e$ and $N \in \mathbb{N}$, admit reflectionless eigenfunctions that are explicitly known. For lack of space we refrain from detailing them here, but we do specify the S-matrices that arise. For $H_s(N+1)$, $s = r, a$, one gets

$$a_r(p) = (-)^N \prod_{n=1}^{N} \frac{\sinh\beta(p - in\kappa)}{\sinh\beta(p + in\kappa)}, \qquad a_a(p) = (-)^N a_r(p), \tag{38}$$

whereas $H_e(N+1)$ yields

$$a_e(p) = \prod_{n=1}^{N} \frac{\cosh\beta(p - in\kappa)}{\cosh\beta(p + in\kappa)}. \tag{39}$$

The relation of the AΔOs $H_s(N+1)$ and their reflectionless eigenfunctions $\Psi_s(N+1)$, $s = r, a, e$, to KdV and Toda solitons is similar to the $N = 1$ case already detailed. Specifically, for $s = a$ the "Toda substitution" (28) yields a Jacobi operator associated to a $2N$-soliton solution in which N solitons move to the right and N to the left. For the operators $H_a(N+1)^2 - 2$ and $H_e(N+1)^2 + 2$ with $\beta \to \beta/2$ (28) yields Jacobi operators encoding N Toda solitons moving to the right and left, resp. Moreover, taking the nonrelativistic limit for $H_a(N+1)$ and $\Psi_a(N+1)$ yields the KdV N-soliton Schrödinger operator with potential $-N(N+1)\kappa^2/\cosh^2(\kappa x)$ and its reflectionless eigenfunctions.

4 Hilbert space aspects

Thus far we have been discussing properties of an algebraic nature. (In essence, the algebraic results we have been using can already be gleaned from Section 2 in our paper Ref. [8].) We now turn to functional-analytic features. (These are established in Ref. [7], cf. also Ref. [9].) For the reflectionless eigenfunctions at hand one would be inclined to expect that the normalized eigenfunction transform

$$\mathcal{F} : L^2(\mathbb{R}, dp) \to L^2(\mathbb{R}, dx), \qquad \phi(p) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} \Psi(x,p)a(p)^{-1/2}\phi(p)dp, \tag{40}$$

yields an isometry onto the orthocomplement of the bound states, if any. That is, setting

$$R_2 \equiv \mathcal{F}^*\mathcal{F} - \mathbf{1}, \qquad R_1 \equiv \mathcal{F}\mathcal{F}^* - \mathbf{1}, \tag{41}$$

one expects

$$R_2 = 0, \qquad R_1 = -P_b, \tag{42}$$

with P_b the orthogonal projection onto the subspace spanned by the L^2-eigenfunctions of the pertinent operator.

It is well known that this holds true for the Schrödinger and (with obvious changes) for the Jacobi case, which is why this expectation seems reasonable in the AΔO case, too. But as it turns out, it is not satisfied unless further restrictions are imposed.

To begin with, for $H_r(N+1)$ and *generic* κ one has $R_2 \neq 0$, so that $\mathcal{F}_r(N+1)$ is not an isometry. The existence of a *discrete* set of κ-values for which $R_2 = 0$ and for which $\mathcal{F}_r(N+1)$ amounts to Fourier transformation can be understood by taking $\kappa = \pi l/(N+1)\beta$, $l \in \mathbb{N}^*$, in $H_r(N+1)$ (33): It then becomes *free*. In particular, one reads off from (36) that for $N = 1$ and $\kappa = \pi l/2\beta$ one obtains the plane wave $\exp(ixp)$ up to phases and sign functions.

Turning to $H_a(N+1)$ and $H_e(N+1)$, let us first take $N = 1$. Then one reads off from (16) and (17) (with $\beta \to 2\beta$) that $\mathcal{F}_a(2)$ and $\mathcal{F}_e(2)$ amount to Fourier transformation for $\kappa = \pi l/2\beta$, $l \in \mathbb{N}^*$, just as in the repulsive case. Excluding these values from now on, the state of affairs is as follows.

For $\kappa \in (0, \pi/2\beta)$ one has $R_2 = 0$ in both cases. For $\mathcal{F}_a(2)$ the operator R_1 equals minus the projection onto the bound state subspace spanned by $(\cosh[\kappa(x-i\beta)]\cosh[\kappa(x+i\beta)])^{-1/2}$, whereas $R_1 = 0$ for $\mathcal{F}_e(2)$. Thus the expectation (42) is fulfilled. For $\kappa > \pi/2\beta$, however, the operators R_1 and R_2 are both nonzero, although they are still of finite rank. (The rank increases to ∞ as $\kappa \to \infty$.)

The situation for general N is similar. Requiring

$$\kappa \in (0, \pi/2N\beta), \qquad N \in \mathbb{N}^*, \tag{43}$$

the expectation (42) is borne out by the facts, with P_b projecting onto an N-dimensional bound state space for $H_a(N+1)$ and with $P_b = 0$ for $H_e(N+1)$. For generic $\kappa > \pi/2N\beta$ one has $R_2 \neq 0$, but R_1 and R_2 are still finite-rank for arbitrary $N \in \mathbb{N}^*$ and $\kappa > 0$.

With these results at our disposal, we can be more precise regarding the notion of "reflectionless self-adjoint AΔO". Indeed, using the eigenfunction transform $\mathcal{F}_a(N+1)$, we can *define* a self-adjoint operator $\hat{H}_a(N+1)$ on $(\mathbf{1} - P_b)L^2(\mathbb{R}, dx)$ as the pull-back of the self-adjoint operator of multiplication by $2\cosh\beta p$ on $L^2(\mathbb{R}, dp)$, *provided* we choose κ in the interval (43). The AΔO $H_a(N+1)$ has eigenvalues $2\cos(n\beta\kappa)$, $n = 1, \ldots, N$, on the N bound states, and so we can extend $\hat{H}_a(N+1)$ to a self-adjoint operator on $L^2(\mathbb{R}, dx)$ by letting its action on $P_b L^2(\mathbb{R}, dx)$ coincide with $H_a(N+1)$. Likewise, the operators $H_e(N+1)$ and $H_e(N+1)^2 + 2$ give rise to self-adjoint operators on $L^2(\mathbb{R}, dx)$ by pulling back the multiplications by $2\sinh\beta p$ and $2\cosh 2\beta p$, resp., with $\mathcal{F}_e(N+1)$, *provided* we impose the restriction (43).

5 Further developments and outlook

Our requirement that a *bona fide* self-adjoint operator $\hat{A}$ on $L^2(\mathbb{R}, dx)$ should be associated to a given *formally* self-adjoint reflectionless AΔO A is not only inspired by the findings reported in the previous section. Indeed, if we drop the self-adjointness requirement, then the three formally self-adjoint AΔO families (33)–(35) can be viewed as reflectionless AΔOs for a *dense set* in the parameter space $\beta, \kappa \in (0, \infty), g \in \mathbb{R}$. For $H_r(g)$ this can be gleaned from our paper Ref. [8], whereas for $H_a(g)$ and $H_e(g)$ it follows by performing a suitable analytic continuation of the $H_r(g)$-eigenfunctions introduced in Ref. [8].

Now as we have already mentioned, for $H_r(N+1)$ the reflectionless eigenfunctions do not give rise to a self-adjoint operator on $L^2(\mathbb{R}, dx)$. On the other hand, for $\kappa < \pi/N\beta$ the operators R_2, R_1 (41) are only nonzero on the even subspace of $L^2(\mathbb{R}, dx)$. (Note that $H_r(g)$ commutes with parity.) Accordingly, it *is* possible to associate to $H_r(N+1)$ a self-adjoint operator $\hat{H}_{r,-}(N+1)$ on the *odd* subspace (once more via pullback), provided κ is restricted to the interval $(0, \pi/N\beta)$ [7].

More generally, this holds true for $H_r(g)$ when the parameters are suitably restricted, and for the dense set mentioned above this involves the reflectionless eigenfunctions from Ref. [8]. But for $H_a(g)$ and $H_e(g)$ one cannot use the reflectionless eigenfunctions obtained by analytic continuation, since then isometry (and hence self-adjointness) breaks down. Rather, quite different eigenfunctions (yielding a nonzero reflection) are needed to obtain self-adjoint operators $\hat{H}_a(g)$ and $\hat{H}_e(g)$ for suitable parameters.

The pertinent eigenfunctions can be obtained via a generalization of the hypergeometric function, which we introduced in Ref. [6]. (See also Ref. [10] and papers to appear.) This generalized hypergeometric function is a joint eigenfunction of four hyperbolic AΔOs of Askey–Wilson type. It is quite likely that a suitable specialization of parameters in this function gives rise to further explicit examples of reflectionless self-adjoint AΔOs, but we have not completed the details thus far.

The class of reflectionless self-adjoint AΔOs whose existence is suggested by the above results is however far larger. Indeed, it should give rise to all functions $a_A(p)$ of the form

$$a_A(p) = \prod_{n=1}^{N} \frac{\sinh[\beta(p - i\kappa_n^+)/2]}{\sinh[\beta(p + i\kappa_n^+)/2]} \cdot \prod_{m=1}^{M} \frac{\cosh[\beta(p - i\kappa_m^-)/2]}{\cosh[\beta(p + i\kappa_m^-)/2]}, \tag{44}$$

with

$$0 < \kappa_N^+ < \cdots < \kappa_1^+ < \pi/2\beta, \qquad 0 < \kappa_M^- < \cdots < \kappa_1^- < \pi/2\beta, \tag{45}$$

and for each of these functions there should exist an $(N+M)$-dimensional family of self-adjoint AΔOs labeled by arbitrary positive parameters $\nu_1^+, \ldots, \nu_N^+, \nu_1^-, \ldots, \nu_M^-$.

From the above explicit examples one deduces, however, that one can only expect to find N (not $N+M$) square-integrable eigenfunctions

$$\phi_n^+(x) \equiv \Psi_A(x, i\kappa_n^+), \qquad \int_{-\infty}^{\infty} |\phi_n^+(x)|^2 dx = 1/\nu_n^+, \qquad n = 1, \ldots, N. \tag{46}$$

The point is that with the requirement (45) in force, the eigenfunctions

$$\phi_m^-(x) \equiv \Psi_A\left(x, \frac{i\pi}{\beta} + i\kappa_m^-\right), \qquad m = 1, \ldots, M, \tag{47}$$

are not square-integrable when one chooses A equal to $H_a(M+1)$ or to $H_e(M+1)^2 + 2$ with $\beta \to 2\beta$. (Note $N = M$ for the first choice, whereas $N = 0$ for the second one.)

At present, the question whether self-adjoint AΔOs on $L^2(\mathbb{R}, dx)$ with all of the properties just delineated exist, is wide open. But when they do exist (as we expect), then it is also natural to conjecture that they will correspond to the soliton solutions of a soliton hierarchy that may be viewed as a generalization of the KdV and Toda lattice hierarchies. To be sure, this is for the time being a quite speculative scenario.

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