June 2003

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Computer Science
University of Manchester

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Department of Computer Science
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Preprint Series
CSPP-23

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Abstract

We prove completeness and decidability results for a family of combinations of propositional dynamic logic and unimodal doxastic logics in which the modalities may interact. The kind of interactions we consider include three forms of commuting axioms, namely, axioms similar to the axiom of perfect recall and the axiom of no learning from temporal logic and a Church-Rosser axiom. We investigate the influence of the substitution rule on the properties of these logics and propose a new semantics for the test operator to avoid unwanted side effects caused by the interaction of the classic test operator with the extra axioms.

1 Introduction

Propositional dynamic logic (*PDL*) [6] is a powerful and convenient logical tool for reasoning about programs or actions. It has found applications in various branches of computer science, ranging from program verification to such areas as agent-based systems. Agent-based systems are formalised and analysed with a variety of logical methods. Often combinations of *PDL* and modal logics form the basis of agent theories which are intended to formalise the behaviour of autonomous agents [2, 7, 10, 12]. While *PDL* provides the apparatus for modelling and reasoning about the dynamic activities of the agents, informational aspects of the agents, in particular, the properties of the agents' belief and knowledge spaces, are typically modelled in the doxastic modal logics *K(D)45* and *S5* [2]. Combinations of *PDL* with *K(D)45* and *S5* are thus meaningful in agent applications.

Generally, there are different ways of combining of modal logics. Certain forms of combinations of modal logics are rather well-behaved, while other forms of combinations are known to pose problems in proving any results about them. The simplest form of combination of two (or more) logics is their fusion, or independent join. It is well-known [8, 9, 16] that fusions of modal logics inherit properties such as soundness, completeness, the finite model property and decidability, from the individual logics.

In modelling intelligent agents, actions and belief and also knowledge are normally linked in some form. Well-known connections between informational attitudes and actions are the properties of no learning and prefect recall (see e.g. [14]). No learning is given by the axiom schema

$$[a]\Box p \to \Box[a]p$$

(where $[a]$ is a *PDL* modality and $\Box$ is the epistemic modality). The schema says that the agent knows the result of her action in advance; in other words, there is no learning. Perfect recall is commonly formulated by the axiom schema

$$\Box[a]p \to [a]\Box p.$$

It expresses the persistence of the agent's knowledge after the execution of an action. In this paper we also allow the interaction of modalities via the Church-Rosser axiom

$$\Diamond[a]p \to [a]\Diamond p.$$

The axiom says that if an agent has a possibility to perform an action a in such way that she will believe p, then in the current state she believes that it is possible to obtain the result p by doing a.

Whereas fusions of well-behaved modal logics inherit the good properties from the individual logics, the situation of fusions of modal logics extended with interaction axioms like the above is not as straightforward. For such combinations there is no preservation theorem of the generality as for fusions. The particular type of interaction between the modal operators and the *PDL* operators makes it more difficult (but not impossible) to obtain positive results regarding completeness, the finite model property and decidability as we will see in this paper.

A particular problem we encounter in the study of combinations of *PDL* and doxastic modal logics with the above interaction axioms is the collapse of certain combinations of logics to one of their constituents. Our analysis shows that there are at least two explanations for this collapse. One explanation is the strength of the standard substitution rule. Normal modal logics are closed under substitution, which means that arbitrary propositional formulae may be substituted for propositional variables in axioms. *PDL* is also closed under substitution. Because *PDL* has a two-sorted language over actions and propositions, this means arbitrary propositional formulae may be substituted for propositional variables in axioms, and likewise, arbitrary action formulae may be substituted for action variables. In the combinations of *PDL* and informational logics such as *K(D)45* and *S5* with any of the axioms of no learning, perfect recall and the Church-Rosser axiom, closure under substitution, particularly, closure under substitution for actions, can have unwanted effects as is shown in this paper. We consider therefore the question as to how substitutivity should be defined in such logics based on *PDL*. The question is the following: In axiom schemata, do we allow substitution of all action formulae into action variables, or do we allow substitution of atomic action formulae into action variables only? If the answer is 'yes' we speak of *full substitutivity*, whereas if the answer is 'no' we speak of *weak substitutivity*. *PDL* is closed under full substitution. It means propositions and actions are treated uniformly in *PDL*. However in extensions of *PDL* and *K(D)45* by any of the axioms no learning, perfect recall and Church-Rosser axiom this (good) property implies unnatural properties of an agent. More dramatically, we show that under full substitutivity the *S5* operator in the analogous combination of *PDL* and *S5* is vacuous.

This paper focuses on the fusion of *PDL* and *K45*, *KD45* or *S5* extended with various choices of the axioms of no learning, perfect recall and the Church-Rosser axiom. We explore all the possible definitions of such extensions with respect to full and weak substitutivity. For each definition we consider the problems of completeness, the small model property, decidability and the admissibility of the full substitution rule (Sections 3 and 4). Our proofs combine the Gabbay-Shehtman filtration method for products of *S5* in [5] and the Fischer-Ladner filtration technique developed for *PDL* [3]. We prove that in combinations of *PDL* with *K45*, *KD45* or *S5* full substitutivity does not follow from weak substitutivity. We prove further that full substitutivity in the extensions of the fusion of *PDL* and *S5* causes this logic to collapse to *PDL*.

This collapse is not witnessed for test-free versions of the logics (Section 5). This is a pointer to the negative influence of the test operator of *PDL* in the interactions. Closer analysis shows indeed a second explanation for the collapse to *PDL* is: the interaction between the test operator of *PDL* with reflexive modalities. This negative result is perhaps not surprising because the test operator includes propositions into the semantics at the level of frames. Since the omission of the test operator from *PDL* would imply the agents have no means to program, or reason about programs, we use a modification of the definition and semantics of the test operator introduced in [13]. The new test operator has informational character rather than universal character; it confirms the belief (or knowledge) of an agent as opposed to confirming the properties of the actual state as the classical test operator does in *PDL*. Completeness and the small model property (therefore also decidability) are shown for the logics with the informational test operator (Section 6). These logics are shown to coincide under both notions of substitutivity, and we show that even though the definition and semantics of the test operator has been changed, classic *PDL* can still be simulated in the fusion of the introduced informational variation of *PDL* and modal logic *S5*.

The next section (Section 2) introduces the logics under consideration and provides the definitions of important concepts needed for the main part of the paper (Sections 3–6). The paper concludes with a mention of some open problems and observations relating to products of the considered logics (Section 7).

2 Main definitions and preliminary observations

The language $\mathcal{L}$ we consider is an extension of the language of *PDL* [6] with a new modal operator $\Box$. Formally, the language $\mathcal{L}$ is defined over the following primitive types: a countable set $\mathrm{Var} = \{p, q, r, \ldots\}$ of propositional variables and a countable set $\mathrm{AtAc} = \{a, b, c, \ldots\}$ of atomic action variables. The connectives in $\mathcal{L}$ are the Boolean connectives, $\rightarrow$, $\bot$, the dynamic logic connectives, $\cup$, ;, $*$, ?, and the modal operators $[\_]$ and $\Box$. The set For of formulae and the set Ac of action formulae in $\mathcal{L}$ are the smallest sets that satisfy the following conditions. (i) $\mathrm{AtAc} \subseteq \mathrm{Ac}$, $\mathrm{Var} \cup \{\bot\} \subseteq \mathrm{For}$. (ii) If ϕ and ψ are formulae in For and α and β are action formulae in Ac then $\phi?$, α^*, $\alpha \cup \beta$, $\alpha;\beta$ are action formulae in Ac, and $\phi \rightarrow \psi$, $\Box\phi$, $[\alpha]\phi$, are formulae in For. The connectives $\neg$, $\vee$, $\wedge$, $\leftrightarrow$, $\langle\_\rangle$ are defined as usual.

By definition, an *atomic action* is an action variable, and a *semi-atomic action* is an atomic action or a test action $\phi?$.

By a *theory* in $\mathcal{L}$ we understand any subset of For closed under the following standard rules:

$$\phi,\ \phi \rightarrow \psi \vdash \psi \qquad\qquad \phi \vdash [\alpha]\phi \qquad\qquad \phi \vdash \Box\phi$$

Generally axioms and theorems of a logic are assumed true for all instantiations for the atomic symbols. However, in this paper we distinguish between two variants of the substitution rule. The *weak substitution rule* allows the substitution of arbitrary formulae for the atomic propositional symbols but does not allow substitution for atomic action symbols. By contrast, the *full substitution rule* allows both kinds of substitutions, i.e. both propositional substitutions and action substitutions.

A *logic* in $\mathcal{L}$ is a theory which is closed under the full substitution rule and a *weak logic* in $\mathcal{L}$ is a theory closed under the weak substitution rule. Weak logics are notationally discerned by a subscript $_w$.

For an axiomatisation of *PDL* we refer to [6], and axiomatisations of *K45*, *KD45* and *S5* can be found in [1, 16].

Let Γ and Δ be any subsets of For. By $\Gamma \oplus \Delta$ (resp. $\Gamma \oplus \Delta_w$) we denote the least logic (resp. the least weak logic) which contains both Γ and Δ. The *fusion*, or *independent*

Axiom		Correspondence property	
(*NL*)	$[a]\Box p \to \Box[a]p$	(com^r)	$R \circ Q(a) \subseteq Q(a) \circ R$
(*PR*)	$\Box[a]p \to [a]\Box p$	(com^l)	$Q(a) \circ R \subseteq R \circ Q(a)$
(*CR*)	$\Diamond[a]p \to [a]\Diamond p$	(cr)	$Q(a)^{\smile} \circ R \subseteq R \circ Q(a)^{\smile}$

Figure 1: The interaction axioms and their correspondence properties.

join, of *PDL* and a unimodal logic L is therefore denoted by $PDL \oplus L$. By an *extension* of a logic L (resp. a weak logic L_w) by a set of axioms Δ we understand the logic $L \oplus \Delta$ (resp. $L \oplus \Delta_w$). We say the full substitution rule is *admissible* in a weak logic L_w iff $L_w = L$.

Proposition 1 Let L be any normal unimodal logic. Then $PDL \oplus L_w = PDL \oplus L$, i.e. the full substitution rule is admissible in $PDL \oplus L_w$.

For any unimodal logic L, standard $PDL \oplus L$ models are combinations of the familiar standard models for *PDL* and Kripke models for L. That is, a *standard* $PDL \oplus L$ *model* is a tuple $\langle S, Q, R, \models\rangle$ such that $\langle S, Q, \models_0\rangle$ is a standard *PDL* model [6] and $\langle S, R, \models_1\rangle$ is an L model [1] where $\models_0$ and $\models_1$ are obvious restrictions of $\models$ on *PDL*-formulae and unimodal formulae respectively. Here, S is a non-empty set of states, R is a binary relation on S, $\models$ is the usual truth relation on the model and Q is a mapping from the set of actions to the set of binary relations on S which satisfies the following conditions.

$$Q(\alpha \cup \beta) = Q(\alpha) \cup Q(\beta) \qquad Q(\alpha;\beta) = Q(\alpha) \circ Q(\beta)$$
$$Q(\alpha^*) = Q(\alpha)^* \qquad Q(\phi?) = \{(s,s) \in S^2 \mid s \models \phi\}$$

($\circ$ denotes relational composition, and $*$ is the reflexive and transitive closure operator on relations.)

This paper investigates the family of all possible extensions of $PDL \oplus L$ and *test-free* $PDL \oplus L$, where L is any of the logics in $\{K45, KD45, S5\}$, by the axioms *NL*, *PR* and *CR*. *NL* is short for no learning, *PR* for perfect recall, and *CR* for Church-Rosser. These axioms are listed in Figure 1 together with their Sahlqvist correspondence properties. The symbol $\smile$ in the figure denotes converse. In the case of logics with full substitution we assume that a ranges over all the actions, whereas for weak logics we assume that a ranges over the set of atomic actions only. In the latter case we refer to the correspondence properties as com^r_w, com^l_w and cr_w.

Our aim is to prove soundness and completeness results, so, the properties com^l, com^r, cr and the weak versions of them determine the intended standard models for the logics. For instance, by definition, a *standard* $PDL \oplus L \oplus \{NL, PR, CR\}$ *model* (resp. a *standard test-free* $PDL \oplus L \oplus \{NL, PR, CR\}$ *model*) is a standard $PDL \oplus L$ model (resp. a standard *test-free* $PDL \oplus L$ model) which satisfies the properties com^r, com^l and cr. The definitions of the models for the other logics and their weak versions are similar and can be obtained by substitution of the appropriate model properties which correspond to the extra axioms of a given logic. Soundness of all these logic with respect to the corresponding classes of standard models can be proved in the usual manner.

For $\Delta \subseteq \{NL, PR, CR\}$, if a structure $\langle S, Q, R, \models \rangle$ satisfies all the properties of a standard $PDL \oplus L \oplus \Delta_w$ model except possibly the property $Q(\alpha^*) = (Q(\alpha))^*$, but still validates all the formulae of $PDL \oplus L \oplus \Delta_w$ then we call this structure a *non-standard model* for $PDL \oplus L \oplus \Delta_w$. This notion is similarly defined for full logics. It is obvious that, for any of the considered logics, the class of all standard model for this logic is included in the class of all non-standard models for the logic. It can be proved that the axioms *NL*, *PR* and *CR* are canonical in the presence of the full substitution rule for the properties com^r, com^l and *cr*, respectively. Similarly, these axioms are canonical for com^r_w, com^l_w and cr_w in the presence of the weak substitution rule. It is also a well-known fact that *4*, *5*, *D* and *T* are canonical, respectively, for transitivity, Euclideaness, seriality and reflexivity of the doxastic accessibility relation. In the canonical model of the logic $PDL \oplus K45 \oplus \{NL, PR\}$, for instance, any action relation commutes with the doxastic relation and the doxastic relation is transitive and Euclidean. Using a standard argument for *PDL*, it can be shown that canonical models for all the considered logics are not standard and belong to the class of non-standard models. In fact, the following (more general) completeness theorem holds.

Theorem 2 Let L be any canonical unimodal logic and suppose $\Delta \subseteq \{NL, PR, CR\}$. Then $PDL \oplus L \oplus \Delta_w$ and $PDL \oplus L \oplus \Delta$ are strongly complete with respect to the corresponding classes of non-standard models.

It is easy to prove the following proposition which states that the axioms *NL* and *CR* are inter-derivable in the presence of the axiom $p \to \Box\Diamond p$.

Proposition 3 Let L_0 be either *PDL* or *test-free PDL*, and let L_1 be any normal unimodal logic extending $KB = K \oplus \{p \to \Box\Diamond p\}$. Then $L_0 \oplus L_1 \oplus \{NL\}_w = L_0 \oplus L_1 \oplus \{CR\}_w$ and $L_0 \oplus L_1 \oplus \{NL\} = L_0 \oplus L_1 \oplus \{CR\}$.

Proof. It is clear that the rule $\phi \to \psi, \psi \to \chi \vdash \phi \to \chi$ is admissible in classical propositional calculus. It is also easy to see that the formula $(p \to q) \to (\Diamond p \to \Diamond q)$ is derivable in the logic K. We use this two facts in the following derivations.

Suppose that *NL* and *B* are both contained in a (weak) logic $\mathfrak{L}$. Then, using axioms *B*, *K* and the necessitation rule for $[a]$ we obtain $[a]p \to [a]\Box\Diamond p \in \mathfrak{L}$. Hence, applying *NL* to the succedent of the implication, we have $[a]p \to \Box[a]\Diamond p \in \mathfrak{L}$, and, consequently, $\Diamond[a]p \to \Diamond\Box[a]\Diamond p \in \mathfrak{L}$. Finally, from the latter, with use of the contraposition of *B*, we obtain $CR \in \mathfrak{L}$.

For the converse, suppose that *CR* and *B* are both contained in a (weak) logic $\mathfrak{L}$. Then, $[a]\Diamond\Box p \to [a]p \in \mathfrak{L}$ by the contraposition of the axiom *B*, necessitation and the axiom *K*. Hence, $\Diamond[a]\Box p \to [a]p \in \mathfrak{L}$ by *CR*. Again, by using the axiom *K* and the necessitation rule for $\Box$ we obtain $\Box\Diamond[a]\Box p \to \Box[a]p \in \mathfrak{L}$. And, finally, $NL \in \mathfrak{L}$ by the *B* axiom. $\Box$

Moreover, a consequence of soundness is the following.

Proposition 4 *PDL* $\oplus$ *S5* $\oplus \{PR\}_w \not\supseteq \not\subseteq$ *PDL* $\oplus$ *S5* $\oplus \{CR\}_w$ and *test-free PDL* $\oplus$ *S5* $\oplus \{PR\}_w \not\supseteq \not\subseteq$ *test-free PDL* $\oplus$ *S5* $\oplus \{CR\}_w$.

Proof. Let $M' = \langle S, Q', R, \models\rangle$ and $M'' = \langle S, Q'', R \models\rangle$ where $S = \{0,1,2\}$, $R = \{(0,0),(0,1),(1,0),(1,1),(2,2)\}$, $Q'(a) = \{(0,2)\}$, $Q''(a) = \{(2,0)\}$ for a fixed atomic action a, $Q'(b) = Q''(b) = \emptyset$ for any atomic action b which differs from a, $M', 0 \models p$, $M', 1 \models p$ and $M', 2 \not\models p$ and $M'', 0 \models p$, $M'', 1 \not\models p$ and $M'', 2 \not\models p$. Then com^l_w is true in M', cr_w is true in M'' but $M', 0 \not\models \Diamond[a]p \to [a]\Diamond p$ and $M'', 2 \not\models \Box[a]p \to [a]\Box p$. Thus, none of the logics can contain the other by the soundness property. □

The result is also true for the combinations with *K45* and *KD45* in place of *S5*. Furthermore, it is easy to prove the next two propositions.

Proposition 5 *PDL* $\oplus$ *K45* $\oplus \{NL, PR\}_w \not\supseteq$ *PDL* $\oplus$ *K45* $\oplus \{CR\}_w$ and *test-free PDL* $\oplus$ *K45* $\oplus \{NL, PR\}_w \not\supseteq$ *test-free PDL* $\oplus$ *K45* $\oplus \{CR\}_w$.

Proof. Let $M = \langle S, Q, R, \models\rangle$ where $S = \{0,1,2\}$, $Q(a) = \{(0,2)\}$ for a fixed atomic action a, $Q(b) = \emptyset$ for any atomic action b which differs from a, $R = \{(0,0),(0,1),(1,1)\}$.Then Euclideaness and transitivity of R, com^l_w and com^r_w are true in M. Trivially, $M, 0 \models \Diamond[a]p$, but $M, 0 \not\models [a]\Diamond p$. Thus, M does not validate *cr*. Now, the statement of the proposition is trivial consequence of the soundness property. □

Proposition 6 *PDL* $\oplus$ *KD45* $\oplus \{NL, PR\}_w \supseteq$ *PDL* $\oplus$ *KD45* $\oplus \{CR\}_w$ and *test-free PDL* $\oplus$ *KD45* $\oplus \{NL, PR\}_w \supseteq$ *test-free PDL* $\oplus$ *KD45* $\oplus \{CR\}_w$.

Proof. Fix an arbitrary atomic action a. Consider a sublogic $\mathfrak{L}$ of the logic *PDL* $\oplus$ *KD45* $\oplus$ $\{NL, PR\}_w$ which is defined as follows. The language of $\mathfrak{L}$ is a sublanguage of $\mathcal{L}$ where the set atomic actions is the singleton set $\{a\}$ and all dynamic operators on actions are forbidden. $\mathfrak{L}$ is axiomatised by the axioms and the inference rules of *PDL* $\oplus$ *KD45* $\oplus$ $\{NL, PR\}_w$ without the dynamic operators. Actually $\mathfrak{L}$ is a copy of an extension of the fusion of *K* and *KD45* by the axioms *NL* and *PR*.

We will show that the logic $\mathfrak{L}$ contains the formula *CR*. The easiest way to show this is by use of semantical methods and the completeness theorem for the extension of the fusion of *K* and *KD45* with the axioms *NL* and *PR*. This completeness theorem follows as usual from Sahlqvist's theorem for standard multi-modal logics.

Let $M = \langle S, Q, R, \models\rangle$ be an arbitrary model for $\mathfrak{L}$. Thus, M satisfies seriality, Euclideaness and transitivity for the relation R and also the commutativity properties com^l_w and com^r_w. We will show that cr_w is also true in M. Suppose $(s,t) \in R$ and $(s,u) \in Q(a)$. Our goal is to find an element s_0 such that $(t, s_0) \in Q(a)$ and $(u, s_0) \in R$. By the seriality of R there exists v such that $(u,v) \in R$. So, by com^l_w, there is a w such that $(s,w) \in R$ and $(w,v) \in Q(a)$. Then, by Euclideaness of R, $(t,w),(w,t) \in R$. Hence, by com^r_w, there exists x such that $(t,x) \in Q(a)$ and $(x,v) \in R$. From $(t,x) \in Q(a)$ and $(w,t) \in R$ by com^r_w there exists y such that $(w,y) \in Q(a)$ and $(y,x) \in R$. By the transitivity of R, from $(x,v) \in R$ and $(y,x) \in R$ we obtain that $(y,v) \in R$. Then, by Euclideaness of R,

$(v, x) \in R$. Finally, by the transitivity of R, we obtain that $(u, x) \in R$. Thus, x is the element s_0 we were aiming to find.

It follows from the above that there exists a derivation of *CR* in $\mathfrak{L}$. It is obvious that any derivation in $\mathfrak{L}$ is a derivation in *test-free PDL* $\oplus$ *KD45* $\oplus$ $\{NL, PR\}_w$ and in *PDL* $\oplus$ *KD45* $\oplus$ $\{NL, PR\}_w$. □

Proposition 7 *PDL* $\oplus$ *KD45* $\oplus$ $\{PR, CR\}_w \not\supseteq$ *PDL* $\oplus$ *KD45* $\oplus$ $\{NL\}_w$ and *test-free PDL* $\oplus$ *KD45* $\oplus$ $\{PR, CR\}_w \not\supseteq$ *test-free PDL* $\oplus$ *KD45* $\oplus$ $\{NL\}_w$.

Proof. Let $M = \langle S, Q, R, \models \rangle$ where $S = \{0, 1, 2\}$, $Q(a) = \{(1, 2)\}$ for a fixed atomic action a, $Q(b) = \emptyset$ for any atomic action b which differs from a, $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$, and $M, 2 \not\models p$. Then seriality, Euclideaness and transitivity of R, com^l_w and cr_w are true in M. Trivially, $M, 0 \models [a]\Box p$, but $M, 0 \not\models \Box[a]p$. Thus, M does not validate *NL*. Now, the statement of the proposition is trivial consequence of the soundness property. □

By these five propositions, all the axioms *PR*, *NL* and *CR* are independent of each other already with respect to the fusion of *PDL* and *K45* closed under the full or weak substitution rule. Thus, we need to consider all combinations of interaction axioms producing extensions of *PDL* $\oplus$ *K45* and their weak versions. In the case of *PDL* $\oplus$ *S5* and *PDL* $\oplus$ $S5_w$, by Propositions 3 and 4, in order to get all the possible extensions by all of the interaction axioms it is enough to consider only two of the axioms, one of which is *PR*, i.e., for instance, *PR* and *CR*.

In the remainder of the paper we will consider all possible extensions of *PDL* $\oplus$ L and *test-free PDL* $\oplus$ L (where $L \in \{K45, KD45, S5\}$) by the axioms *NL*, *PR* and *CR* for completeness (w.r.t. classes of the intended standard models), the admissibility of full substitution rule and the *small model property* which will be stated in the following form:

> If a formula ϕ is satisfiable in a *non-standard* model for a logic then it is satisfiable in a finite *standard* model for this logic and the upper bound for the size of the latter is effectively computable from the length of ϕ.

It is well-known and quite easy to see that the small model property implies the existence of a decision algorithm for a logic. Thus, any logic with the small model property is decidable. Immediate consequences of the small model theorems proved in this paper are therefore decidability.

Because the canonical model is not standard, completeness with respect to the classes of standard models satisfying the frame correspondence properties of the additional axioms is not a straightforward consequence of the canonicity of the logic L and the extra axioms. To get completeness we modify the canonical models constructed by the method of Fischer and Ladner [3, 6], while taking care that left and right commutativity and the Church-Rosser property are preserved.

Subsequently when we talk about a model without explicit use of the words 'standard' or 'non-standard', we mean a standard model.

3 Weak extensions of $\boldsymbol{PDL} \oplus \boldsymbol{L}$

This section proves results about extensions $PDL \oplus L_w$ by axioms *NL*, *PR* and *CR* closed under weak substitutivity. The first results show that in combinations of *PDL* with extensions of *K45*, it is not the case that weak substitutivity implies full substitutivity.

Theorem 8 Let L be any normal unimodal logic which is contained in the logic of two-element clusters. Then any weak extension of the logic $PDL \oplus L$ by any combination of the axioms *NL*, *PR* and *CR* do not admit the rule of full substitution.

Proof. Let M_0 be a model $\langle S, Q, R, \models \rangle$ with $S = \{0, 1\}$, $Q(a) = \emptyset$ for any atomic action a, R is the universal relation on S, $0 \models p$ and $1 \models \neg p$. It is easy to see that *NL* and *PR* are true in M_0 for all atomic actions. But all the formulae $[\neg p?]\Box p \rightarrow \Box[\neg p?]p$, $\Box[p?]p \rightarrow [p?]\Box p$ and $\Diamond[p?]\bot \rightarrow [p?]\Diamond\bot$ are false in the state 0. $\Box$

Standard examples of logics contained in the logic of two-element clusters are *KB*, *KT*, *KD*, *K4*, *K5*, *S5*, *KD45*, *K45*, etc. Thus, in particular:

Corollary 9 None of the following logics admit the rule of full substitution: $PDL \oplus K45 \oplus \{NL, PR, CR\}_w$, $PDL \oplus KD45 \oplus \{NL, PR, CR\}_w$ and $PDL \oplus S5 \oplus \{NL, PR\}_w$.

By the way we note that the example in the proof of Theorem 8 is also suitable for proving that the full substitution rule is not admissible in the product of *PDL* and L (as defined in [15]).

As said in the previous section, the canonical model for any of the considered logics is a non-standard model. Following the common line of argument in proving completeness results our goal is to modify non-standard models to obtain standard models while preserving the satisfiability of the given *finite* set of formulae. Our method essentially repeats the filtration method of Fischer and Ladner [3], but because of the modal interaction axioms, special considerations are needed which depart from the standard route to ensure the preservation of the corresponding properties through the filtration. What is required, as we show, is a combination of the filtration method of Fischer and Ladner with the filtration method of Gabbay and Shehtman [5] for products of modal logics with *S5*.

The following definition is standard.

Definition 12 Let $\prec$ be the smallest transitive relation over $\{0, 1\} \times For$ satisfying the following.

$$(0, \phi), (0, \psi) \prec (0, \phi \rightarrow \psi) \qquad (1, [\alpha]\phi), (1, [\beta]\phi) \prec (1, [\alpha \cup \beta]\phi)$$
$$(0, \phi) \prec (0, \Box\phi) \qquad (1, [\alpha][\beta]\phi), (1, [\beta]\phi) \prec (1, [\alpha;\beta]\phi)$$
$$(0, \phi), (1, [\alpha]\phi) \prec (0, [\alpha]\phi) \qquad (1, [\alpha][\alpha^*]\phi) \prec (1, [\alpha^*]\phi)$$
$$(0, \phi) \prec (1, [\phi?]\psi)$$

Let $\preceq$ be the reflexive closure of $\prec$. For a set Σ of formulae, the *Fischer-Ladner closure* of Σ is defined by

$$FL(\Sigma) =^{\text{def}} \{\psi \mid \exists j \in \{0,1\}\, \exists \phi \in \Sigma\ (j,\psi) \preceq (0,\phi)\}.$$

Subsequently, we write $FL(\phi)$ instead of $FL(\{\phi\})$.

It is easy to see that $\prec$ is well-founded. Consequently, $FL(\Sigma)$ is finite whenever Σ is finite. The next lemma allows us to find an upper bound for the cardinality of $FL(\Sigma)$ in terms of the lengths of formulae in Σ.

We denote by $|\phi|$ and $|\alpha|$ the lengths of ϕ and α (as number of the symbols excluding parentheses), respectively.

Lemma 13 $\mathsf{Card}(FL(\phi)) \leq |\phi|$.

The next definition is inspired by [5, Proposition 12.5].

Definition 14 Let $M = \langle S, Q, R, \models \rangle$ be a (non-standard) model for *PDL* $\oplus$ *L*, let Σ be a finite set of formulae. For any s in M we denote by $\Sigma(s)$ the set $\{\phi \in FL(\Sigma) \mid s \models \phi\}$.

Define a relation $\sim_\Sigma$ on S by:

$$\begin{aligned} s \sim_\Sigma t \iff & \Sigma(s) = \Sigma(t) \text{ and} \\ & \{\Sigma(u) \mid (s,u) \in R\} = \{\Sigma(u) \mid (t,u) \in R\}. \end{aligned}$$

Let $\|s\|_\Sigma$ (or, simply, $\|s\|$) denote the equivalence class for s, i.e. $\{t \mid s \sim_\Sigma t\}$. Let S^Σ be the set of equivalence classes over S, i.e. $S^\Sigma = \{\|s\| \mid s \in S\}$.

Define a filtrated model $M^\Sigma = \langle S^\Sigma, Q^\Sigma, R^\Sigma, \models \rangle$ by the *least filtration* of M, that is, the relations on the M^Σ are defined by the following. For any atomic action a,

$$\begin{aligned} Q^\Sigma(a) &=^{\text{def}} \{(\|s\|, \|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s',t') \in Q(a)\}, \\ R^\Sigma &=^{\text{def}} \{(\|s\|, \|t\|) \mid \exists s' \in \|s\|\ \exists t' \in \|t\|\ (s',t') \in R\} \end{aligned}$$

Furthermore, for any propositional variable p,

$$\|s\| \models p \iff s \models p \text{ and } p \in FL(\Sigma).$$

The definition of the truth relation is extended to all formulae and the definition of Q^Σ is extended to all actions so that all the conditions on actions for standard *PDL* $\oplus$ *L* model are satisfied. This completes the definition of the filtrated model M^Σ.

The following lemma follows immediately from the definition of R^Σ.

Lemma 15 If R is reflexive then R^Σ is reflexive and if R is serial then R^Σ is serial.

The proofs of the next two lemmata follow the proof of Proposition 12.5 in [5].

Lemma 16 If R is transitive and Euclidean then $\sim_\Sigma \circ R \subseteq R \circ \sim_\Sigma$.

Proof. Suppose that $s \sim_\Sigma t$ and $(t,u) \in R$. Then, by the definition of $\sim_\Sigma$, there is some t' with $(s,t') \in R$ and $\Sigma(t') = \Sigma(u)$. We need to show that $\{\Sigma(v) \mid (t',v) \in R\} = \{\Sigma(v) \mid (u,v) \in R\}$. Let $(t',v) \in R$. Hence, $(s,v) \in R$ by the transitivity of R. It follows from $s \sim_\Sigma t$ that there exists v' such that $(t,v') \in R$ and $\Sigma(v) = \Sigma(v')$. And from $(t,u),(t,v') \in R$ we obtain $(u,v') \in R$ because R is a Euclidean relation. Thus, $\{\Sigma(v) \mid (t',v) \in R\} \subseteq \{\Sigma(v) \mid (u,v) \in R\}$. For the backward inclusion fix some v and assume that $(u,v) \in R$. Hence, $(t,v) \in R$ by the transitivity of R and, consequently, there exists v' such that $(s,v') \in R$ and $\Sigma(v') = \Sigma(v)$. And again, because R is Euclidean we get (t',v'). Thus, $t' \sim_\Sigma u$ and because s,t,u were picked arbitrarily, $\sim_\Sigma \circ R \subseteq R \circ \sim_\Sigma$. $\square$

Lemma 17 If R is transitive and Euclidean then R^Σ is transitive and Euclidean.

Proof. To prove R^Σ is transitive suppose that $(\|s\|,\|t\|),(\|t\|,\|u\|) \in R^\Sigma$. Without loss of the generality we can assume that $(s,t) \in R$, $t \sim_\Sigma t'$ and $(t',u) \in R$. By Lemma 16 there exists a u' such that $(t,u') \in R$ and $\|u'\| = \|u\|$. From the transitivity of R we have $(s,u') \in R$ and, hence, $(\|s\|,\|u\|) \in R^\Sigma$.

To prove that R^Σ is Euclidean suppose that $(\|s\|,\|t\|),(\|s\|,\|u\|) \in R^\Sigma$. Again, without loss of the generality we can assume that $(s,t) \in R$, $s \sim_\Sigma s'$ and $(s',u) \in R$. By Lemma 16 there exists t' such that $(s',t') \in R$ and $\|t'\| = \|t\|$. Because R is Euclidean, we obtain $(t',u') \in R$ and, hence, $(\|t\|,\|u\|) \in R^\Sigma$. $\square$

Lemma 18 If R is transitive and Euclidean then the properties com^l_w and cr_w persist under the filtration of Definition 14.

Proof. Let $(\|s\|,\|t\|) \in Q(a)^\Sigma$ and $(\|t\|,\|u\|) \in R^\Sigma$, where a is an atomic action. By definition of R^Σ and $Q^\Sigma(a)$, there exist s',t',t'',u' such that $s' \sim_\Sigma s$, $t' \sim_\Sigma t'' \sim_\Sigma t$, $u' \sim_\Sigma u$, $(s',t') \in Q(a)$ and $(t'',u') \in R$. By Lemma 16 there is a u'' such that $u' \sim_\Sigma u''$ and $(t',u'') \in R$. Thus, because com^l_w is true in M, we obtain that (s',u'') belongs to $R \circ Q(a)$ and consequently, $(\|s\|,\|u\|) \in R^\Sigma \circ Q^\Sigma(a)$.

To prove the statement of the lemma for cr_w assume that $(\|s\|,\|t\|) \in Q(a)^\Sigma$ and $(\|s\|,\|u\|) \in R^\Sigma$. That is, without loss of the generality, $(s,t) \in Q(a)$, $(s',u) \in R$ and $s' \sim_\Sigma s$. By Lemma 16 there exists a u' such that $(s,u') \in R$ and $u' \sim_\Sigma u$. Because cr_w holds in M, there exists a v such that $(u',v) \in Q(a)$ and $(t,v) \in R$. Thus, $(\|u\|,\|v\|) \in Q^\Sigma(a)$ and $(\|t\|,\|u\|) \in R^\Sigma$. $\square$

Proposition 19 Let L be any of the logics *K45*, *KD45*, or *S5*, and assume $\Delta \subseteq \{PR, CR\}$. Suppose that M is a non-standard model for $PDL \oplus L \oplus \Delta_w$. Then M^Σ is a standard model for $PDL \oplus L \oplus \Delta_w$.

Proof. By construction M^Σ is a standard model for $PDL \oplus K_w$. Hence, we only need to prove that M^Σ satisfies all the semantical properties which correspond to the extra axioms of the logic $PDL \oplus L \oplus \Delta_w$. But M satisfies these properties because it is assumed to be a non-standard model for the logic, and all the properties which correspond to *4*, *5*, *D*, *T*, *PR* and *CR* are persistent under the filtration by Lemmata 15, 17 and 18. $\square$

It is important in the filtration method of Fischer and Ladner that for a finite set Σ any equivalence class $\|s\|$ is formula definable in the original model M. This is where the argument in this paper crucially departs from the argument usually used in proving the completeness of *PDL*. Usually, what is required in proving the completeness of *PDL* and *PDL* related logics is that any union of the equivalence classes in the original model is formula definable. This allows to apply the standard syntactical argument [6] in Lemma 21 below. So, we need to prove the following lemma to make the method work.

Lemma 20 If Σ is finite then each equivalence class $\|s\|$ is associated with a formula $\psi(\|s\|)$ which uniquely determines $\|s\|$ in the original model M.

Proof. Let $\Gamma_0 = \Sigma(s)$ and $\{\Gamma_1, \ldots, \Gamma_k\} = \{\Sigma(u) \mid (s,u) \in R\}$. The set $\mathcal{P}(FL(\Sigma))$ of all subsets of $FL(\Sigma)$ is finite, because Σ is finite. Let $\{\Delta_1, \ldots, \Delta_l\} = \mathcal{P}(FL(\Sigma)) \setminus \{\Gamma_1, \ldots, \Gamma_k\}$. Let γ_0 be the conjunction of formulae from Γ_0 together with the negations of the formulae in $FL(\Sigma) \setminus \Gamma_0$. Similarly, formulae γ_j and δ_j are introduced for Γ_j and Δ_j, respectively. Define $\psi(\|s\|)$ as follows.

$$\psi(\|s\|) =^{\text{def}} \gamma_0 \wedge \bigwedge_{1 \le j \le k} \Diamond\gamma_j \wedge \bigwedge_{1 \le j \le l} \Box\neg\delta_j$$

Now it is routine to check that $\psi(\|s\|)$ is defined correctly and the only equivalence class in which it is true is $\|s\|$, i.e. $\|s\|$ coincides with $\{t \mid t \models \psi(\|s\|)\}$. □

Lemma 21 Let Σ be a finite set of formulae.

1. For all $(0, \phi)$ such that $(0, \phi) \preceq (0, \sigma)$ for some $\sigma \in \Sigma$,

   $\|s\| \models \phi \iff s \models \phi.$

2. For all $(0, \Box\phi)$ such that $(0, \Box\phi) \preceq (0, \sigma)$ for some $\sigma \in \Sigma$,

   if $(\|s\|, \|t\|) \in R^{\Sigma}$ and $s \models \Box\phi$ then $t \models \phi$.

3. For all $(1, [\alpha]\eta)$ such that $(1, [\alpha]\eta) \prec (0, \sigma)$ for some $\sigma \in \Sigma$,

   (a) if $(s,t) \in Q(\alpha)$ then $(\|s\|, \|t\|) \in Q^{\Sigma}(\alpha)$;

   (b) if $(\|s\|, \|t\|) \in Q^{\Sigma}(\alpha)$ and $s \models [\alpha]\eta$ then $t \models \eta$.

This lemma can be proved by induction on the well-founded relation $\prec$ (see [6]), using Lemma 20 in the case 3a for $\alpha = \beta^*$ which is actually the most difficult case in the induction.

Now the Filtration Lemma follows easily.

Corollary 22 (Filtration Lemma) If the set Σ of formulae is finite then for all $\phi \in FL(\Sigma)$, $\|s\| \models \phi \iff s \models \phi$.

As a consequence, with the help of Lemmata 15, 17 and 18, we obtain the following theorem.

Theorem 23 (Small Model Theorem) Let L be any of the logics *K45*, *KD45*, or *S5*, and assume $\Delta \subseteq \{PR, CR\}$. Let ϕ be a formula and $|\phi| = n$. If ϕ is satisfiable in a possibly non-standard model for the logic $PDL \oplus L \oplus \Delta_w$ then it is satisfiable in a standard model for this logic with no more than $2^n \cdot 2^{2^n}$ states.

Proof. Suppose that M is a (non-standard) model for the given logic. By the Filtration Lemma, ϕ is satisfiable in a (non-standard) model M if and only if it is satisfiable in the filtrated model M^Σ whenever $\phi \in FL(\Sigma)$. By Proposition 19, M^Σ is a standard model for the logic. Further, it is easy to see that every state of the filtrated model, i.e. any equivalence class $\|s\|$ is uniquely determined by the pair of sets $\langle \Sigma(s), \{\Sigma(u) \mid (s,u) \in R\}\rangle$. Then there are 2^k possibilities for $\Sigma(s)$ and 2^{2^k} possibilities for $\{\Sigma(u) \mid (s,u) \in R\}$, where $k = \mathsf{Card}(FL(\Sigma))$. Therefore, $\mathsf{Card}(S^\Sigma) \leq 2^k \cdot 2^{2^k}$. Thus, if $\Sigma = \{\phi\}$, then $\mathsf{Card}(S^\Sigma) \leq 2^n \cdot 2^{2^n}$, by Lemma 13. □

Theorem 24 (Completeness) Let L be any of the logics *K45*, *KD45*, or *S5*, and assume $\Delta \subseteq \{PR, CR\}$. Then the logic $PDL \oplus L \oplus \Delta_w$ is complete with respect to the corresponding class of standard models.

Proof. If a formula ϕ is not derivable in any of the logics then the set $\{\neg\phi\}$ is consistent with respect to this logic and, consequently, $\neg\phi$ is satisfiable in the canonical model M for the logic. The canonical model is a non-standard model for the logic and, hence, by the Small Model Theorem $\neg\phi$ is satisfiable in a standard model for the logic. Thus, ϕ is not valid in any standard model of the logic. □

Note that the logics $PDL \oplus K45 \oplus \{PR, NL, CR\}_w$ and $PDL \oplus KD45 \oplus \{PR, NL, CR\}_w$ are not covered by the above theorems. It is open whether these extensions with *NL* have the small model property and are complete with respect to a class of standard models.

4 Extensions of $PDL \oplus L$ with full substitution

This section considers the logics closed under the full substitution rule.

Lemma 25 Let L be a normal unimodal logic. If one of the formulae $\Box(p \to q) \to (p \to \Box q)$, $(p \to \Box q) \to \Box(p \to q)$, or $\Diamond(p \to q) \to (p \to \Diamond q)$ belongs to L, then $p \to \Box p$ belongs to L.

Proof. (i) Assume the first formula belongs to L. As L is closed under the substitution rule, the formula $\Box(p \to p) \to (p \to \Box p)$ belongs to L. The left side of the implication is always true. Thus, this formula is classically equivalent to $p \to \Box p$. (ii) If the second formula holds in L then substitute $\neg q$ for p. So, $(\neg q \to \Box q) \to \Box(\neg q \to q)$ is in L. This formula is equivalent to $(\neg q \to \Box q) \to \Box q$. The following equivalences complete the proof.

$((\neg q \to \Box q) \to \Box q) \leftrightarrow (\neg(q \vee \Box q) \vee \Box q) \leftrightarrow ((\neg q \wedge \neg\Box q) \vee \Box q) \leftrightarrow ((\neg q \vee \Box q) \wedge (\neg\Box q \vee \Box q)) \leftrightarrow (q \to \Box q)$. (iii) Let the third formula be in L. Then $\Diamond(p \to \bot) \to (p \to \Diamond\bot)$ too belongs to L. Hence, we have $\Diamond\neg p \to \neg p$, because $\Diamond\bot \leftrightarrow \bot$ and $(p \to \bot) \leftrightarrow \neg p$. Now observe that $p \to \Box p$ is the contrapositive of $\Diamond\neg p \to \neg p$. □

Corollary 26 Let L be any normal unimodal logic. Then $p \to \Box p$ belongs to $PDL \oplus L \oplus \{NL\}$, $PDL \oplus L \oplus \{PR\}$, and $PDL \oplus L \oplus \{CR\}$.

Proof. Take the axiom NL, $[a]\Box p \to \Box[a]p$, and substitute a test expression $q?$ for a. This gives $[q?]\Box p \to \Box[q?]p$. Since $[q?]p \leftrightarrow (q \to p)$ belongs to PDL, $(q \to \Box p) \to \Box(q \to p)$ is in the logic. In the cases of PR or CR being valid $\Box(q \to p) \to (q \to \Box p)$, resp. $\Diamond(p \to q) \to (p \to \Diamond q)$, is valid in the logic. Now apply Lemma 25. □

From a philosophical perspective the axiom $p \to \Box p$ is not appropriate as an axiom for belief or knowledge. Informally, this axiom says that an agent believes that all properties are true in the current state. But, in general, agents are not aware of everything that is happening in the world and, hence, cannot build a complete picture about the current state. Thus, this axiom expresses some divine property which is uncharacteristic of a typical agent. In the combinations with *S5* the situation is even more dramatic.

We say that a modal logic L *admits the elimination of* $\Box$, if the formula $p \leftrightarrow \Box p$ belongs to L. Let *KT* be the weakest reflexive normal unimodal logic.

Theorem 27 If an extension L of $PDL \oplus KT$ contains any of the axioms *NL*, *PR* or *CR* and the full substitution rule is admissible in L, then L admits the elimination of the $\Box$ modality of L.

Proof. For the proof it is enough to apply Corollary 26 and note that $\Box p \to p$ belongs to *KT*. □

Consequently:

Theorem 28 All possible extensions of $PDL \oplus S5$ by the axioms *NL*, *PR*, and *CR* with the full substitution rule are equal to $PDL \oplus K \oplus \{\Box p \leftrightarrow p\}$ and, hence, deductively equivalent to *PDL*.

Proof. All the axioms of *S5* are derivable from $\Box p \leftrightarrow p$. Thus, $S5 \oplus \{\Box p \leftrightarrow p\} = K \oplus \{\Box p \leftrightarrow p\}$. For the rest it is enough to refer to Theorem 27. □

It follows from this theorem that an analogue of Proposition 4 for the logics under full substitutivity does not hold.

It is not hard to see that in any model for the logic $PDL \oplus K \oplus \{\Box p \leftrightarrow p\}$ the relation R corresponding to the $\Box$-operator is an identity relation on the set of states. Denote by $\mathcal{C}_{triv}$ the class of $PDL \oplus K$ models in which the relation R is an identity relation on the set of states. Then the following theorems can be easily proved.

Theorem 29 (Small Model Theorem) Let ϕ be a formula of $\mathcal{L}$ and let n be the number of symbols in ϕ. If ϕ is satisfiable in a $\mathcal{C}_{triv}$ model then it is satisfiable in a $\mathcal{C}_{triv}$ model with no more than 2^n states.

Theorem 30 (Completeness) $PDL \oplus K \oplus \{\Box p \leftrightarrow p\}$ and, consequently, any extension of $PDL \oplus S5$ by the axioms *NL*, *PR*, and/or *CR* is complete with respect to the class $\mathcal{C}_{triv}$.

That is, all the considered extensions of the fusion of *PDL* and *S5* closed under the full substitution rule collapse to *PDL* and, hence, inherit all the good properties of *PDL* such as completeness, small model property and decidability.

5 Test-free extensions

Omitting the test operator from the considered logics we can prove the following results.

Lemma 32 Let L be any normal unimodal logic. Let P denote one of the properties com^r, com^l, cr. Let M be a standard $PDL \oplus L$ model which satisfies P_w. Then, P is true in M for all test-free actions.

Proof. All the cases can be proved by induction on the length of an action formula α. We only prove the case for $P = com^r$. The base case holds by the assumptions of the lemma. We skip the easy case where $\alpha = \beta \cup \gamma$. For the case $\alpha = \beta;\gamma$, by taking into account the induction hypothesis, we obtain $Q(\beta;\gamma) \circ R = Q(\beta) \circ Q(\gamma) \circ R \subseteq Q(\beta) \circ R \circ Q(\gamma) \subseteq R \circ Q(\beta) \circ Q(\gamma) = R \circ Q(\beta;\gamma)$. For $\alpha = \beta^*$ the argument is similar. $\Box$

By Lemma 32 the two classes of the intended standard models for the test-free weak logics and the corresponding test-free full logics coincide. Thus, repeating the arguments of Section 3 we obtain the following results.

Theorem 36 (Small Model Theorem) Let L be any of the logics *K45*, *KD45*, or *S5*, and let $\Delta \subseteq \{PR, CR\}$. Let ϕ be a formula and $|\phi| = n$. If ϕ is satisfiable in a (non-standard) model for *test-free* $PDL \oplus L \oplus \Delta$ then it is satisfiable in a standard model for this logic with no more than $2^n \cdot 2^{2^n}$ states.

Theorem 37 (Completeness) Let L be any of the logics *K45*, *KD45*, or *S5*, and let $\Delta \subseteq \{PR, CR\}$. Then *test-free* $PDL \oplus L \oplus \Delta$ is complete with respect to the corresponding class of standard models.

Theorem 38 Let L be any of the logics *K45*, *KD45*, or *S5*, and let $\Delta \subseteq \{PR, CR\}$. Then *test-free* $PDL \oplus L \oplus \Delta_w$ admits the rule of full substitution.

6 Logics with informational test

The results of Section 4 show that the formula $p \to \Box p$ is valid in full-substitutional extensions of the fusion of *PDL* and any normal unimodal logic by any of the axioms *NL*, *PR* and *CR*. As pointed out above, for agent applications this property is inappropriate for modelling and reasoning about beliefs of agents. Also, from a logical perspective the elimination of the *S5* operator in the combination of *PDL* and *S5* is unsatisfactory. The reason for the elimination of the *S5* operator is the implicit connection between the test operator and $\Box$ in the commutativity axioms under full substitutivity. This raises the question, whether, and how *PDL* and *S5* can be integrated in a formalism without any of the connectives becoming *trivially* superfluous. One possibility is to rely on the weaker form of substitutivity. Inspection of the proof of Lemma 25 suggests assuming full substitutivity is inappropriate for the commutativity axioms. However, full substitution gives us the possibility to reason uniformly about all actions, in the same way as we reason about all propositions in any logic. *PDL* is closed under the full substitution rule and, thus, fits this paradigm. So, perhaps, the problem is with the definition of the semantics of the test operator when interacting with $\Box$. A solution we propose here is to define an alternative semantics for test such that in the resulting logic, $\Box$ and test interact in a way so that weak substitutivity implies full substitutivity.

Therefore, as replacement for the standard test operator we use a new operator, denoted by ⁇, first defined in [13]. This operator is intended to remedy the problem of the standard test in the presence of an informational modal operator. The new operator is called the *informational test* operator. The intuition of p⁇ is an action which can be successfully accomplished only if p is *believed* (or *known*) in the current state. Thus, p⁇ is the action of confirming the agent's own beliefs (or knowledge). In contrast, with the usual test operator the agent has the capability to confirm truths rather than beliefs (knowledge), which is a strong property of an agent. Thus in agent based applications the new interpretation of the test operator appears more suitable than the classic interpretation.

The logical apparatus is the same as previously with the obvious changes. The symbol ⁇ is used in the superscript to indicate the replacement of the operator ? by ⁇. Let $(PDL \oplus L)^{⁇}$ be the smallest logic in the language $\mathcal{L}^{⁇}$ containing L and the axioms of *PDL*, but the usual test axiom $[p?]q \leftrightarrow (p \to q)$ is replaced by the axiom

$$(*) \qquad [p⁇]q \leftrightarrow \Box(\Box p \to q).$$

In accordance with this axiom, the formula $[p⁇]q$ can be read as 'q is believed with respect to p being believed' (or with 'known' replacing 'believed' as appropriate). Consequently, we may think of the operator $[\_⁇]$ as the modal operator of *relative information*.

A standard (resp. non-standard) $(PDL \oplus L)^{⁇}$ model is a tuple $\langle S, Q, R, \models \rangle$ satisfying all the properties of a standard (resp. non-standard) $PDL \oplus L$ model, except the meaning of ⁇ is specified by:

$$(**) \qquad Q(\phi⁇) =^{\text{def}} \{(s,t) \in R \mid t \models \Box\phi\}.$$

(This property can be easily found from $(*)$ using the SCAN tool [11].) The definitions of standard and non-standard models for the extensions of $(PDL \oplus L)^{⁇}$ by the axioms *NL*, *PR*, and *CR* (with the full or weak substitution rule) are the expected generalisations of the definitions of standard and non-standard $PDL \oplus L$ models, respectively.

The definition of ⁇ still allows the elimination of $\Box$ but this time the elimination is not trivial.

Proposition 39 $\Box p \leftrightarrow [\top ⁇]p \in (PDL \oplus L)^{⁇}$ for any normal unimodal logic L.

Proposition 40 The following formulae are derivable from the axioms of $(PDL \oplus K45)^{⁇}$ with the full or weak substitution rule.

$$[p⁇]\Box q \to \Box[p⁇]q \qquad \Box[p⁇]q \to [p⁇]\Box q \qquad \Diamond[p⁇]q \to [p⁇]\Diamond q$$

To prove the completeness theorem we need to show that the canonical model for the given logic is a non-standard model for this logic. We did not prove this previously because the proof is standard. Because the semantics of the test operator has been modified, it is necessary to show that the canonical models for the considered logics with the informational test operator satisfy the property $(**)$.

Let $\mathfrak{L}$ be a (weak) logic in the language $\mathcal{L}^{⁇}$. The canonical model $M_{\mathfrak{L}} = \langle S_{\mathfrak{L}}, Q_{\mathfrak{L}}, R_{\mathfrak{L}}, \models_{\mathfrak{L}} \rangle$ for the logic $\mathfrak{L}$ is defined as follows. $S_{\mathfrak{L}}$ is a set of all maximal theories Γ in the language $\mathcal{L}^{⁇}$ such that $\Gamma \supseteq \mathfrak{L}$ and $\bot \notin \Gamma$. The accessibility relations $R_{\mathfrak{L}}$ and $Q_{\mathfrak{L}}(\alpha)$ for any action α are defined by

$$(\Gamma, \Delta) \in R_{\mathfrak{L}} \iff \forall \phi\, (\Box\phi \in \Gamma \Rightarrow \phi \in \Delta)$$
$$(\Gamma, \Delta) \in Q_{\mathfrak{L}}(\alpha) \iff \forall \phi\, ([\alpha]\phi \in \Gamma \Rightarrow \phi \in \Delta)$$

The truth relation $\models_{\mathfrak{L}}$ is a membership relation: $\Gamma \models_{\mathfrak{L}} \phi \iff \phi \in \Gamma$.

Lemma 41 Let $\mathfrak{L}$ be arbitrary extension of $(PDL \oplus K)^{⁇}_{w}$. Then the canonical model $M_{\mathfrak{L}}$ satisfies $(**)$, i.e. $Q_{\mathfrak{L}}(\phi⁇) = \{(\Gamma, \Delta) \in R_{\mathfrak{L}} \mid \Delta \models_{\mathfrak{L}} \Box\phi\}$ for any formula ϕ.

Proof. Suppose $(\Gamma, \Delta) \in Q_{\mathfrak{L}}(\phi⁇)$, i.e. $[\phi⁇]\psi \in \Gamma$ implies $\psi \in \Delta$ for any formula ψ. Take an arbitrary ψ such that $\Box\psi \in \Gamma$. The formula $\Box\psi \to \Box(\Box\phi \to \psi)$ is derivable in modal logic K and, therefore, in $\mathfrak{L}$. Hence, from the axiom $(*)$ we obtain that $\Box\psi \to [\phi⁇]\psi$ is in $\mathfrak{L}$. Therefore, Γ contains $[\phi⁇]\psi$ and, consequently, $\Delta \ni \psi$. Thus, $(\Gamma, \Delta) \in R_{\mathfrak{L}}$. Further, $\Box(\Box\phi \to \Box\phi)$ is valid in K, and, hence, $\Box(\Box\phi \to \Box\phi) \in \mathfrak{L} \subseteq \Gamma$. Hence, by the axiom $(*)$ $[\phi⁇]\Box\phi \in \Gamma$ and, therefore, $\Delta \ni \Box\phi$. Thus, $Q_{\mathfrak{L}}(\phi⁇) \subseteq \{(\Gamma, \Delta) \in R_{\mathfrak{L}} \mid \Delta \models_{\mathfrak{L}} \Box\phi\}$.

For the backward inclusion, suppose that $(\Gamma, \Delta) \in R_{\mathfrak{L}}$ and $\Box\phi \in \Delta$. Assume that $[\phi⁇]\psi \in \Gamma$. By use of $(*)$ we obtain $\Box(\Box\phi \to \psi) \in \Gamma$. Therefore, $\Box\phi \to \psi$ is in Δ because $(\Gamma, \Delta) \in R_{\mathfrak{L}}$. Hence, $\psi \in \Delta$ because $\Box\phi \in \Delta$. Consequently, $(\Gamma, \Delta) \in Q_{\mathfrak{L}}(\phi⁇)$. $\Box$

It turns out, we do not need to modify the definition of the Fischer-Ladner closure to apply the filtration method described previously. So, all the definitions of Section 3 are preserved. The proof of the Filtration Lemma is the same as earlier with a slight modification of the induction step for the test actions in the proof of Lemma 21.3 which can be formulated as the following lemma.

Lemma 42 Let $[\xi\text{⁇}]\eta \in FL(\Sigma)$ and $s \models \xi \iff \|s\| \models \xi$ for all $s \in S$. Then

1. if $(s,t) \in Q(\xi\text{⁇})$ then $(\|s\|,\|t\|) \in Q^{\Sigma}(\xi\text{⁇})$;
2. if $(\|s\|,\|t\|) \in Q^{\Sigma}(\xi\text{⁇})$ and $s \models [\xi\text{⁇}]\eta$ then $t \models \eta$.

Proof. For 1, suppose $(s,t) \in Q(\xi\text{⁇})$, then by the definition of the semantics of the new test operator, $(s,t) \in R$ and $t \models \Box\xi$. Hence, $(\|s\|,\|t\|) \in R^{\Sigma}$ by the definition of R^{Σ}. We need to show that $\|t\| \models \Box\xi$. Suppose that $(\|t\|,\|u\|) \in R^{\Sigma}$ for some $\|u\|$. This means that there exist $t' \in \|t\|$ and $u' \in \|u\|$ such that $(t',u') \in R$. By Lemma 16 there exists $u'' \in \|u'\| = \|u\|$ such that $(t,u'') \in R$. Hence, $u'' \models \xi$ and by the assumptions $\|u\| \models \xi$. Thus, $\|t\| \models \Box\xi$ and, consequently, $(\|s\|,\|t\|) \in Q^{\Sigma}(\xi\text{⁇})$.

For 2, let $(\|s\|,\|t\|) \in Q^{\Sigma}(\xi\text{⁇})$ and $s \models [\xi\text{⁇}]\eta$. Hence, $(\|s\|,\|t\|) \in R^{\Sigma}$ and $\|t\| \models \Box\xi$ by the definition of the semantics of the new test operator. Consequently, there are $s' \in \|s\|$ and $t' \in \|t\|$ such that $(s',t') \in R$. Suppose that $(t',u) \in R$ for some u. Then, $(\|t\|,\|u\|) \in R^{\Sigma}$ and, consequently, $\|u\| \models \xi$. By the lemma assumptions, $u \models \xi$, and, hence, $t' \models \Box\xi$. So, we have $(s',t') \in Q(\xi\text{⁇})$. Because $[\xi\text{⁇}]\eta \in FL(\Sigma)$, η also belongs to $FL(\Sigma)$. The rest is simple. We have $s' \models [\xi\text{⁇}]\eta$ because $s \sim_{\Sigma} s'$ and, hence, $t' \models \eta$. And from $t \sim_{\Sigma} t'$ we conclude that $t \models \eta$. ☐

Applying the filtration technique and using Lemma 32 and Proposition 40 we can obtain completeness, the small model property and the admissibility of the full substitution rule.

Theorem 43 (Small Model Theorem) Let L be *K45*, *KD45*, or *S5*. Let ϕ be a formula and $|\phi| = n$. If ϕ is satisfiable in a (non-standard) model for the logic $(PDL \oplus L)^{\text{⁇}}$, $(PDL \oplus L \oplus \{PR\})^{\text{⁇}}$, $(PDL \oplus L \oplus \{CR\})^{\text{⁇}}$ or $(PDL \oplus L \oplus \{PR, CR\})^{\text{⁇}}$ then it is satisfiable in a standard model for this logic with no more than $2^n \cdot 2^{2^n}$ states.

Theorem 44 (Completeness) Let L be *K45*, *KD45*, or *S5*. Then $(PDL \oplus L)^{\text{⁇}}$ and any its extension by the axioms *PR* and/or *CR* is complete with respect to the corresponding class of standard models.

Theorem 45 Let L be *K45*, *KD45*, or *S5*. Then any extension of $(PDL \oplus L)^{\text{⁇}}$ by the axioms *PR* and/or *CR* with the weak substitution rule admits the rule of full substitution.

Finally let us consider how the standard test operator of *PDL* relates to the informational test operator. It turns out that there is a simulation of *PDL* in $(PDL \oplus S5)^{\text{⁇}}$. Define the translation mapping σ from formulae of *PDL* to $\mathrm{For}^{\text{⁇}}$ by the following:

$$\sigma p = \Box p \qquad \sigma\bot = \bot \qquad \sigma a = a$$

$$\sigma(\psi?) = (\sigma\psi)\text{⁇} \qquad \sigma(\alpha \cup \beta) = \sigma\alpha \cup \sigma\beta \qquad \sigma(\alpha;\beta) = \sigma\alpha;\top\text{⁇};\sigma\beta$$

$$\sigma(\alpha^*) = (\sigma\alpha;\top\text{⁇})^* \qquad \sigma(\phi \to \psi) = \Box(\sigma\phi \to \sigma\psi) \qquad \sigma([\alpha]\psi) = \Box[\sigma\alpha]\sigma\psi$$

Theorem 46 For any $\Box$-free formula ϕ in $\mathcal{L}$, $\phi \in PDL$ iff $\sigma\phi \in (PDL \oplus S5)^{\text{⁇}}$.

Proof. Only the idea of the proof is given. It is easy to see that any model of *PDL* can be extended to a $(PDL \oplus \mathrm{S5})^{?}$ model such that R is the identity relation on the set of states. In any state s of such a model, any formula ϕ is valid if and only if $\Box\phi$ is valid in s. This proves the right to left direction of the theorem.

For the opposite direction, we observe that in any $(PDL \oplus \mathrm{S5})^{?}$ model $M = \langle S, Q, R, \models \rangle$, R is an equivalence relation. Now, define a *PDL* model $M' = \langle S', Q', \models' \rangle$ as follows. Let $\|s\| =^{\mathrm{def}} \{t \in S \mid (s,t) \in R\}$, $S' =^{\mathrm{def}} \{\|s\| \mid s \in S\}$, and

$$(\|s\|, \|t\|) \in Q(\alpha) \iff \exists s_0 \in \|s\| \, \exists t_0 \in \|t\| \, (s_0, t_0) \in Q(\alpha),$$
$$M', \|s\| \models' p \iff M, s \models \Box p \quad \text{for any } p \in \mathrm{Var}.$$

Now it is routine to check that $M', \|s\| \models' \phi$ iff $M, s \models \sigma\phi$, for any *PDL*-formula ϕ. $\Box$

7 Conclusion

The analysis in this paper of logics from agent-based systems with commuting dynamic and informational modalities provides some new results of completeness, the small model property and decidability for extensions of fusions of *PDL* with normal modal logics by the axioms *PR* and *CR*. The work also provides new insights into the interplay between the substitution rule, the test operator, the informational modalities and their interaction with action modalities. We conclude by mentioning some open problems arising from this work. One open problem already mentioned, is the completeness, the small model property and decidability of combinations of *PDL* and *K(D)45* with the *NL* axiom. A completeness result for the logics with all of the axioms *NL*, *PR* and *CR* has special significance in connection with the axiomatisation of products of *PDL* and modal logics. The threesome of the axioms *NL*, *PR* and *CR* is normally needed for axiomatising a product of two modal logics [4, 5]. For the axiomatisation of the product of *PDL* and *S5* it is however enough to consider any of the following pairs of the axioms $\{NL, PR\}$ or $\{CR, PR\}$. This is a consequence of Proposition 3. Now it follows from [4, Section 5.1] and from the completeness theorem for $PDL \oplus S5 \oplus \{PR, CR\}_w$ (Theorem 24) that the logic $PDL \oplus S5 \oplus \{PR, CR\}_w$ coincides with the corresponding product of the logics, in other words, the pair $(PDL, S5)$ is product matching [5].[1] Whether the pair $(PDL, K(D)45)$ is also product matching is still an open question. To prove this it would be enough to apply the same argument from [4, Section 5.1] as for $PDL \oplus S5 \oplus \{PR, CR\}_w$, provided it is possible to show the logic $PDL \oplus K(D)45 \oplus \{NL, PR, CR\}_w$ is complete with respect to the corresponding class of models satisfying com^r, com^l and *cr*.

The products of *PDL* (with converse) and normal modal logics have also been investigated in [15] where the quasi-model technique is applied to show the decidability of the logics. Later in [4], this method is used to show the decidability of the products of *PDL* with *S5* and *KD45*. This gives another proof of the decidability of $PDL \oplus S5 \oplus \{PR, CR\}_w$.

[1] Agnes Kurucz pointed this out to us and showed us a proof.

Acknowledgements

We are grateful to the anonymous referee of the short version of this report (which is to appear in *Advances in Modal Logic*, Volume 4, World Scientific Publishing) for detailed comments and thank Ullrich Hustadt and Agnes Kurucz for valuable discussions and suggestions. The work is supported by research grant GR/M88761 from the UK Engineering and Physical Sciences Research Council.

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