Layout Area of the Hypercube

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Abstract

In this paper we study the square grid area required for laying out H_l, the Boolean hypercube of $N = 2^l$ vertices. It is shown that this area is $\frac{4}{9}N^2 + o(N^2)$. We describe a layout which occupies this much area and prove that no layout of less area exists.

1 Introduction

The Boolean hypercube is one of the most important network architectures for parallel computing; see, for example, Leighton [7]. For implementing this architecture one needs to lay it on a chip, and the less area it occupies the faster the system works. A common model for this problem is that of Thompson [10, 11], in which one looks for a way to lay out the network on a square grid using as little are as possible.

In this paper we study the square grid area required for laying out H_l, the Boolean hypercube of $N = 2^l$ vertices. It is shown that this area is $\frac{4}{9}N^2 + o(N^2)$. We describe a layout which occupies this much area and prove that no layout of less area exists.

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In the grid layouts we consider, each hypercube vertex is represented by a square box of l rows and l columns, and every two such boxes are disjoint. Each hypercube edge is represented by a grid path which starts on the boundary of a box representing one endpoint of the hypercube edge and ends on the boundary of the box representing the other endpoint. Two grid paths which represent two hypercube edges are disjoint; they share no grid edges. They may share a grid point, but in this case they cross at this point without bending; i.e. no knock-knees are allowed.

The suggested layout uses a square array of the boxes. Each row or column represents the vertices of a subcube of $\sqrt{N}$ vertices, and in between the rows and columns the paths representing the edges of that subcube are routed. Our layout is similar to those in [12, 13] and [8], except that in their arrangements the vertices in each row and column appear in the natural order of the vertex labels, while in ours the order of the labels is in Gray code. Our layout requires, essentially, the same area as the ones quoted above, and we include its description for the sake of self-containment.

Contrary to previous publications, we provide a tight lower bound. The proof of the lower bound on the area uses concepts suggested by Thompson [10, 11]. However, instead of using a bisection, in which the set of vertices is divided into two equal parts, we use a dissection into two parts, with the ratio 1:2. We prove that for the 1:2 ratio a minimum cut contains the maximum number of edges, and this number is $\frac{2}{3}N$. Thus, the area of a rectangle encompassing the layout is at least of area $\frac{4}{9}N^2$. The previously known lower bound was $\frac{1}{4}N^2$.

The remainder of the paper is organized as follows. In Section 2 our layout of the hypercube is presented and the upper bound is derived. In Section 6 our lower bound is proved. In this proof we use results developed in the sections in between:

- In Section 3 some definitions are presented, including that of completely decomposable graphs, of which the hypercube is a special case.
- In Section 4 an upper bound of the number of edges in completely decomposable graphs is shown.
- In Section 5 the size of a max-min cut of the hypercube is studied.

2 A Compact Layout of The Hypercube

In this section we present a compact grid layout of the l-dimensional binary hypercube H_l. Following Thompson [10, 11], each vertex occupies a square box of area $l \times l$ and its l incident edges are represented by grid paths; each of these paths has one end on the boundary of the box which represents the vertex.

We begin our description with a layout in which all boxes are laid out in a linear array, side by side, and each of the paths representing the edges has two bends; it starts with a vertical section, continues with a horizontal one and ends with a vertical section. Each of the 2^l vertices (boxes) has a label, which is a binary word of length l, and every two vertices whose labels differ in one bit are connected by an edge (path).

A Gray code [3, 2] is an ordering of the 2^l binary words of length l such that two adjacent words have only one bit which is different. One particular such code, C_l, is called the *reflected Gray code* and it can be defined recursively as follows.

$$C_1 = (0, 1)$$

If $C_i = (w_1^i, w_2^i, \ldots, w_{2^i}^i)$ then

$$C_{i+1} = (0w_1^i, 0w_2^i, \ldots, 0w_{2^i}^i, 1w_{2^i}^i, 1w_{2^i-1}^i, \ldots, 1w_2^i, 1w_1^i).$$

Thus,

$$C_2 = (00, 01, 11, 10),$$
$$C_3 = (000, 001, 011, 010, 110, 111, 101, 100),$$

etc.

Next, we describe a recursive (linear array) layout of H_l for even l. A similar construction exists for odd l.

The grid layout of H_2, in a linear fashion, is shown in Figure 1. Note that this layout uses two horizontal tracks to wire all four paths (edges).

Now, given a layout of H_i, we get a layout of H_{i+2} by the following steps (the layout of H_4 is shown in Figure 2):

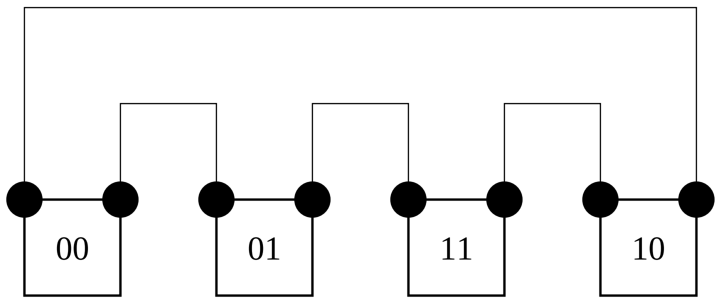

Figure 1: The Linear Layout of H_2

- Draw 2^{i+2} square boxes, side by side, where each side of a square extends over $i+2$ grid-lines. Thus, the top side of a box has $i+2$ terminals and can accommodate $i+2$ incident edges (paths).

- In the first section of 2^i boxes, use in each box the first i terminals on its left hand side, and copy the connections from the layout of H_i. In the second section of 2^i boxes, use in each box the i middle terminals, leaving the first and the last free for the edges of the $(i+1)$'st and $(i+2)$'nd dimensions, respectively, and for the middle terminals copy the layout of H_i. The connections in the third section are a reflection of those of the second section, and the connections in the fourth section are a reflection of the first section.

- For the connections of the $(i+1)$'st dimension (those which correspond to connections between two boxes whose names differ in the second most significant bit) use 2^i new layers. In the first section, in each box, use the second terminal from the right hand side of the box. In the second section, use the left most terminal of each box. In the first new layer connect the terminal of box 0^{i+2} with that of 010^i. In the second new layer, connect the terminal of $0^{i+1}1$ with that of $010^{i-1}1$, etc. In the 2^i'th new layer connect box 0010^{i-1} with its neighbor 0110^{i-1}. (In fact, this last connection can be made on an old layer, but this greediness, in the end, will not yield any saving.) The connections of the $(i+1)$'st dimension, in the second half of the array, is a reflection of those in the first. Note that in each half, the connections of the $(i+1)$'st dimension

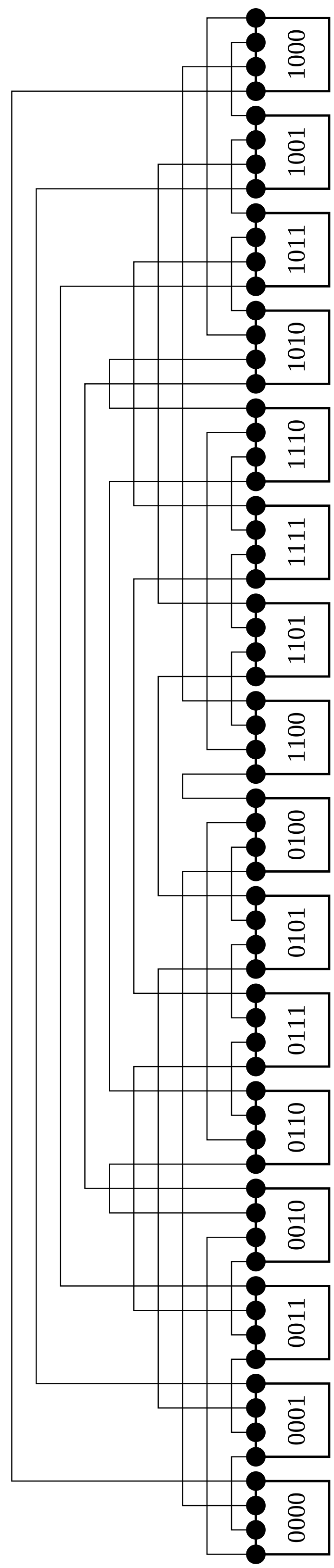

Figure 2: The Linear Layout of H_4

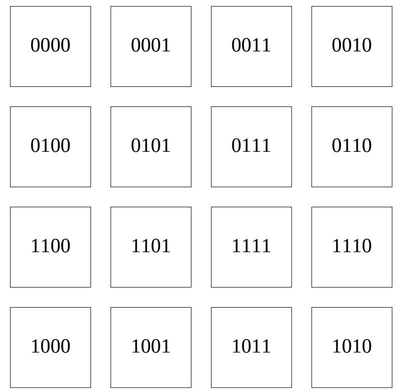

Figure 3: The Vertices Placement of H_4

form an upright pyramid.

- For the connections of the $(i+2)$'nd dimension, in each of the first 2^{i+1} boxes use the right most terminal and in the last 2^{i+1} boxes use the left most terminals. For the 2^{i+1} connections use one layer for each connection. The first 2^i layers will be shared with the connections of the $(i+1)$'st dimension, and 2^i additional new layers will be used. These 2^{i+1} connections form an inverted pyramid: In the first layer connect box 010^i with 110^i. In the second, box $010^{i-1}1$ with $110^{i-1}1$, etc. In the last layer connect box 0^{i+2} with 10^{i+1}.

Next, a two dimensional arrangement of the boxes is described. To simplify our presentation, assume l is divisible by 4. In our layout the $N = 2^l$ vertices of H_l are arranged in a square matrix; $2^{\frac{l}{2}}$ rows and $2^{\frac{l}{2}}$ columns. Gaps are left between the rows and the columns which are used for routing the paths which represent the edges of H_l. (see Figure 3 for $l = 4$.)

The vertices of H_l are labeled with binary words of length l. We will refer to the vertices by these labels. Clearly, two vertices are connected by an edge if and only if their labels differ in exactly one of the l bits.

In each row, all $2^{\frac{l}{2}}$ vertices have the same prefix (of their labels) of length $\frac{l}{2}$, and all vertices in each column have the same suffix of length $\frac{l}{2}$. The order the vertices are arranged in each row is such that the suffixes form a reflected

Gray code. The order the vertices are arranged in each column is such that the prefixes form a reflected Gray code.

Between every two rows of boxes we need to leave enough rows, required in the linear arrangement of $H_{\frac{l}{2}}$, and similarly, between every two columns of boxes we need to leave enough columns which serve as tracks for the connections required for the linear arrangement of $H_{\frac{l}{2}}$, only rotated 90^o. Note that we could have used boxes of size $\frac{l}{2} \times \frac{l}{2}$, since we need only $\frac{l}{2}$ edges on top of each box and on one of its sides. However, this saving of area is negligible in the asymptotic computation, and we shall stay with the recommendation of Thompson, and use $l \times l$ boxes.

Now, let us compute the area this layout requires. We begin by calculating the number of rows used in the linear arrangement of H_l.

There are l rows occupied by the boxes. Let us denote the number of tracks (rows) used for laying out the paths (edges of H_l) by $r(l)$. Clearly, $r(2) = 2$ and

$$r(i+2) = r(i) + 2^{i+1}.$$

Thus,

$$r(i) = \frac{2}{3}(2^i - 1).$$

The two dimensional arrangement of H_l, where l is divisible by 4, uses an area $A(l)$, computed as follows: The total number of rows is

$$2^{\frac{l}{2}}[l + \frac{2}{3}(2^{\frac{l}{2}} - 1)],$$

and therefore,

$$A(l) = 2^l[l + \frac{2}{3}(2^{\frac{l}{2}} - 1)]^2.$$

which is easily shown to yield

$$A(l) = \frac{4}{9}2^{2l} + o(2^{2l}).$$

The discussion above proves the following Theorem:

Theorem 1 *The area required for laying out the hypercube of N vertices in the square grid is bounded from above by*

$$\frac{4}{9}N^2 + o(N^2). \tag{1}$$

3 Definitions

The graphs we deal with are all undirected, finite and have no self-loops.

- Let $G(V, E)$ be a graph. Let $S \subset V$. The set of edges $C \subseteq E$, which have one endpoint in S and the other in $V - S$, is called a *cut*.
- A cut C is *matched* if no two edges of C share an endpoint.
- A cut C is *minimal* if no proper subset of C is a cut.
- A graph G is *decomposable* if it has a matched cut.
- A graph G is *completely decomposable* if every subgraph of G (including G itself) is decomposable. It follows that a completely decomposable graph has no parallel edges.
- For a nonnegative integer i, $\omega(i)$ is the number of ones in the binary representation, $B(i)$, of i.
- For a nonnegative integer k, $g(k)$ is defined by:

  $$g(0) = 0$$

  and for $k > 0$,

  $$g(k) = \omega(0) + \omega(1) + \cdots + \omega(k-1). \tag{2}$$

- Let α and β be binary words of length l. β *covers* α if

  $$\forall 0 \leq i < l, \quad \alpha[i] = 1 \;\Rightarrow\; \beta[i] = 1.$$

- Let α be a binary word of length l. The *inverse* word of α, $\bar{\alpha}$, is defined by:

  $$\forall 0 \leq i < l, \quad \bar{\alpha}[i] = 1 - \alpha[i].$$

  Notice that if β covers α then $\bar{\alpha}$ covers $\bar{\beta}$.

- Let M and M' be sets of binary words of length l such that $|M| = |M'|$. M' *covers* M if there is a bijection $\varphi : M \to M'$ such that for every $\alpha \in M$, $\varphi(\alpha)$ covers α.

- For a set of words M, the *inverse* set $\bar{M}$ of M is defined by:

$$\bar{M} = \{\bar{\alpha} \;\mid\; \alpha \in M\}.$$

Notice that if M' covers M then $\bar{M}$ covers $\bar{M}'$.

4 An Upper Bound on the Number of Edges of a Completely Decomposable Graph

The material of this section is based on the work of R. L. Graham [4] and on the Master's thesis of Kupershtok [6]. An alternate paper on the subject is due to Hart [5]. For the sake of self-containment, we present complete proofs of all claims. (We believe that there is a mistake in [4], specifically, in the proof of his Lemma 2.)

Theorem 2 *For every completely decomposable graph G of n vertices*

$$|E(G)| \leq g(n).$$

Before we present the proof of Theorem 2 we prove two lemmas which are used in that proof.

Lemma 1 *Let r and s be nonnegative integers, l defined by $2^{l-1} \leq s+r < 2^l$ and for $0 \leq k \leq s+r$, $B(k)$ is the binary representation of k in l bits. If*

$$M = \{B(0), B(1), \ldots, B(r)\} \quad \textit{and} \quad M' = \{B(s), B(s+1), \ldots, B(s+r)\}$$

then M' covers M.

Proof: The proof is by induction on r. For $r = 0$ the claim is trivial. Suppose the lemma holds for values less than r.

If $M \cap M' \neq \Phi$ then for every $B(i) \in M \cap M'$ let $\varphi(B(i)) = B(i)$ and trivially, $\varphi(B(i))$ covers $B(i)$. By the induction hypothesis one can complete the definition of φ so that $M' - M$ covers $M - M'$. Thus, we may assume that $M \cap M' = \Phi$.

Let p be a positive integer such that $2^p \leq r+1 < 2^{p+1}$.

Case 1: $r+1 = 2^p$
Every binary word of length p occurs exactly once as a suffix of a word in M and as a suffix of a word in M'. Let us define a bijection $\varphi : M \to M'$ such that for every $B(i) \in M$, $\varphi(B(i))$ and $B(i)$ have the same suffix of length p. Every word in M has a prefix 0^{l-p}. Thus $\varphi(B(i))$ covers $B(i)$.

Case 2: $r+1 > 2^p$

Case 2a: There exist two consecutive words $B(s+i)$ and $B(s+i+1)$ $(0 \leq i < r)$ in M' such that $B(s+i)[p] = 1$ and $B(s+i+1)[p] = 0$.
Notice that the suffix of length $p+1$ of $B(s+i)$ is 1^{p+1}. The suffix of length $p+1$ of $B(s+i+1)$ is 0^{p+1}.
Let N be the subset of M consisting of $B(0), B(1), \ldots, B(r-i-1)$ and N' be the subset of M' consisting of $B(s+i+1), B(s+i+2), \ldots, B(s+r)$. Notice that for every $0 \leq j \leq r-i-1$, the suffix of length $p+1$ of $B(j)$ is equal to that of $B(s+i+1+j)$. Thus, N' covers N.
Let Q denote the set of suffixes of length $p+1$ of words in $M-N$, and Q' the suffixes of length $p+1$ of words in $M'-N'$. Notice that as $\bar{Q}'$ contains the word 0^{p+1}. By the inductive hypothesis $\bar{Q}$ covers $\bar{Q}'$. Thus Q' covers Q. Since the prefix of every word in $M-N$ is 0^{l-p-1}, it follows that $M'-N'$ covers $M-N$.

Case 2b: $B(s)[p] = B(s+1)[p] = \cdots = B(s+i)[p] = 0$
while $B(s+i+1)[p] = 1$.
Clearly, $i < 2^p$, the suffix of length $p+1$ of $B(s+i)$ is 01^p and that of $B(s+i+1)$ is 10^p.

Let $k = 2^p - i - 1$. Define N to be the subset of M consisting of $B(k), B(k+1), \ldots, B(r)$ and N' be the subset of M' consisting of $B(s), B(s+1), \ldots, B(s+r-k)$. Notice that the suffix of length $p+1$ of $B(k+j)$ and $B(s+j)$, for $0 \leq j \leq r-k$ are identical. Since the prefix of every word in N is 0^{l-p-1}, N' covers N.

By the inductive hypothesis, $M'-N'$ covers $M-N$. ■

Let r, s, M and M' be as in Lemma 1. Let us extend the definition of φ: For a nonnegative integer k, let $\varphi(k)$ stand for the integer j such that

$\varphi(B(k)) = B(j)$. For a bijection $\varphi : M \to M'$ define $\delta(\varphi)$ by

$$\delta(\varphi) = \min_{0 \le k \le r} \{\omega(\varphi(k)) - \omega(k)\}. \tag{3}$$

Lemma 2 *For r, s, M and M' as above, the following holds:*

- *There exists a φ such that $\delta(\varphi) \ge 0$.*
- *If $s > r$ then there exists a φ such that $\delta(\varphi) \ge 1$.*

Proof: By choosing φ as in Lemma 1 the first part of the Lemma is trivial.

If $s > r$ then $M \cap M' = \Phi$. Again, choose φ as in Lemma 1. Since the integer represented by $\varphi(i)$ is greater than i, it follows that $\omega(\varphi(i)) > \omega(i)$. Hence, the second part holds. ■

Proof: (Theorem 2) The proof is by induction on n. For $n = 1, 2$ the theorem holds. For $n = 3$, K_3 is the only connected graph with 3 vertices and 3 edges. Since K_3 is not decomposable the theorem holds for $n = 3$.

Suppose the theorem holds for all graphs with less than n vertices. Let G be a completely decomposable graph with n vertices and m edges. G has a matched cut C. After removing C from G, we are left with two nonempty and disjoint subgraphs, G_1 and G_2. For $i = 1, 2$ let $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$. W.l.o.g. suppose that $n_1 \le n_2$. Since C is a matching between $V(G_1)$ and $V(G_2)$, $|C| \le n_1$. Thus,

$$m_1 + m_2 + n_1 \ge m. \tag{4}$$

G_1 and G_2 are completely decomposable. By the inductive hypothesis, $g(n_1) \ge m_1$ and $g(n_2) \ge m_2$. By Equation 4 one gets

$$g(n_1) + g(n_2) + n_1 \ge m. \tag{5}$$

By Lemma 2, there exists a bijection:

$$\varphi : \{B(0), B(1), \ldots, B(n_1 - 1)\} \to \{B(n_2), B(n_2 + 1), \ldots, B(n_2 + n_1 - 1)\}$$

such that $\delta(\varphi) \ge 1$. Thus, for every $0 \le k \le n_1 - 1$

$$\omega(\varphi(k)) - \omega(k) \ge 1. \tag{6}$$

By Equation 2

$$g(n_1+n_2)-g(n_2)=\sum_{k=n_2}^{n_2+n_1-1}\omega(k)=\sum_{k=0}^{n_1-1}\omega(\varphi(k)).$$

By Equation 6

$$\sum_{k=0}^{n_1-1}\omega(\varphi(k))\geq\sum_{k=0}^{n_1-1}(\omega(k)+1)=g(n_1)+n_1.$$

Thus,

$$g(n_1+n_2)-g(n_2)\geq g(n_1)+n_1$$

and, using Equation 5,

$$g(n)=g(n_1+n_2)\geq g(n_1)+g(n_2)+n_1\geq m.$$

■

Let H_l denote the Boolean hypercube of dimension l. Clearly, H_l is completely decomposable.

Theorem 3 *For every $1\leq n\leq 2^l$ there exists a subgraph of H_l, $G(V,E)$, of n vertices, such that*

$$|E|=g(n).$$

Proof: Let

$$N=\{B(0),B(1),....,B(n-1)\}.$$

By definition, $g(n)$ is equal to the sum total of ones in the words of N. Also, N denotes a set of vertices of H_l. Let $G(V,E)$ be the subgraph of H_l induced by N. Let us show a bijection between the ones which appear in words of N and E.

1. Consider $B(k)$, $1\leq k<n$, and assume the i-th bit of $B(k)$ is 1. Clearly, the i-th bit of $B(k-2^i)$ is 0, while the remaining bits are the same in $B(k)$ and $B(k-2^i)$. Thus, in H_l there is an edge between the vertices $B(k)$ and $B(k-2^i)$. Thus, every one which appears in a word in N corresponds to an edge in G.

2. Assume there is an edge between $B(p)$ and $B(q)$ in G, and w.l.o.g. $p > q$. There is exactly one bit, say in the i-th place, in which the words differ; in $B(p)$ it is 1 and in $B(q)$ it is 0. Thus, this edge is the image of the one in the i-th bit of $B(p)$, as in 1.

It follows that

$$|E| = g(n).$$

■

5 On the Size of a Max-Min Cut of the Hypercube

We remind the reader that H_l is the hypercube of dimension l, with $N = 2^l$ vertices. If $S \subseteq V(H_l)$ then $\bar{S} = V(H_l) - S$. Let $(S : \bar{S})$ denote the set of edges which have one endpoint in S and the other in $\bar{S}$. We also refer to $(S : \bar{S})$ as the cut induced by S.

For proving our lower bound on the area required for laying out the hypercube, following Thompson's ideas, we need to know the following: For a given $1 \leq n \leq 2^l$, what is the minimum number of edges in a cut $(S : \bar{S})$ such that $|S| = n$, and for which n this minimum is maximum? Surprisingly, the value for which the maximum is achieved is $n \approx \frac{N}{3}$, in contrast to the classical use of a bisection (a cut which splits the set of vertices into two equal parts). For the case of even l, the value of the max-min is stated in Theorem 4.[1]

Theorem 4 *If l is even then*

$$\max_{1 \leq n \leq 2^l} \min_S \{|(S : \bar{S})| : S \subseteq V(H_l) \wedge |S| = n\} = \frac{2}{3}(2^l - 1).$$

For the proof of Theorem 4 we need a few lemmas.

[1]These results are equivalent to the *cutwidth* of the hypercube; see Bezrukov et al. [1]. Our exposition, based on Kupershtok [6], is different and is presented here for the sake of completeness.

Lemma 3 *Let $l \geq 2$, $0 \leq n < 2^l$ and $B(n) = b_{l-1} \cdots b_1 b_0$.*

$$g(n) = \sum_{i=0}^{l-1} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l-1} b_j]. \tag{7}$$

Proof: The proof is by induction on l. For $l = 2$ it is easily verified that Equation 7 yields the right values for $0 \leq n < 4$. Suppose the lemma is true for all n where $0 \leq n < 2^l$. Let $0 \leq \tilde{n} < 2^{l+1}$ and $B(\tilde{n}) = b_l b_{l-1} \cdots b_1 b_0$. We need to prove that

$$g(\tilde{n}) = \sum_{i=0}^{l} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l} b_j]. \tag{8}$$

Notice that for $0 \leq \tilde{n} < 2^l \quad b_l = 0$ and Equation 8 holds by the inductive hypothesis.

Now assume $2^l \leq \tilde{n} < 2^{l+1}$.
Since $0 \leq \tilde{n} - 2^l < 2^l$, $B(\tilde{n} - 2^l) = 0 b_{l-1} \cdots b_1 b_0$. By the inductive hypothesis,

$$g(\tilde{n} - 2^l) = \sum_{i=0}^{l-1} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l-1} b_j]. \tag{9}$$

Notice that for $2^l \leq k \leq \tilde{n}$, the most significant bit in $B(k)$ (in $l + 1$ bits) is one. Thus,

$$g(\tilde{n}) = g(2^l) + g(\tilde{n} - 2^l) + (\tilde{n} - 2^l).$$

Since $g(2^l) = l \cdot 2^{l-1}$,

$$g(\tilde{n}) = l \cdot 2^{l-1} + g(\tilde{n} - 2^l) + (\tilde{n} - 2^l).$$

By Equation 9,

$$g(\tilde{n}) = l \cdot 2^{l-1} + \sum_{i=0}^{l-1} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l-1} b_j] + (\tilde{n} - 2^l).$$

Clearly

$$\tilde{n} - 2^l = \sum_{i=0}^{l-1} b_i \cdot 2^i,$$

and since $b_l = 1$

$$g(\tilde{n}) = l \cdot 2^{l-1} + \sum_{i=0}^{l-1} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l} b_j],$$

and hence Equation 8 holds. ■

Let us define the function $e(n)$ by

$$e(n) = n \cdot l - 2 \cdot g(n). \tag{10}$$

Lemma 4 *If $S \subset V(H_l)$ and $n = |S|$ then $|(S : \bar{S})| \geq e(n)$.*

Proof: Let $G(S, E)$ be the subgraph of H_l induced by S. In H_l, the number of edge incidences on vertices of S is $n \cdot l$. However, each edge of E consumes two such incidences. Thus,

$$|(S : \bar{S})| = n \cdot l - 2 \cdot |E|.$$

Since G is completely decomposable, by Theorem 2, $|E| \leq g(n)$, and the lemma follows. ■

Lemma 5 *Let $B(n) = b_{l-1} \cdots b_1 b_0$.*

$$e(n) = \sum_{i=0}^{l-1} b_i \cdot 2^i[l - i - 2 \cdot \sum_{j=i+1}^{l-1} b_j]. \tag{11}$$

Proof:

By Equation 10

$$e(n) = n \cdot l - 2 \cdot g(n).$$

By Equation 7

$$\begin{aligned} e(n) &= l \cdot \sum_{i=0}^{l-1} b_i \cdot 2^i - 2 \cdot \sum_{i=0}^{l-1} b_i \cdot 2^{i-1}[i + 2 \cdot \sum_{j=i+1}^{l-1} b_j] \\ &= \sum_{i=0}^{l-1} b_i \cdot [l \cdot 2^i - 2^i(i + 2 \cdot \sum_{j=i+1}^{l-1} b_j)] \\ &= \sum_{i=0}^{l-1} b_i \cdot 2^i[l - i - 2 \cdot \sum_{j=i+1}^{l-1} b_j)]. \end{aligned}$$

■

Lemma 6 *For an even l and $n = \frac{1}{3}(2^l - 1)$, $e(n) = \frac{2}{3}(2^l - 1)$.*

Proof: Observe that $B(n) = 0101 \cdots 01$. Thus, the values of i for which $b_i = 1$ are given by $i = 2k$, where $k = 0, 1, \ldots, \frac{l-2}{2}$. Substituting into Equation 11, one gets

$$\begin{aligned} e(n) &= \sum_{k=0}^{\frac{l-2}{2}} 2^{2k}[l - 2k - 2\frac{l-2k-2}{2}] \\ &= \sum_{k=0}^{\frac{l-2}{2}} 2^{2k} \cdot 2 \\ &= 2 \cdot \sum_{k=0}^{\frac{l-2}{2}} 4^k \\ &= 2 \cdot \frac{4^{\frac{l}{2}} - 1}{4 - 1} \\ &= \frac{2}{3}(2^l - 1). \end{aligned}$$

■

Lemma 7 *For an even l and $0 \leq n < 2^l$*

$$e(n) \leq \frac{2}{3}(2^l - 1).$$

Proof: The proof is by induction on l. For $l = 2$, $\frac{2}{3}(2^l - 1) = 2$. And indeed, for $n = 0, 1, 2, 3$, $e(n) \leq 2$.

Now suppose the lemma holds for l, and let us show it holds for $l + 2$. Let $0 \leq \tilde{n} < 2^{l+2}$, where $B(\tilde{n}) = b_{l+1}b_l \cdots b_1 b_0$. We need to show that

$$e(\tilde{n}) \leq \frac{2}{3}(2^{l+2} - 1).$$

By Lemma 5

$$e(\tilde{n}) = \sum_{i=0}^{l+1} b_i \cdot 2^i \cdot [l + 2 - i - 2 \cdot \sum_{j=i+1}^{l+1} b_j]$$

$$
\begin{aligned}
&= b_{l+1} \cdot 2^{l+1} \cdot [l + 2 - (l+1) - 2 \cdot 0] \\
&\quad + b_l \cdot 2^l \cdot [l + 2 - l - 2 \cdot b_{l+1}] \\
&\quad + \sum_{i=0}^{l-1} b_i \cdot 2^i \cdot [l + 2 - i - 2 \cdot (b_{l+1} + b_l + \sum_{j=i+1}^{l-1} b_j] \\
&= b_{l+1} \cdot 2^{l+1} + b_l \cdot 2^l \cdot (2 - 2b_{l+1}) \\
&\quad + \sum_{i=0}^{l-1} b_i \cdot 2^i \cdot [l - i - 2 \cdot \sum_{j=i+1}^{l-1} b_j] \\
&\quad + \sum_{i=0}^{l-1} b_i \cdot 2^i \cdot [2 - 2 \cdot (b_{l+1} + b_l)].
\end{aligned}
$$

Thus, by the inductive hypothesis,

$$
\begin{aligned}
e(\tilde{n}) &\leq 2^{l+1}(b_{l+1} + b_l(1 - b_{l+1})) + \frac{2}{3}(2^l - 1) \\
&\quad + \sum_{i=0}^{l-1} b_i \cdot 2^{i+1} \cdot (1 - (b_{l+1} + b_l)). \qquad (12)
\end{aligned}
$$

Case 1: $b_{l+1} = b_l = 0$.
In this case

$$
\begin{aligned}
e(\tilde{n}) &\leq \frac{2}{3}(2^l - 1) + 2n \\
&\leq \frac{2}{3}(2^l - 1) + 2(2^l - 1) \\
&= \frac{8}{3}(2^l - 1) \\
&< \frac{2}{3}(2^{l+2} - 1).
\end{aligned}
$$

Case 2: $b_{l+1} = 1$ or $b_l = 1$, or both.
Notice that, in this case, the last term of Equation 12 is less than or equal to zero, and can be ignored. Also, $(b_{l+1} + b_l(1 - b_{l+1})) = 1$. Thus,

$$
e(\tilde{n}) \leq 2^{l+1} + \frac{2}{3}(2^l - 1) = \frac{2}{3}(2^{l+2} - 1).
$$

■

Now we are ready to prove Theorem 4.

Proof: (Theorem 4) By Lemma 4, for every $1 \leq n < 2^l$, if $S \subset V(H_l)$ and $|S| = n$ then

$$|(S:\bar{S})| \geq e(n).$$

By Theorem 3, there exists an S for which $|(S:\bar{S})| = e(n)$. Thus, for every $1 \leq n < 2^l$,

$$\min_S\{|(S:\bar{S})| : S \subset V(H_l) \wedge |S| = n\} = e(n).$$

By Lemma 7, for every $1 \leq n < 2^l$, $e(n) \leq \frac{2}{3}(2^l - 1)$, proving that

$$\max_{1 \leq n \leq 2^l} \min_S\{|(S:\bar{S})| : S \subset V(H_l) \wedge |S| = n\} \leq \frac{2}{3}(2^l - 1).$$

However, by Lemma 6, equality holds (for the value of $n = \frac{1}{3}(2^l - 1)$). ∎

Finally, let us consider the case of odd l.

Theorem 5 *If l is odd then*

$$\max_{1 \leq n \leq 2^l} \min_S\{|(S:\bar{S})| : S \subseteq V(H_l) \wedge |S| = n\} = \frac{1}{3}(2^{l+1} - 1).$$

Again, one can use in the proof the following two lemmas:

Lemma 8 *For an odd l and $n = \frac{1}{3}(2^l + 1)$, $e(n) = \frac{1}{3}(2^{l+1} - 1)$.*

Lemma 9 *For an odd l and $0 \leq n < 2^l$*

$$e(n) \leq \frac{1}{3}(2^{l+1} - 1).$$

The proofs of Lemmas 8 and 9 are similar to those of Lemmas 6 and 7, respectively. The proof of Theorem 5 is similar to that of Theorem 4.

6 A Lower Bound on the Area Required by the Hypercube

In this section we show that the upper bound on the area required to layout the hypercube of N vertices, derived in Section 2, Equation 1, is also a lower bound. For the sake of completeness we repeat some of the ideas of Thompson [10, 11], but modify them, since unlike Thompson, we are interested in the coefficient of N^2 in the bound.

At first, let us consider the problem of laying out a graph $G(V, E)$ on the square grid, where the degree of the vertices of G is bounded by 4, and every vertex of G is represented by a single grid point. Let $0 < r < 1$ and define $c(r)$ in the following way. Assume that in every cut $C = (S : \bar{S})$ defined by $S \subset V$ such that $|S| = r|V|$, the number of edges, $|C|$, is at least $c(r)$.

Lemma 10 *The grid-area of a layout of G is at least* $(c(r) - 1)^2$.

Proof: Assume G is laid out in an $a \times b$ (upright) rectangle R. W.l.o.g. assume $a \leq b$ and the side of length a is vertical.

A *jogged cut* is an edge cut of the part of the grid encompassed by R, and is defined by a vertical line which may have a one step jog. See Figure 4. In this example $a = 5$ and the jogged line cuts (crosses) 6 grid edges inside R or on it. In the following discussion the jogged cut will slide, step by step, to include each time one more grid point on its l.h.s.[2]The process starts with a vertical cut, abutting R immediately on its l.h.s., so that all R's grid points are on the r.h.s. of the cut. Next, a jog is introduced, so that the upper left grid point of R is now on the l.h.s. of the cut. The jog is lowered to include one more grid point on the l.h.s. of the cut. This is continued until all grid points of this column are included, and the cut is again a straight line. The process is now repeated on the next column, etc., until the sliding cut passes the whole rectangle, from left to right.

Now, let us stop the sliding of the jogged cut when exactly $r|V|$ vertices of G are laid out on the l.h.s. of the jogged cut, while the remaining vertices of G are on its r.h.s. This defines a cut C of G, and its number of edges is at least $c(r)$. However, each graph edge is represented in the grid layout by a

[2]Left hand side.

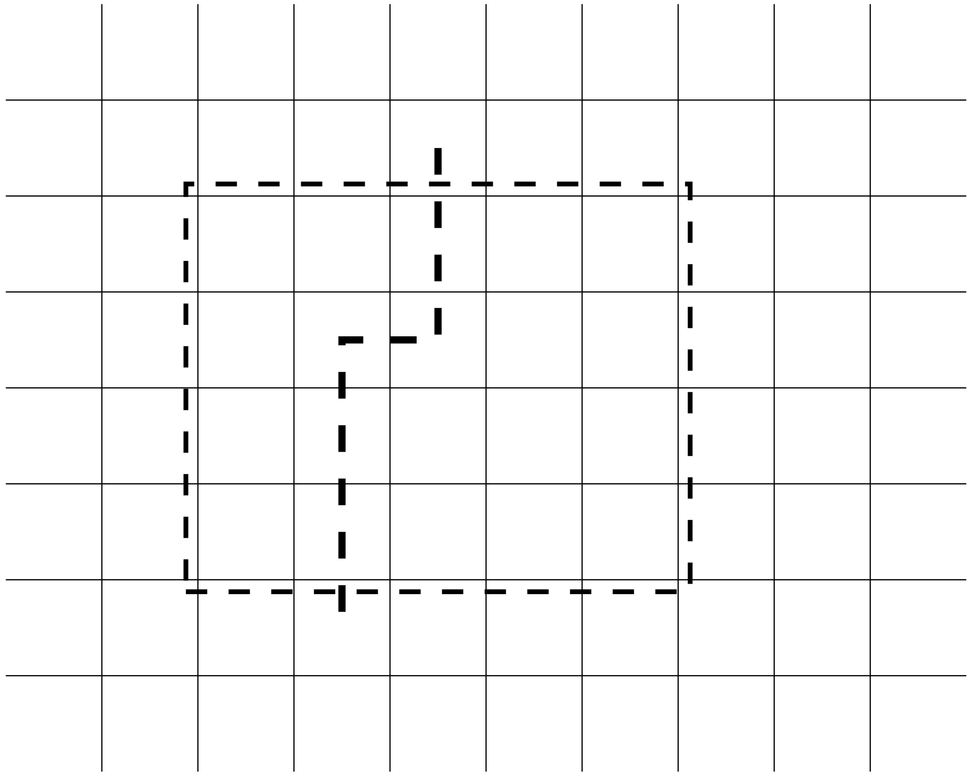

Figure 4: An edge cut defined by a jogged line

grid path, and no two such paths share a grid edge. Thus, the number of grid edges crossed by the jogged cut is at least $c(r)$. It follows that $a + 1 \geq c(r)$, and $a \cdot b \geq (c(r) - 1)^2$. ■

Next, the scope of the lower bound is extended to the case of general graphs, where it is assumed that a vertex of degree d is represented by a square box of d rows and d columns. This was done by Raspaud et al. [9].

Lemma 11 *If there is a grid layout of G in a rectangle R, where each vertex of degree d is represented by a $d \times d$ box, then there is a grid layout of G in R where each vertex of degree d is represented by a vertical line segment of d grid points.*

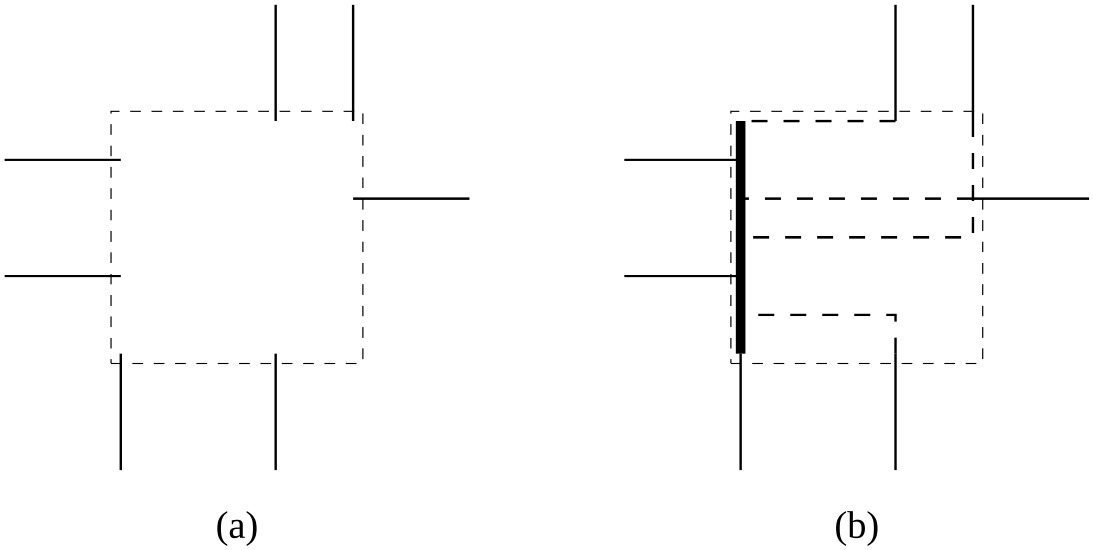

Figure 5: Replacing a 7×7 box by a vertical segment

Proof: Consider Figure 5, where it is demonstrated how a 7×7 box, representing a graph vertex of degree 7, is replaced by a vertical segment of 7 grid points. It is easy to see that for every d this transformation is possible, and that nothing is changed in the layout except "inside" the boxes. ■

Now it is easy to see that the proof of Lemma 10 is extendible to layouts of general graphs: Given any layout in a grid rectangle R, we first use Lemma 11 to change the layout to use vertical segments to represent the graph's vertices. Next, we use the sliding of the jogged cut, as in the proof of Lemma 10, except that we consider only the cases when the jogged cut does not cross any segment representing a graph vertex. The remainder of the proof is applicable.

Theorem 6 *The area required for laying out the hypercube of N vertices in the square grid is bounded from below by*

$$\frac{4}{9}N^2 + o(N^2). \tag{13}$$

Proof: Theorems 4 and 5 imply that for the ratio $r = \frac{1}{3} + o(1)$ one gets the minimum cut size of $c(\frac{1}{3}) = \frac{2}{3}N + o(N)$. By Lemma 10, in its extended context, the area required for laying out the hypercube of N vertices is bounded from below by formula 13. ■

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