Division of Informatics, University of Edinburgh

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by

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Keywords :

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Selection and Uniformization in Generalized Product

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Abstract

We investigate the selection and uniformization properties for classes of structures constructed by the Feferman-Vaught generalized product. We show that if classes K_1 and K_2 have the selection (respectively, uniformization) property, then the generalized product of these classes has the selection (respectively, uniformization) property.

1 Introduction

De£nition 1 (Selectors) *A class K of structures has the selection property if for every formula $\varphi(\vec{t})$ there is a formulas $\psi(\vec{t})$ such that*

1. $K \models \exists^{\leq 1}\vec{t}.\ \psi(\vec{t})$
2. $K \models \forall\vec{t}\psi(\vec{t}) \to \varphi(\vec{t})$ *and*
3. $K \models \exists\vec{t}.\ \varphi(\vec{t}) \to \exists\vec{t}.\psi(\vec{t})$.

Here and below "$\exists^{\leq 1}$" stands for "there is at most one", and "$\exists!$" stands for "there is a unique" and $\vec{t}$ is a list of distinct variables. We say that $\psi(\vec{t})$ is a *selector* for $\varphi(\vec{t})$ over K if (1)-(3) of the above de£nition hold.

De£nition 2 *A class of structures K has the uniformization property if for every formula $\beta(\vec{t}, \vec{u})$ there is a formula $\alpha(\vec{t}, \vec{u})$ such that*

1. $K \models \forall \vec{t}. \exists^{\leq 1} \vec{u}.\ \alpha(\vec{t}, \vec{u})$.
2. $K \models \forall \vec{t}. \forall \vec{u}.\ \alpha(\vec{t}, \vec{u}) \rightarrow \beta(\vec{t}, \vec{u})$ *and*
3. $K \models \forall \vec{t}.\ (\exists \vec{u}.\ \beta(\vec{t}, \vec{u}) \rightarrow \exists \vec{u} \alpha(\vec{t}, \vec{u}))$.

We say that $\alpha(\vec{t}, \vec{u})$ *uniformizes* $\beta(\vec{t}, \vec{u})$ over K if (1)-(3) of De£nition 2 hold.

The selection and uniformization properties were studied for some important structures [6, 1, 3, 4, 5].

In [2] Feferman-Vaught introduced a generalized product construct which encompasses a wide variety of algebraic constructions. This construct associates with a structure Ind (index structure) and a family A_i $(i \in Ind)$ of factor structures the generalized product structure of A_i over Ind. The Feferman-Vaught theorem reduces the £rst-order theory of the generalized product to the £rst-order theory of the factors and the £rst-order theory of an algebra of subsets over indices.

We show that if classes K_1 and K_2 of structures have the selection (respectively, uniformization) property, then the generalized product of these classes has the selection (respectively, uniformization) property. Our proofs are constructive. They reduce a construction of a selector for φ in the generalized product of K_1 over K_2 to a construction of (a £nite set of) selectors in the class K_1 and a construction of (a £nite set of) selectors in the class K_2.

The paper is organized as follows. In Section 2 we £x notations, recall the de£nition of the generalized product construct and state the Feferman-Vaught theorem. In Section 3 we show that the selection property is inherited by the generalized product construct. In Section 4 we show that the uniformization property is inherited by the generalized product construct.

2 Generalized Product

Notations. Let A be a structure. We use $|A|$ for the universe of A and R^A for the interpretation of the relational symbol R in A. However, whenever there is no confusion we will also use A for the universe of A; sometimes we use "$a \in A$" instead of "$a \in |A|$". We write $\varphi(t_1, \dots, t_k)$ to indicate that the free variables of φ are among $\{t_1, \dots, t_k\}$. Let $\varphi(t_1, \dots t_k)$ be a formula in a language L. We write $A, a_1, \dots a_k \models \varphi(t_1, \dots, t_k)$ if the formula $\varphi(t_1, \dots t_k)$ is true in the structure A for L when the elements $a_1, \dots, a_k$ in A are assigned to the variables names $t_1 \dots t_k$. We also abbreviate these to $A \models \varphi(a_1 \dots a_k)$. or to $A \models \varphi(\bar{a})$ where the bar denotes a tuple of the appropriate length; whenever $\bar{a}$ or A are clear from the context we even will use $A \models \varphi$ or $\varphi(\bar{a})$ respectively.

In a seminal paper [2] Feferman-Vaught introduced a generalized product; their composition theorem reduces £rst-order theory of the generalized product to the £rst-order theory of the factors and the £rst-order theory of an algebra of subsets over indices.

For the reader's convenience we recall the relevant de£nitions.

Let L_{ind} be a signature. A structure A is called an L_{ind} algebra of subsets over a set I if

1. The universe of A is the set of subsets of I,

2. A is a structure for the signature $L_{ind} \cup \{\Lambda, \ ^-, \ \cup, \ \cap, \ \subseteq\}$, where Λ is a constant symbol interpreted as the empty set, $^-$ is a unary function symbol interpreted as the complementation relative to I, $\cup$ and $\cap$ are binary function symbols interpreted as the binary union and intersection operators and $\subseteq$ is a binary relation symbol interpreted as the inclusion relation. There is no restriction on the interpretation of L_{ind} symbols.

We will use upper case letters for the variables ranging over the universe of L_{ind} algebras over I. Note that every such variable is interpreted as a subset of I.

The generalized product construct will deal with L_{ind} - a language and Ind - an L_{ind} algebra of subset over $|Ind|$; L_{factor} - a language and a family A_i (for $i \in |Ind|$) of factor structures for L_{factor} and L_{result} - a language for a generalized product and A - a structure for L_{result}

The universe $|G|$ of the structure for the generalized product of A_i ($i \in |Ind|$) will be the Cartesian product $\Pi_{i \in |Ind|}|A_i|$ of the family $|A_i|$ ($i \in |Ind|$); hence, the universe is the set of all functions g with the domain $|Ind|$ such that for each $i \in |Ind|$, $g(i)$ is an element of $|A_i|$. The interpretation of symbols in L_{result} is de£ned by the Feferman-Vaught determining sequences. We recall the syntax and the semantics of these sequences.

De£nition 3 (Determining sequence for an n-nary predicate -Syntax) *An n-ary determining sequence consists of*

1. *A £nite sequence $\varphi_1(x_1, \dots x_n), \varphi_2(x_1, \dots, x_n), \dots \varphi_m(x_1, \dots, x_n)$ of L_{factor} formulas (the free variables of these formulas are among $\{x_1, \dots, x_n\}$) and*

2. *A formula $\psi(Y_1, \dots, Y_m)$ in the language for L_{ind} algebras (the number of the free variables is at most as the length of the sequence of L_{factor} formulas).*

Now we are ready to de£ne the semantics of the generalized product. Let Ind be an L_{ind} algebra of subset and let A_i ($i \in |Ind|$) be a family of L_{factor} structures. The generalized product of A_i over Ind is denoted by $\Pi_{Ind} A_i$ and is de£ned as the following structure:

Signature: L_{result} has exactly one n-ary relational symbol for every n-ary determining sequence.

Universe: The Cartesian product of $|A_i|$ - the universes of A_i ($i \in |Ind|$).

Interpretation of relational symbols: To de£ne the interpretation of a relational symbol that corresponds to a determining sequence

$$\langle \psi(Y_1, \dots, Y_m), \varphi_1(x_1, \dots x_n), \varphi_2(x_1, \dots, x_n), \dots, \varphi_m(x_1, \dots, x_n) \rangle$$

we need to introduce some notations.

Let $\langle a_1, \dots, a_n \rangle$ be an n-tuple of elements in $\Pi_{i \in |Ind|} |A_i|$. With such n-tuple and a formula $\varphi_l(x_1, \dots, x_n)$ in L_{factor} a subset P_l of $|Ind|$ (which depends on $\langle a_1, \dots, a_n \rangle$ and φ_l) is de£ned as follows:

$$P_l \triangleq \{i \in |Ind| \ : \ A_i, a_1(i), \dots a_n(i) \models \varphi_l(x_1, \dots, x_n)\}$$

The n-tuple $\langle a_1, \dots, a_n \rangle$ satis£es the determining sequence described above iff

$$Ind, P_1, \dots, P_m \models \psi(Y_1, \dots, Y_m)$$

Finally, a relational symbol that corresponds to the above determining sequence is interpreted as the set of tuples that satis£es the determining sequence.

For a class K_1 of L_{factor} structures and for a class K_2 of L_{ind} algebras, the generalized product of K_1 over K_2 is the class of structures $\{\Pi_{Ind} A_i \ : \ A_i \in K_1 \text{ and } Ind \in K_2\}$; we denote this class by $\Pi_{K_2} K_1$

De£nition 4 (Partitions) *1. A sequence $\alpha_1(\vec{t}), \dots, \alpha_m(\vec{t})$ of formulas is called a partitioning sequence if $\bigvee_{i=1}^{m} \alpha_i$ is valid and $\alpha_i \wedge \alpha_j$ is unsatis£able for $i \neq j$.*

2. A determining sequence $\langle \beta(X_1, \dots X_k), \alpha_1(\vec{t}) \dots \alpha_k(\vec{t}) \rangle$ is called a partitioning determining sequence if $\alpha_1, \dots \alpha_k$ is a partitioning sequence.

3. *Subsets $P_1, \dots P_m$ of a set I partition I iff they are disjoint and their union is I.*

Note that for every $m > 0$ there is a formula $Part_m(X_1, \dots, X_m)$ in the signature $\{\Lambda, \ ^-, \cup, \cap, \subseteq\}$ that says "$X_1, \dots, X_m$ partitions the universe".

Theorem 5 (Composition Theorem for Generalized Product [2]) *For every formula $\varphi(u_1, \dots, u_n)$ (with n free variables) in L_{result} there is a partitioning n-ary determining sequence σ such that for every L_{ind} algebra Ind and every family of structures A_i ($i \in |Ind|$) for L_{factor} the relation de£nable in $A = \Pi_{Ind} A_i$ by φ is the same as the relation de£nable in A by σ. Moreover, σ is computable from φ.*

3 Selectors

Observe that

Lemma 6 *K has the selection property if and only if for every formula $\varphi(t)$ with one free variable there exists $\psi(t)$ such that $\psi(t)$ is a selector for $\varphi(t)$ over K.*

Proof By induction on n we will show that for every $\varphi(t_0, \dots, t_n)$ there exists $\psi(t_0, \dots, t_n)$ such that ψ is a selector for φ over K.

The base $n = 0$ is just the assumption of the lemma.

The inductive step $n \to n+1$. Let $\varphi(t_0, \dots, t_{n+1})$ be a formula. By the inductive assumption there is a selector $\alpha(t_0, \dots, t_n)$ for $\exists t_{n+1} \varphi(t_0, \dots, t_{n+1})$. By the assumption of the lemma there exists a selector $\beta(t_{n+1})$ for $\exists t_0 \dots \exists t_n (\alpha(t_0, \dots t_n) \wedge \varphi(t_0, \dots, t_{n+1})$. One can easy verify that $\alpha(t_0, \dots, t_n) \wedge \beta(t_{n+1})$ is a selector for φ over K. □

In this section we prove

Theorem 7 *Let K_1 be a class of L_{factor} structures and let K_2 be a class of L_{ind} algebras. If K_1 has the selection property and K_2 has the selection property, then the generalized product of K_1 over K_2 has the selection property.*

Proof By lemma 6 it is suf£cient to show that for every $\varphi(t)$ there is exists $\psi(t)$ which is a selector for φ over the generalized product of K_1 over K_2.

Let $\varphi(t)$ be a formula in L_{result} and let $\langle \beta(X_1, \dots X_k), \alpha_1(t) \dots \alpha_k(t) \rangle$ be the partitioning determining sequence that corresponds to $\varphi(t)$ by Theorem 5.

Let $\gamma_i(t)$ be selectors for $\alpha_i(t)$ over K_1.

Let $\delta(X_1, \ldots, X_k)$ be a selector for $Part(X_1, \ldots, X_k) \wedge \beta(X_1, \ldots, X_k)$ over the class K_2. Let $\psi(t)$ be de£ned by the determining sequence $\langle \delta, \gamma_1(t), \ldots \gamma_k(t) \rangle$. In the rest of this section we show that $\psi(t)$ is a selector for $\varphi(t)$ over the generalized product of K_1 and K_2.

Let Ind be in K_2 and let A_i ($i \in |Ind|$) be a family of structures from K_2. Let M be the generalized product of A_i over Ind. First, we will show that $M \models \exists^{\leq 1} t \psi(t)$. Assume that for some $a, b \in M$

$$M, a \models \psi(t) \text{ and } M, b \models \psi(t) \tag{1}$$

Let

$$Q_i^a \triangleq \{j \in |Ind| \ : \ A_j, a_j \models \gamma_i(t)\} \tag{2}$$

and

$$Q_i^b \triangleq \{j \in |Ind| \ : \ A_j, b_j \models \gamma_i(t)\}. \tag{3}$$

Observe that by the de£nition of $\psi(t)$ and (1) we obtain

$$Ind \models \delta(Q_1^a, \ldots, Q_k^a) \tag{4}$$

and

$$Ind \models \delta(Q_1^b, \ldots, Q_k^b) \tag{5}$$

Since, δ is a selector, it follows

$$Ind \models \exists^{\leq 1} X_1 \ldots X_k. \ \delta(X_1, \ldots, X_k) \tag{6}$$

Hence, by (4)-(6) we obtain

$$Q_i^a = Q_i^b. \tag{7}$$

Moreover, since $K_1 \models \exists^{\leq 1} t. \ \gamma_i(t)$, it follows by (2), (3) and (7)

$$a_j = b_j \text{ for } j \in Q_i^a \text{ and } i = 1, \ldots, k \tag{8}$$

Recall that δ is a selector for $Part(X_1, \ldots, X_k) \wedge \beta$. Therefore from (4) and (8) we obtain that Q_i^a ($i = 1, \ldots, k$) partitions $|Ind|$ and

$$a_j = b_j \text{ for } j \in |Ind| \tag{9}$$

Hence $a = b$ and this establishes $K \models \exists^{\leq 1} t. \ \psi(t)$.

Now we will show that $K \models \forall t. \psi(t) \rightarrow \varphi(t)$. Let $M = \Pi_{i \in Ind} A_i$ for $Ind \in K_2$ and $A_i \in K_1$. Assume that

$$M, a \models \psi(t) \tag{10}$$

Let Q_i^a for $i = 1, \ldots, k$ be de£ned as in (2). Then by (10) and by the de£nitions of $\psi(t)$ and of δ

$$Ind \models Part(Q_1^a, \ldots, Q_k^a) \wedge \beta(Q_1^a, \ldots, Q_k^a) \tag{11}$$

Let P_i^a for $i = 1, \ldots, k$ be de£ned as

$$P_i^a \triangleq \{j \in |Ind| \ : \ A_j, a_j \models \alpha_i(t)\} \tag{12}$$

Since, $\gamma_i(t)$ is a selector for $\alpha_i(t)$, it follows from (2) and (12) that $Q_i^a \subseteq P_i^a$ for $i = 1, \ldots, k$. Observe that P_i^a is a partition of $|Ind|$, because $\alpha_i(t)$ is a partitioning sequence. Q_i^a is a partition of $|Ind|$ by (11). Therefore, none of the above inclusions can be proper. Hence, $Q_i^a = P_i^a$ for $i = 1, \ldots, k$. This together with (11), (12) and the de£nition of φ imply that $M, a \models \varphi(t)$. This completes the proof of $K \models \forall t.\psi(t) \rightarrow \varphi(t)$.

It remains to show that $K \models \exists t\varphi(t) \rightarrow \exists t\psi(t)$.

Assume that $M, a \models \varphi(t)$. We are going to de£ne $b \in M$ such that $M, b \models \psi(t)$. Let P_i^a be de£ned as in (12). By the de£nition of φ we have

$$Ind \models \beta(P_1^a, \ldots, P_k^a) \tag{13}$$

Recall that $\alpha_i(t)$ is a partitioning sequence, Hence, by (12) we obtain

$$Ind \models Part(P_1^a, \ldots, P_k^a) \tag{14}$$

Since $\gamma_i(t)$ is a selector for $\alpha_i(t)$ it follows from (12) that for every $j \in P_i^a$ there is a unique $b_j \in A_j$ such that

$$A_j, b_j \models \gamma_i(t) \tag{15}$$

De£ne b as $\langle b_j \ : \ j \in |Ind|\rangle$. Note that this is well de£ned because P_i^a is a partition of $|Ind|$. Let Q_i^b be de£ned as in (3). From the de£nition of b it follows that $P_i^a \subseteq Q_i^b$ for $i = 1, \ldots, k$. None of the inclusion can be proper, because P_i^a is a partition and Q_i^b are disjoint (indeed, if $j \in Q_{i_1}^b \cap Q_{i_2}^b$ then $A_j, b_j \models \gamma_{i_1}(t) \wedge \gamma_{i_2}(t)$; this implies that $A_j, b_j \models \alpha_{i_1}(t) \wedge \alpha_{i_2}(t)$; since $\alpha_i(t)$ is a partitioning sequence we will obtain that $i_1 = i_2$.) Therefore, $P_i^a = Q_i^b$, hence, by (13) and (14) we obtain that

$$Ind \models Part(Q_1^b, \ldots, Q_k^b) \wedge \beta(Q_1^b, \ldots, Q_k^b) \tag{16}$$

The last equations together with the de£nition of ψ imply $M, b \models \psi(t)$. This completes the proof of $\exists t\varphi(t) \rightarrow \exists t\psi(t)$.

Therefore $\psi(t)$ is a selector for $\varphi(t)$. $\square$

4 Uniformization

Theorem 8 *Let K_1 be a class of L_{factor} structures and let K_2 be a class of L_{ind} algebras. If K_1 has the uniformization property and K_2 has the uniformization property, then the generalized product of K_1 over K_2 has the uniformization property.*

Proof We will show that for every $\varphi(t, u)$ there exists $\psi(t, u)$ which uniformizes $\varphi(t, u)$ over the generalized product. The general case when t and u are lists of variables is proved exactly in the same way.

Let $\varphi(t, u)$ be a formula in L_{result} and let $\langle \beta(X_1, \ldots X_k), \alpha_1(t, u) \ldots \alpha_k(t, u) \rangle$ be the partitioning determining sequence that corresponds to $\varphi(t, u)$ by Theorem 5. We have to construct $\psi(t, u)$ that uniformizes φ. Below we £rst describe the construction of ψ and then prove that the construction is correct.

Construction. For every $M \subseteq \{1, \ldots, k\}$ we de£ne a formula $\theta_M(t)$ as

$$\theta_M(t) \triangleq \bigwedge_{i \in M} \exists u.\alpha_i(t, u) \wedge \bigwedge_{i \notin M} \neg \exists u.\alpha_i(t, u) \tag{17}$$

The sequence $\theta_M(t)$ ($M \subseteq \{1, \ldots, k\}$) is a partitioning sequence.

Let $\gamma_i(t, u)$ be a formula that uniformizes $\alpha_i(t, u)$ ($i = 1, \ldots, k$) over K_1.

Let $\overline{Z}$ be the set of variable $\{Z_M \ : \ M \subseteq \{1, \ldots, k\}\}$. Let $\Delta(\overline{Z}, Y_1, \ldots, Y_k)$ be de£ned as

$$\Delta(\overline{Z}, Y_1, \ldots Y_k) \triangleq Part(\overline{Z}) \rightarrow \begin{array}{l} (\beta(Y_1, \ldots, Y_k) \wedge Part(Y_1, \ldots, Y_k) \wedge \\ \wedge \bigwedge_{m=1}^{k} (\, Y_m \subseteq \bigcup_{m \in M} Z_M). \end{array} \tag{18}$$

Let $\delta(\overline{Z}, Y_1, \ldots, Y_k)$ be a formula that uniformizes Δ over K_2, i.e.,

$$K_2 \models \forall \overline{Z}.\exists^{\leq 1} \overline{Y}.\ \delta(\overline{Z}, \overline{Y}). \tag{19}$$

$$K_2 \models \forall \overline{Z}.\forall \overline{Y}.\ \delta(\overline{Z}, \overline{Y}) \rightarrow \Delta(\overline{Z}, \overline{Y}) \tag{20}$$

$$K_2 \models \forall \overline{Z}.\ (\exists \overline{Y}.\ \Delta(\overline{Z}, \overline{Y}) \rightarrow \exists \overline{Y} \delta(\overline{Z}, \overline{Y})). \tag{21}$$

Let $\psi(t, u)$ be de£ned by the determining sequence

$$\langle \delta(\overline{Z}, \overline{Y}), \ldots \theta_M(t) \ldots, \gamma_1(t, u), \ldots, \gamma_k(t, u) \rangle.$$

In the rest of this section we show that $\psi(t, u)$ uniformizes $\varphi(t, u)$ over the generalized product of K_1 and K_2.

Let Ind be in K_2 and let A_i ($i \in |Ind|$) be a family of structures from K_2. Let A be the generalized product of A_i over Ind. First, we will show that $A \models \forall t \exists^{\leq 1} u.\ \psi(t,u)$. Assume that for some $a, b, c \in A$

$$A \models \psi(a,b) \tag{22}$$

$$A \models \psi(a,c) \tag{23}$$

For $M \subseteq \{1, \dots, k\}$ let

$$P^a_M = \{ i \in Ind\ :\ A_i \models \theta_M(a)\ \} \tag{24}$$

For $m \in \{1, \dots, k\}$ let

$$Q^{a,b}_m = \{ i \in Ind\ :\ A_i \models \gamma_m(a_i, b_i)\ \} \tag{25}$$

$$Q^{a,c}_m = \{ i \in Ind\ :\ A_i \models \gamma_m(a_i, c_i)\ \} \tag{26}$$

From the de£nition of ψ and (22)-(26), it follows that

$$Ind \models \delta(\overline{P^a}, Q^{a,b}_1, \dots Q^{a,b}_k) \tag{27}$$

$$Ind \models \delta(\overline{P^a}, Q^{a,c}_1, \dots Q^{a,c}_k) \tag{28}$$

(Here and below we use $\overline{P^a}$ for the sequence $\langle P^a_M\ :\ M \subseteq \{1, \dots, k\} \rangle$.) Hence, by (19), for $m \in \{1, \dots, k\}$

$$Q^{a,b}_m = Q^{a,c}_m \tag{29}$$

Since $\gamma_m(t,u)$ uniformizes $\alpha_m(t,u)$, by (25), (26) and (29), we obtain

$$b_i = c_i \text{ for } i \in Q^{a,b}_m \tag{30}$$

$Q^{a,b}_m$ ($m = 1, \dots, k$) is a partition of $|Ind|$, by (27) and the de£nitions of Δ and δ. Therefore (30) implies that $b = c$. Hence, $A \models \forall t \exists^{\leq 1} u.\ \psi(t,u)$.

Inclusion of ψ in φ. Let us show that $A \models \forall t \forall u \psi(t,u) \to \varphi(t,u)$. Assume that

$$A \models \psi(a,b) \tag{31}$$

Let P^a_M for $M \subseteq \{1, \dots, k\}$. be de£ned like in (24) and let $Q^{a,b}_m$ for $m \in \{1, \dots, k\}$ be de£ned like in (25). From (31) and the de£nition of ψ we obtain

$$\begin{aligned} Part(\overline{P^a}) \to \quad & (\beta(Q^{a,b}_1, \dots, Q^{a,b}_k) \wedge Part(Q^{a,b}_1, \dots, Q^{a,b}_k) \wedge \\ & \wedge \textstyle\bigwedge_{m=1}^{k} (\ Q^{a,b}_m \subseteq \bigcup_{m \in M} P^a_M). \end{aligned} \tag{32}$$

Recall that θ_M is a partitioning sequence. Therefore, $Part(\overline{P^a})$, and by (32), we obtain

$$Ind \models \beta(Q_1^{a,b}, \ldots, Q_k^{a,b}) \tag{33}$$

Since $\gamma_m(t, u)$ uniformizes $\alpha_m(t, u)$ we obtain

$$A_i \models \alpha_m(a_i, b_i) \text{ for } i \in Q_m^{a,b} \tag{34}$$

For $m \in \{1, \ldots, k\}$ let

$$R_m^{a,b} = \{i \in Ind \;:\; A_i \models \alpha_m(a_i, b_i)\} \tag{35}$$

By (34) and (35) we obtain that $Q_m^{a,b} \subseteq R_m^{a,b}$ for $m \in \{1, \ldots, k\}$. Both $R_m^{a,b}$ ($m \in \{1, \ldots, k\}$) and $Q_m^{a,b}$ ($m \in \{1, \ldots, k\}$) are partitions ($R_m^{a,b}$, because α_m is a partition sequence, and $Q_m^{a,b}$ by (32)). Hence none of the above inclusions can be proper. Therefore, $R_m^{a,b} = Q_m^{a,b}$ for $m = 1, \ldots, k$. This together with (33), (35) and the de£nition of φ implies that $A \models \varphi(a, b)$. This establishes that $\forall t \forall u \psi(t, u) \rightarrow \varphi(t, u)$ holds over the generalized product.

It remains to show that $\forall t\, .(\exists u\; \varphi(t, u) \rightarrow \exists u\; \psi(t, u))$ holds over the generalized product. Assume that

$$A \models \varphi(a, b) \tag{36}$$

We are going to construct $c \in A$ such that $A \models \psi(a, c)$

Let $R_m^{a,b}$ for $m \in \{1, \ldots, k\}$ be de£ned like in (35). From the de£nition of φ and (36), it follows that

$$Ind \models \beta(R_1^{a,b}, \ldots, R_k^{a,b}). \tag{37}$$

Since α_m is a partitioning sequence, it follows that $Part(R_1^{a,b}, \ldots, R_k^{a,b})$. Let P_M^a for $M \subseteq \{1, \ldots, k\}$. be de£ned like in (24). From the de£nition of θ_M, it follows that $R_m^{a,b} \subseteq \bigcup_{m \in M} P_M^a$. Therefore, $\Delta(\overline{P^a}, R_1^{a,b}, \ldots, R_k^{a,b})$ holds. Since, δ uniformizes Δ it follows by (21) that there are $S_m^{a,b}$ ($m = 1, \ldots, k$) such that

$$Ind \models \delta(\overline{P^a}, S_1^{a,b}, \ldots, S_k^{a,b}) \tag{38}$$

Since P_M^a partitions $|Ind|$, it follows from (18) and (38) that

$$Ind \models \bigwedge_{m=1}^{k} (\; S_m^{a,b} \subseteq \bigcup_{m \in M} P_M^a). \tag{39}$$

$$Ind \models \beta(S_1^{a,b}, \ldots, S_k^{a,b}) \wedge Part(S_1^{a,b}, \ldots, S_k^{a,b}) \tag{40}$$

Note that, by (24), $A_i \models \theta_M(a_i)$ for $i \in P^a_M$. Therefore by the de£nition of θ_M, for every $m \in M$ there is d_i such that $A_i \models \alpha_m(a_i, d_i)$. Since γ_m uniformizes α_m, this implies

For every $i \in P^a_M$ and $m \in M$ there is $e_{i,m}$ such that $A_i \models \gamma_m(a_i, e_{i,m})$ (41)

Let c_i be de£ned as follows

$$c_i = e_{i,m} \text{ for } i \in S^{a,b}_m \tag{42}$$

Since P^a_M partitions $|Ind|$, it follows from (39)-(41) that the c_i in (42) are well de£ned, moreover, (42) de£nes c_i for every $i \in |Ind|$. We are going to show that $A \models \psi(a, c)$.

For $m \in \{1, \dots, k\}$ let $Q^{a,c}_m$ be de£ned as in (26). Note that (26), (41) and (42) imply that $Q^{a,c}_m \supseteq S^{a,b}_m$. None of these inclusions can be proper. Indeed, γ_m uniformizes α_m and α_m ($m = 1, \dots, k$) is a partitioning sequence, therefore $Q^{a,c}_m$ are disjoint sets. Hence, if one of the inclusion is proper, it would contradict (40). Therefore, $Q^{a,c}_m = S^{a,b}_m$ for $m \in \{1, \dots, k\}$. This together with (38) imply that $Ind \models \delta(\overline{P^a}, Q^{a,c}_1, \dots, Q^{a,c}_k)$. Therefore, by the de£nition of ψ, it follows that $A \models \psi(a, c)$. This completes the proof that $\forall t\, .(\exists u\, \varphi(t, u) \rightarrow \exists u\, \psi(t, u))$ holds

Therefore, $\psi(t, u)$ uniformizes $\varphi(t, u)$. □

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