Developing a Multigrid Solver for Standing Wave Equation

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Abstract

In this paper multigrid technique is adapted for solving standing one-dimensional wave equation with radiation boundary conditions. Solver, consisting of wave cycle and ray cycle, uses Gauss-Seidel and Kaczmarz relaxation sweeps and is aimed to work efficiently for all error components.

1 Introduction

Multigrid technique is an efficient method for solving differential and integral equations, eigenvalues problems, etc. [1]. In this paper the multigrid technique is used to solve the discrete problems, obtained by discretization of standing wave equations.

1.1 Wave Equation

If we consider solution of the one dimensional wave equation

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{1}{c^2(x)} \frac{\partial^2 v(x,t)}{\partial t^2} \tag{1}$$

in the form $v(x,t) = g(t)u(x)$ and substitute it in (1), we get $g(t)\frac{d^2u(x)}{dx^2} = \frac{u(x)}{c^2(x)}\frac{d^2g(t)}{dt^2}$ which after separation of variables can be written in the form $\frac{c^2(x)}{u(x)}\frac{d^2u(x)}{dx^2} = \frac{1}{g(t)}\frac{d^2g(t)}{dt^2}$. Substituting $-\omega^2 = \frac{\frac{d^2g(t)}{dt^2}}{g(t)}$ and $k^2(x) = \frac{\omega^2}{c^2(x)}$ leads to the standing wave equation

$$\frac{d^2u(x)}{dx^2} + k^2(x)u(x) = 0. \tag{2}$$

We shall look for the solution in the so-called ray form

$$u(x) = a(x)e^{ikx} + b(x)e^{-ikx} \tag{3}$$

where $a(x)$ and $b(x)$ are smooth. Functions $a(x)$ and $b(x)$ are called ray functions. We impose the radiation boundary conditions as $a(-\infty) = a_0$, $b(+\infty) = b_0$. Substituting (3) in (2) we obtain a system of equations

$$a''(x) + 2ika'(x) = 0 \tag{4}$$

$$b''(x) - 2ikb'(x) = 0. \tag{5}$$

1.2 Discretization

In order to solve equation (2) on interval $[a, b]$, we approximate it by a system of linear equations. Solution of the one-dimensional discrete problem is a vector of equidistant points whose values approximate the continuous solution of the differential problem. Solution $u(x)$ is approximated by $u_l = u(hl)$, where $h = \frac{b-a}{N+1}$ is a meshsize and N is number of interior points.
The discrete operator for (2) is of the form

$$(L^h u^h)_l = \frac{u_{l-1} - 2u_l + u_{l+1}}{h^2} + k_l^2 u_l \tag{6}$$

where $l = 1, .., N$. The solution of the discrete problem satisfies the system of linear equations

$$L^h u^h = 0. \tag{7}$$

The exact solution of the discrete problem differs from the solution of differential problem due to discretization error. For (7) the most important aspect of the discretization error is phase error. Solution of (7) is of the form

$$u_l^h = a_l e^{ik^h hl} + b_l e^{-ik^h hl} \tag{8}$$

where $k^h = \frac{\arccos(1 - \frac{k^2 h^2}{2})}{h}$. To enable discrete solution to be an accurate approximation to the differential one along the interval $[a, b]$, the meshsize h should be chosen to provide $|\frac{2\pi}{k} - \frac{2\pi}{k^h}|(b-a)\frac{k}{2\pi} \ll \frac{\pi}{k}$.

2 Methods

2.1 Relaxation Methods

Discretization of the differential problem leads to a system of N equations with N unknowns. Exact solution of the system can be obtained by matrix inversion or Gauss elimination. Such methods, however, require a lot of work. Work can be decreased by using relaxation techniques, which produce solution with required accuracy and can be used as part of powerful and efficient multigrid techniques. By this, we can significantly reduce computation cost, especially in higher-dimensional problems.
Consider a system of linear equations $Au = b$. Let $\tilde{u}$ be a current solution, $v = u - \tilde{u}$ error and $r = b - Au$ a residual. Our solver uses Gauss-Seidel relaxation with update rule

$$\tilde{u}_l^{new} = \frac{1}{a_{ll}}(b_l - \sum_{i=1}^{l-1} a_{li}\tilde{u}_i^{new} - \sum_{i=l+1}^{N} a_{li}\tilde{u}_i^{old})$$

and Kaczmarz relaxation which uses update rule

$$\tilde{u}_l^{new} = \tilde{u}_l^{old} - \overline{a}_{il}(b_i - \sum_{j=1}^{N} a_{ij}\tilde{u}_j^{old}) / \sum_{j=1}^{N} a_{ij}\overline{a}_{ij}$$

for each $l = 1, .., N$ to force i-th equation residual to zero.
Relaxation efficiency can be estimated by analysis of Fourier error components reduction. Analysis shows that, generally, relaxation methods efficiently reduce error components that are high-frequency on current grid. For smooth error components both Kaczmarz and Gauss-Seidel relaxation may have very small convergence (Gauss-Seidel can even diverge in some cases).

2.2 Multigrid Cycles

The idea of the multigrid is to transfer smooth errors to coarser grid, where they become higher-frequency and can be efficiently reduced, investing a little work.
We shall enumerate grid levels by d ($d = 0$ for coarsest grid). If we choose each coarser grid meshsize $H = 2h$ total amount of work will be proportional to amount of work on finest level (which will be equal to the cost of a few relaxation sweeps). According to Fourier analysis, solver decides which relaxation to use and calculates number of sweeps for each grid so that error reduction is approximately 10 for each level. FAS V-cycle algorithm used by our solver is a special case of multigrid cycle algorithm which uses FAS (Full Approximation Scheme) intergrid transfer scheme and can be described by following pseudo-code:

FUNCTION VCycle($d, h, \tilde{u}_{old}^h, b^h$) RETURNS $\tilde{u}_{new}^h$
  $\tilde{u}_{new}^h = \tilde{u}_{old}^h$
  α_d times DO IF GaussSeidel is efficient
    THEN $\tilde{u}_{new}^h = GaussSeidelRelaxation(\tilde{u}_{new}^h, h, b^h)$
    ELSE $\tilde{u}_{new}^h = KaczmarzRelaxation(\tilde{u}_{new}^h, h, b^h)$
  IF $d > 0$ THEN
    $H = 2h$
    Use FAS transfer scheme to obtain approximate initial guess $\tilde{u}_{old}^H$ and
    right-hand side for coarse grid b^H:
    $\tilde{u}_{old}^H = I_h^H \tilde{u}_{new}^h$
    $b^H = I_h^H r^h + L^H \tilde{u}_{old}^H$
    $\tilde{u}_{new}^H = VCycle(d-1, H, \tilde{u}_{old}^H, b^H)$
    $\tilde{u}_{new}^h = \tilde{u}_{new}^h + I_H^h(\tilde{u}_{new}^H - \tilde{u}_{old}^H)$
  β_d times DO IF GaussSeidel is efficient
    THEN $\tilde{u}_{new}^h = GaussSeidelRelaxation(\tilde{u}_{new}^h, h, b^h)$
    ELSE $\tilde{u}_{new}^h = KaczmarzRelaxation(\tilde{u}_{new}^h, h, b^h)$
END VCycle.

The I_h^H operator transfers from a finer to a coarser grid either by averaging or injec-

tion, and I_H^h denotes interpolation from a coarser to a finer grid (which in our case is linear but can be cubic if necessary).

2.3 Multigrid Adaptation for Specific Problem

Our solver consists of the wave cycle and the ray cycle functions. The wave cycle is a V-cycle applied to (7). It uses Kaczmarz and Gauss-Seidel relaxation schemes. For error of the form $v_l = A_\theta e^{i\theta l}$, where $|\theta| \in (\frac{\pi}{2}, \pi)$ and A_θ is Fourier component amplitude, Gauss-Seidel has convergence rate $\frac{A_\theta^{new}}{A_\theta^{old}} = \mu(\theta) = \frac{1}{\sqrt{p^2+2p\cos\theta+1}}$ and Kaczmarz has convergence rate $\mu(\theta) = \frac{|e^{i\theta}+2p|}{|(e^{-i\theta}+p)^2+2|}$ where $p = k^2h^2 - 2$. It can be verified that this cycle is efficient for all error frequencies except for $|\theta| \in (\frac{kh}{2}, \frac{3kh}{2})$.
The approximate solution and the unreduced error after the wave cycle can be represented in the ray form (3). The ray approach is used to improve an approximation to $a(x)$ and $b(x)$ on the coarse grids, and, by this, to improve the wave solution. After discretization of (4) and (5) we get two systems of equations

$$(L_+^h a^h)_l = \frac{a_{l-2} - 2a_{l-1} + a_l}{h^2} + 2ik\frac{a_l - a_{l-1}}{h} = 0 \tag{9}$$

$$(L_-^h b^h)_l = \frac{b_{l+2} - 2b_{l+1} + b_l}{h^2} - 2ik\frac{b_{l+1} - b_l}{h} = 0 \tag{10}$$

Ray cycle is a V-cycle applied to (9), (10). Unlike wave cycle, ray cycle uses only Kaczmarz relaxation. For each problem it uses one boundary condition which is introduced on the coarsest level. Since ray cycle is efficient enough for error frequencies $|\theta| \in (\frac{kh}{2}, \frac{3kh}{2})$, we can obtain an efficient solver for the standing wave equation by combining wave cycle with ray cycle. Having obtained approximate solution by the wave cycle, we can use separation process described in [2] to find initial approximations for a and b and apply ray cycle to improve them. Improved $\tilde{a}$ and $\tilde{b}$ values are used to correct the wave solution $\tilde{u}$. For efficient work of ray cycle finest meshsize h is chosen to be $h \approx \frac{\pi}{k}$ so that amount of work is as little as possible, and error components which need the ray representation are high-frequency, but still approximated well enough on this grid.

3 Results

In order to avoid as much program coding errors as possible, programs were developed and tested stage by stage. In order to make process efficient we used MatLab language. Jacobi, Gauss-Seidel (lexicographical and red-black order) and Kaczmarz relaxation functions were implemented. Correction scheme V-cycle for simpler discrete operator was coded and its results were compared to FAS V-cycle. To check multigrid efficiency FMG cycle (see [1]) was written. Wave cycle was implemented. Its approximative solution and residual factors were checked. Ray cycle was written, and its efficiency was tested for frequencies where wave cycle is inefficient.

3.1 Wave Cycle Results

We tested our wave cycle, varying k, N, h and the initial guess in order to check its behaviour in different cases. Efficiency measure used was residual factor f defined as $f = \frac{||r_{old}||}{||r_{new}||}$ where r_{old} and r_{new} are the residuals, calculated on the finest wave level, *before* and *after* V-cycle. Residual norms are given by:

$$||r||^2 = \frac{1}{N}\sum_{i=1}^{N}|r_i|^2.$$

To verify efficiency of our solver we choose $e^{i\omega hl}$ as the initial guess, setting the initial error frequency to ω, and tested $f(\omega)$ dependency. A representative graph for $N = 63$, $h_{finest} = \frac{1}{64}$ and $k = 6$ is printed below. For this example the solver decided to perform 3 Gauss-Seidel sweeps on the finest level and 5 Kaczmarz sweeps on each of the coarser levels.

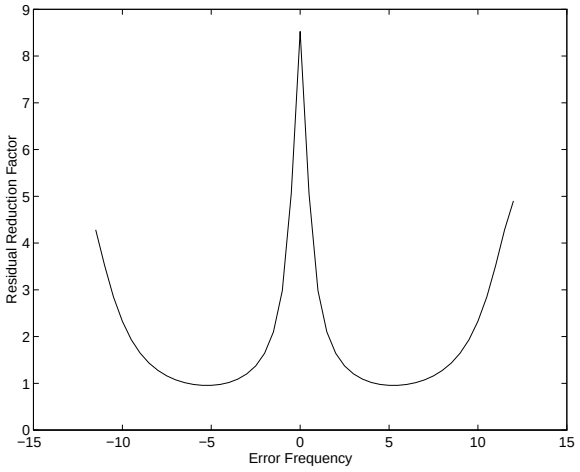

Figure 1

As expected from the analysis, nearly no convergence was obtained for $|\omega| \in (\frac{k}{2}, \frac{3k}{2})$, whereas for all other frequencies there is acceptable residual factor.

3.2 Ray Cycle Results

The ray cycle was tested in a similar manner as wave cycle. However, residual factors obtained are about 10^2. Since residual factors do not take into account boundary condition which is not introduced yet (because it is introduced on the coarsest level) in order to determine convergence rate we used error norm. This is of course possible only when we know the exact solution, but is useful for testing purposes. We used again $N = 63$, $h_{finest} = \frac{1}{64}$ and $k = 6$. For ray cycle the program chose 5 Kaczmarz sweeps for each of the three finest levels and 4 sweeps on the remaining coarser levels.

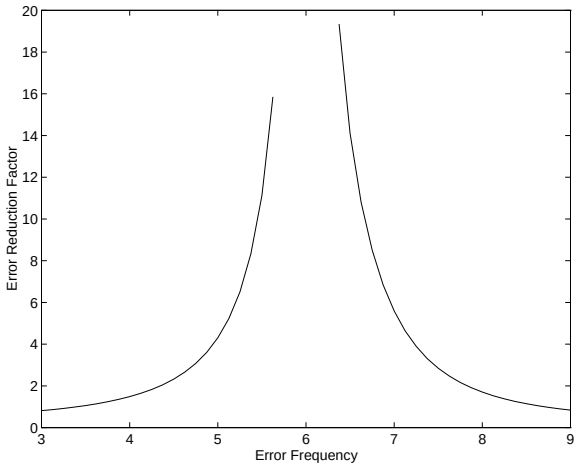

Figure 2

Figure 2 shows efficiency for $a(x)$ ray cycle. Points very close to k are not plotted because their value is extremely high due to nearly constant initial guess. It can be seen that good convergence is obtained for interval $|\omega| \in (\frac{k}{2}, \frac{3k}{2})$ which is the only part where wave cycle is not efficient.

4 Discussion

By combination of wave cycle and ray cycle an efficient solver can be produced that reduces all error frequencies and is of linear complexity. The last step in producing the solver would be implementation of separation procedure which would merge the two cycles. For the particular one-dimensional problem there are algorithms of comparable efficiency, but the multigrid approach can be adapted for multidimensional problem where its efficiency is the best possible.

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References

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