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Polynomials of Bounded Tree-Width

(Extended Abstract)

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Abstract. We introduce a new sparsity conditions on multivariate polynomials in n variables (over some ring R) and show that under this condition many otherwise intractable computational problems involving these polynomials become solvable in polynomial (even linear) time in n (in the BSS-model over R). To define our sparsity condition we associate with these polynomials a hypergraph and study classes of polynomials where this hypergraph has tree width at most k for some fixed $k \in \mathbb{N}$.
We are interested in three cases:
(1) The evaluation of multivariate polynomials where the number of monomials is $O(2^n)$.
(2) The question whether a system of n polynomials $p_i(\bar{x}) \in R[\bar{x}]$ of fixed degree d in n variables has a root in R^n.
(3) For infinite ordered rings R_{ord}, a polynomial of fixed degree d in n variables $p(\bar{x}) \in R_{ord}[\bar{x}]$ and a finite subset $A \subset R_{ord}$ we want to know whether $p(\bar{a}) > 0$ for all $\bar{a} \in R^n_{ord}$.
Our method uses graph theoretic and model theoretic tools developed in the last 15 years and applies them to the algebraic setting. This work is an extension of work by B. Courcelle, J.A. Makowsky and U. Rotics.

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1 Introduction and results

Sparsity conditions were used previously in analyzing the location of zeros of multivariate polynomials. The most spectacular stems from the *Newton polytope* associated with a system of multivariate polynomials over the complex numbers. Bernstein's theorem then relates the number of isolated zeros of this system to the *mixed volume of the Newton polytope*, cf. [CLO97,GKZ94].

We introduce a new sparsity condition on multivariate polynomials in n variables (over some ring R) and show that under this condition many otherwise intractable computational problems involving these polynomials become solvable in polynomial time in n (in the BSS-model over R).

We associate with these polynomials a hypergraph and study classes of polynomials where this hypergraph has tree width at most k for some fixed $k \in \mathbb{N}$. Tree width of graphs is a useful concept with a long history and a plethora of results, cf. [Die96]. A definition for hypergraphs is given in section 2 below.

We are interested in three cases:

(i) The evaluation of multivariate polynomials where the number of monomials is $O(2^n)$, such as the permanent, the hamiltonian or many other *generating functions of graph properties*. In general most of these polynomials are not known to allow evaluation in polynomial time. Here the sparsity condition we impose is the bound k on the tree width of the underlying graph. We show that all these generating functions can be evaluated in time $O(n)$ where the constant depends (superexponentially) on k.

(ii) The question whether a system of n polynomials $p_i(\bar{x}) \in R[\bar{x}]$ of fixed degree d in n variables has a root in R^n. This problem is NP_R hard for large enough degree d. Here the sparsity condition we impose is the bound k on the tree width of the d-hypergraph of non-vanishing monomials. We show that for *finite rings* the problem is solvable in time $O(n)$ where the constant depends superexponentially on k and the size of the ring R. This approach does not work for infinite rings.

(iii) For infinite ordered rings R_{ord}, a polynomial of fixed degree d in n variables $p(\bar{x}) \in R_{ord}[\bar{x}]$ and a finite subset $A \subset R_{ord}$ we want to know whether $p(\bar{a}) > 0$ for all $\bar{a} \in R^n_{ord}$. This problem is co-$NP_{R_{ord}}$ hard in general but solvable in time $O(n)$ for polynomials of tree width at most k. Our method can be extended to check whether $p(\bar{x})$ has a zero in A^n only if we impose some further restriction on the coefficients of p and A.

Our method uses graph theoretic and model theoretic tools developed in the last 15 years and applies them to the algebraic setting. This work is an extension of work by B. Courcelle, J.A. Makowsky and U. Rotics [CMR00].

In this extended abstract we emphasize the results. In section 2 we introduce tree width of matrices, polynomials and systems of polynomials. In section 3 we state a result from [CMR00] to illustrate the definition. In section 4 we state the new results concerning feasibility of polynomial systems. In section 5 we sketch the logical framework in which we express the problems. In section 6 we discuss further research.

2 Tree width of polynomials and matrices

Let $V = \{0, 1, 2, \ldots, n\}$ be the index set of the variables of

$$p(x) = p(x_0, x_1, \ldots, x_n) = \sum_{(i_1,\ldots i_d) \in E} c_{i_1,\ldots,i_d} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_d}$$

with $E \subseteq V^d$ and $x_0 = 1$. E is the set of d-tuples of indices $(i_1, \ldots i_d)$ such that the coefficient $c_{i_1,\ldots,i_d} \neq 0$.

Definition 1. *With $p(x)$ we associate the d-hypergraph $G = \langle V, E\rangle$ and define the tree width of $p(x)$ as the tree width of G.*

For systems of polynomials p_j of degree d we look at the hypergraph of the non-vanishing coefficients $c_{j,\alpha}$, i.e. the induced $d+1$-hypergraph.

Definition 2 (Tree width of a d-hypergraph). *A k-tree decomposition of G is defined as follows:*

(i) $\mathcal{T} = \langle T, <_T\rangle$ is a tree with $t <_T s$ expressing that t is a child of s.
(ii) For each $t \in T$ we have a subset $V_t \subseteq V$ of size at most $k+1$.
(iii) For each hyperedge $(i_1, \dots i_d) \in E$ there is a $t \in T$ such that $\{i_1, \dots i_d\} \subseteq V_t$.
(iv) For each $i \in V$ the set $V(i) = \{t \in T \mid i \in V_t\}$ forms a (connected) subtree of $\mathcal{T}$.

G has tree width at most k if there exists a k-tree decomposition of G.

Examples 1

(i) The polynomial $p_1(x) = \sum_{i=1}^n x_i^4$ has tree width 0. Note that the Newton polytope of p_1 is maximal with respect to polynomials of degree 4 in n variables.
(ii) The polynomials

$$p_2(x) = \sum_{i=1}^n c_{i,i+_3 1, i+_3 2} x_i x_{i+_3 1} x_{i+_3 2}$$

and

$$p_3(x) = \sum_{i=1}^n d_{i,i+_3 1, i+_3 2} x_i^3 x_{i+_3 1}^5 x_{i+_3 2}$$

(where $+_3$ is addition $\pmod 3$) have tree width 2.
(iii) In general, if $p(x)$ in n variables of degree d has tree width k, the number of monomials is $O(n)$ with a constant depending on k, d only.
(iv) For $p_1(x)$ from above, the polynomial $p_1^2(x)$ has tree width $n-1$. This is so because all monomials appear and hence the 4-hypergraph associated with the polynomial is a hyperclique.

Similarly, we can define the tree width of an $(n \times n)$ matrix $M = (m_{i,j})$.

Definition 3. *The tree width of an $(n \times n)$ matrix $M = (m_{i,j})$ is the tree width of the graph $G_M = \langle V_M, E_M\rangle$ with $V_M = \{1, 2, \dots, n\}$ and $(i,j) \in E_M$ iff $m_{i,j} \neq 0$.*

Examples 2
(1) The $(n \times n)$ matrix $M_1 = (m_{i,j})$ with $m_{i,j} = 1$ for all i, j has tree width $n-1$. Note that M_1 has linear rank 1.
(2) The $(n \times n)$ matrix $\mathbf{1} = (m_{i,j})$ with $m_{i,i} = 1$ and $m_{i,j} = 0$ for $i \neq j$ has tree width 0. Note that $\mathbf{1}$ has linear rank n.

Theorem 3 (Bodländer). *There is a linear time algorithm (with bad constants) which decides, given a hypergraph G whether it has a k-tree decomposition, and if yes, constructs one.*

Theorem 4 (Bodländer). *If a hypergraph G over n vertices has a k-tree decomposition, then one can construct in linear time a balanced $O(k)$-tree decomposition of depth $O(logn)$.*

A survey of such results may be found in [Bod98,Bod97].

3 Generating functions of graph properties

The first use of tree width of a matrix was presented in [CMR00]. There the computational complexity of computing the *permanent* and *hamiltonian* of a matrix was studied.

Let $M = \{m_{i,j}\}$ be an $(n \times n)$ matrix over a field K. The *permanent* $per(M)$ of M is defined as

$$\sum_{\pi \in \mathcal{S}_n} \prod_i m_{i,\pi(i)}$$

The *hamiltonian* $ham(M)$ of M is defined as

$$\sum_{\pi \in \mathcal{H}_n} \prod_i m_{i,\pi(i)}$$

where $\mathcal{H}_n$ is the set of hamiltonian permutations of $\{1,\ldots,n\}$. Recall that a permutation $\pi \in \mathcal{S}_n$ is *hamiltonian* if the relation $\{(i,\pi(i)) : i \leq n\}$ is connected and forms a cycle.

In general, both the *permanent* and the *hamiltonian* are hard to compute and the best algorithms known so far are exponential in n, [BCS97,Bür99]. This applies also for the computational model due to Blum, Shub and Smale (BSS model), cf. [BCSS98]. Barvinok in [Bar96] has shown that if the (linear) rank of the matrix is bounded by r both the *permanent* and the *hamiltonian* can be computed in polynomial (not linear) time. Hence these problems are parametrically tractable in the sense of [DF99]. Linear rank and tree width are independent notions: The $(n \times n)$ matrix consisting of 1's only has rank 1 but tree width $n-1$ (it is a clique). The corresponding unit matrix has rank n but tree width 1, as the graph consists of isolated points. Tree width of a matrix also makes the *permanent* and the *hamiltonian* parametrically tractable.

Theorem 5 (Courcelle, Makowsky, Rotics, 1998). *Let M be a real $(n \times n)$ matrix of tree width k. Then $per(M)$ and $ham(M)$ can be computed in time $O(n)$.*

The same technique can also be applied to other families of multivariate polynomials such as *cycle format polynomials*, and, more generally, *generating functions of graph properties*, cf. [Jer87,Bür99].

Let $G = \langle V, E, w \rangle$ be an edge weighted graph with weights in a field K and $\mathcal{E}$ be a class of (unweighted) graphs closed under isomorphisms. We extend w to subsets of E by defining $w(E') = \prod_{e \in E'} w(e)$. The *generating function corresponding to G and $\mathcal{E}$* is defined by

$$GF(G,\mathcal{E}) =_{def} \sum \{w(E') : \langle V, E' \rangle \in \mathcal{E} \text{ and } E' \subseteq E\}$$

Strictly speaking $GF(G,\mathcal{E})$ is a function with argument w and value in K. Furthermore, w is a function

$$w : \{1,\ldots,n\}^2 \to K$$

which can be interpreted as an $(n \times n)$ matrix over K. If we view $w(i,j) = u_{i,j}$ as indeterminates, $GF(G,\mathcal{E})$ is a multivariate polynomial in $K[u_{i,j} : i,j \leq n]$.

The *permanent* is the generating function for $G = K_n$, the clique on n vertices, and $\mathcal{E}_{per}$ the perfect matchings. The *hamiltonian*, similarly, is the generating function for $\mathcal{E}_{ham}$, the class of n-cycles.

The proof of theorem 5 relies on the observation that in these two (and many more) cases $\mathcal{E}$ is definable in Monadic Second Order Logic. This allows us to apply techniques first used in a more restrictive framework in [ALS91] and extended in [CMR00]. For more details, cf. section 5.

4 Feasibility and positivity of polynomial systems

We now want to explore how far these techniques can be pushed further. Is the tree width of a system of polynomials an appropriate tool to decide the existence of zero's?

We look at the following problems:

Definition 4. *Let $\mathbb{F}$ be a field (finite or infinite) Let $A \subseteq \mathbb{F}$ be finite of cardinality a. Let $p(x)$ be a polynomial in $\mathbb{F}[x]$ in the variables $x = (x_1, \ldots, x_n)$ of degree d and tree-width k, and Σ be a system of such polynomials, whose $d+1$-hypergraph is of tree width at most k.*

$(d,k) - FEAS_{\mathbb{F}}$: *Does p have a zero in $\mathbb{F}^n$?*
$(d,k) - FEAS(A)_{\mathbb{F}}$: *Does p have a zero in A^n?*
$(d,k) - HN_{\mathbb{F}}$: *Does Σ have a common zero in $\mathbb{F}^n$?*
$(d,k) - HN(A)_{\mathbb{F}}$: *Does Σ have a common zero in A^n?*

For $\mathbb{F}$ an ordered field, we also have

$(d,k) - POS_{\mathbb{F}}$: *Is $p(r) > 0$ for all $r \in \mathbb{R}^n$?*
$(d,k) - POS(A)_{\mathbb{F}}$: *Is $p(r) > 0$ for all $r \in A^n$?*

If the tree width is not bounded we write $(d,\infty) - FEAS(A)_{\mathbb{F}}$, etc. If R is an (finite, infinite, ordered) ring rather than a field, we use the analogous notation $(d,k) - FEAS(A)_R$, etc.

$(d,\infty) - FEAS_{\mathbb{F}}$, $(d,\infty) - HN_{\mathbb{F}}$ and, for ordered fields $(d,\infty) - POS_{\mathbb{F}}$ are discussed in [BCSS98,MM97]. In general they are $\mathbf{NP}_{\mathbb{F}}$ resp. co-$\mathbf{NP}_{\mathbb{F}}$ hard, and no subexponetial algorithms are known for their solution. If we relativize the problems to $(d,\infty) - FEAS(A)_{\mathbb{F}}$ and $(d,\infty) - POS(A)_{\mathbb{F}}$ they are in $\mathbf{DNP}_{\mathbb{F}}$ resp. co-$\mathbf{DNP}_{\mathbb{F}}$, the classes *digital* $\mathbf{NP}$ *resp. digital co-*$\mathbf{NP}$ *over* $\mathbb{F}$. It is known that $\mathbf{P}_{\mathbb{F}} \subseteq \mathbf{DNP}_{\mathbb{F}} \subseteq \mathbf{NP}_{\mathbb{F}}$ for any field $\mathbb{F}$ but it is not known whether the inclusions are proper, cf. [Poi95,MM97].

Conjecture 6. *The problems $(d,\infty) - FEAS(A)_{\mathbb{R}}$ and $(d,\infty) - POS(A)_{\mathbb{R}}$ are $\mathbf{DNP}_{\mathbb{R}}$ resp. co-$\mathbf{DNP}_{\mathbb{F}}$ complete over the reals and $(d,\infty) - HN_{\mathbb{C}}(A)$ is $\mathbf{DNP}_{\mathbb{C}}$ over the complex numbers.*

Consider the following variation of our problems.

Definition 5. *Given a set of n polynomial inequalities P of degree d and of tree width at most k in n variables over the reals and $A \subset \mathbb{R}$ a finite set.*

$(d,k) - PIS(A)_{\mathbb{R}}$: *Does P have a solution in A^n?*

The following partial result can easily be obtained from [Poi95,CucMat]

Proposition 1. *$(d,\infty) - PIS(A)_{\mathbb{R}}$ is $\mathbf{DNP}_{\mathbb{R}}$ complete over the reals for $d \geq 2$.*

Theorem 7. *For finite fields $\mathbb{F}$ of size f, $(d,k) - FEAS_{\mathbb{F}}$ and $(d,k) - HN_{\mathbb{F}}$ can be solved in time $O(n)$ where the constant depends on k, d, f.*
The same holds for finite rings.

The proof exploits the finiteness of the field (ring) by making all elements of it part of the underlying logic. In this way the problem becomes a problem of the weighted hypergraph of the non-vanishing monomials (with the coefficients as weights). Once the problem is coded like this, we can apply the methods of [CMR00,ALS91]. The proof has no particular algebraic content, but shows that the notion of tree width of systems of polynomials has surprising algebraic applications.

With the same techniques we also get

Theorem 8. *For ordered fields $\mathbb{F}$, $(d,k) - POS(A)_{\mathbb{F}}$ can be solved in time $O(n)$ where the constant depends on k, d, a.*
The same holds for any ordered ring R.

Here we can afford infinite fields (rings), but restrict the set of tuples for which we want to evaluate the polynomial. Without bounded tree width no polynomial algorithm is know for this problem.

Using the results stated in [ALS91] we get

Theorem 9. *For arbitrary fields $\mathbb{F}$ $(d,k) - FEAS(A)$ can be solved in time $O(n)$ where the constant depends on k, d, a and a condition concerning the number of different values the monomials take over A.*

The condition concerning the coefficients and A basically requests that the set of partial sums of the monomials evaluated in A be bounded by a function $O(\log n)$.

A similar theorem can be proved also for $(d,k) - PIS(A)_R$, systems of polynomial inequalities of polynomials of bounded degree $d \geq 2$ over an ordered ring R.

Using very effective versions of quantifier elimination over the reals $\mathbb{R}$ one might be able to prove:

Conjecture 10. *Over the reals $\mathbb{R}$ $(d,k) - POS_{\mathbb{R}}$ and $(d,k) - FEAS_{\mathbb{R}}$ can be solved in time $O(n)$ where the constant depends on k, d, a.*

For the best known algorithms to solve $(d,\infty) - FEAS_{\mathbb{R}}$ and for quantifier elimination the reader should consult the surveys [BPR96,Roy96].

5 Meta-finite Monadic Second Order Logic

In this section we sketch the framework in which our approach is settled. In a first step problem instances are defined as specific finite, relational structures together with real-valued weight functions. The latter are called $\mathbb{R}$-structures in [GG98,GM96].

Problems are then given as conjunction of two formulas; one is expressed in monadic second order logic over the underlying finite structure and the other is given in existential monadic second order logic over the corresponding $\mathbb{R}$-structure (a logic to be defined). This generalizes the framework of *Extended Monadic Second order Logic EMSOL* proposed in [ALS91] and unifies it with the framework of *Meta-finite Model Theory* of [GG98].

We shall show how our problems of sections 3 and 4 can be expressed in this framework.

5.1 Monadic Second Order Logic over $\mathbb{R}$-structures

We consider problem instances as logical structures representing particular hypergraphs. To capture the combinatorial aspects of a problem we start with a finite relational structure $(V, E, R_1, \ldots, R_\ell)$ of signature τ. Here, (V, E) is a hypergraph, i.e. $V := \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and any element of E is a (non-void) subset of V (of arbitrary but fixed arity). Every relation R_i is a subset of V^{n_i}, where $n_i \in \mathbb{N}$ denotes its corresponding arity.

We consider $(V, E, R_1, \ldots, R_\ell)$ as a two-sorted structure with universe $V \uplus E$; by convention, there is a relation $R_{inc} \subseteq V \times E$ among the symbols in τ which gives the incidence relation between vertices and edges.

The monadic second order logic $MSOL(\tau)$ over hypergraphs is defined as sublogic of second order logic, where we allow quantified and free second order variables of arity 1 only. In addition, set variables range over subsets of V or E (see for example [CMR00]).

Besides the combinatorial part of our structures the use of weights in some algebraic structure (ring, field, ordered ring, etc.) has to be incorporated. To simplify our notation we assume here that weights are in the ordered field of real numbers $\mathbb{R}$. Structures of this kind are particular $\mathbb{R}$-structures in the sense of [GG98,GM96].

Towards this aim weight functions of arity 1 are added to the structure. For a second vocabulary σ let $w \in \sigma$ be a weight function either of form $w : V \rightarrow \mathbb{R}$ or $w : E \rightarrow \mathbb{R}$. The structures we are interested in are of signature (τ, σ) and have the form $(V, E, R_1, \ldots, R_\ell, w_1, \ldots, w_m)$ together with the ordered structure $(\mathbb{R}, +, *, \leq, 1, 0, -1, r_1, \ldots, r_s)$. Here, the r_i are fixed real constants.

The properties which will be checked on such $\mathbb{R}$-structures are twofold. One is combinatorial and expressed by a $MSOL(\tau)$ formula. The other involves the weight functions and the real number part. It is given as well as a specific monadic second order property, this time defined for the real number part of the structure.

Let us inductively define terms and formulas for obtaining this monadic second order logic over weighted hypergraphs:

Simple terms are of the following form:
- for any weight function $w \in \sigma, w : V \rightarrow \mathbb{R}$ and any $v \in V$ the expression $w(v)$ is a simple term. Similarly, if $w \in \sigma, w : E \rightarrow \mathbb{R}$ and $e \in E$ then $w(e)$ is a simple term;
- the constants $r_1, \ldots, r_s \in \mathbb{R}$ are simple terms;
- if t_1 and t_2 are simple terms so are $t_1 + t_2$ and $t_1 \cdot t_2$.

Summation and product terms are expressions of the following form:
- all simple terms are summation and product terms;
- if $t(v)$ is a simple term depending on an argument $v \in V$ or $v \in E$ respectively, and if $U \subseteq V$ or $U \subseteq E$ resp., then

$$T(U) := \sum_{v \in U} t(v) \text{ and } T(U) := \prod_{v \in U} t(v)$$

  both are summation and product terms;
- if T_1 and T_2 are summation and product terms so are $T_1 + T_2$ and $T_1 \cdot T_2$.

Min/max terms are defined as summation and product terms but using

$$T(U) := \max_{v \in U} t(v) \text{ and } T(U) := \min_{v \in U} t(v)$$

instead of $\sum$ and $\prod$.

In particular, terms of the following type are <u>not</u> allowed in the above construction: $T(U_1, v_2, \ldots, v_k) := \sum\limits_{v_1 \in U_1} t(v_1, \ldots, v_k)$. It is not possible to apply the summation or product rule to arbitrary terms.

Monadic second order terms are all terms above together with terms of the following form:
- if $T(U_1, \ldots, U_t)$ is a monadic second order term with set variables $U_1, \ldots, U_t$ and if $\rho(U_1, \ldots, U_t)$ is a $MSOL(\tau)$ formula then both $\sum\limits_{U_1, \ldots, U_t, \rho(U_1, \ldots, U_t)} T(U_1, \ldots, U_t)$ and $\prod\limits_{U_1, \ldots, U_t, \rho(U_1, \ldots, U_t)} T(U_1, \ldots, U_t)$ are monadic second order terms.
- if T_1 and T_2 are monadic second order terms then so are $T_1 + T_2$ and $T_1 \cdot T_2$.

After having introduced monadic second order terms we turn to monadic second order formulas built from these terms and denoted by $\exists$-$\mathrm{MSO}_{\mathbb{R}}$.

Basic $\text{MSO}_{\mathbb{R}}$ **formulas** are expressions of the form $T(v_1 \dots, v_k, U_1, \dots, U_t)\Delta 0$ with $\Delta \in \{=, >, \geq\}$; here, T is a monadic second order term, the v_i are elements of the universe and the U_i are subsets of the universe (either of V or of E).

$\text{MSO}_{\mathbb{R}}$ **formulas** are all basic $\text{MSO}_{\mathbb{R}}$ formulas together with

- if $\psi(v_1, \dots, v_k, V_1, \dots, V_t)$ is a $\text{MSO}_{\mathbb{R}}$ formula and $U_1, \dots, U_k$ are subsets of the universe, then $\rho(V_1, \dots, V_t) \equiv \bigwedge_{v_i \in U_i} \psi(v_1, \dots, v_k, V_1, \dots, V_t)$ is a $\text{MSO}_{\mathbb{R}}$ formula;
- if $\psi(v_1, \dots, v_k, U_1, \dots, U_s, V_1, \dots, V_t)$ is a $\text{MSO}_{\mathbb{R}}$ formula and $\rho(v_1, \dots, v_k, U_1, \dots, U_s)$ is a $MSOL(\tau)$ formula then $\phi(V_1, \dots, V_t) \equiv \bigwedge_{\underline{v}, \underline{U}, \rho(\underline{v}, \underline{U})} \psi(\underline{v}, \underline{U}, V_1, \dots, V_t)$ is a $\text{MSO}_{\mathbb{R}}$ formula;
- the closure of the above construction scheme under logical conjunction, disjunction and negation gives the set of $\text{MSO}_{\mathbb{R}}$ formulas.

$\exists$-$\text{MSO}_{\mathbb{R}}$ (Existential monadic second order logic) over $\mathbb{R}$-structures is obtained from $\text{MSO}_{\mathbb{R}}$ logic by defining

- all formulas in $\text{MSO}_{\mathbb{R}}$ to belong to $\exists$-$\text{MSO}_{\mathbb{R}}$
- if $\psi(V_1, \dots, V_t) \in \exists$-$\text{MSO}_{\mathbb{R}}$ then $\exists V_1, \dots, V_t \; \psi(V_1, \dots, V_t) \in \exists$-$\text{MSO}_{\mathbb{R}}$.

The problems considered in this paper are of the following type:

Decision problems: For a fixed $\text{MSO}_{\mathbb{R}}$ formula ψ, decide whether $\mathcal{G} \models \psi$.
Evaluation problem: For a fixed $\text{MSO}_{\mathbb{R}}$ term T, compute its value over $\mathcal{G}$.
Extended decision problem: Given a $MSOL(\tau)$ formula ψ as well as a $\exists$-$\text{MSO}_{\mathbb{R}}$ formula Φ we want to decide whether $\mathcal{G} \models \psi \wedge \Phi$.
Optimization problems: These are like the extended decision problems, but with Φ quantifier free, but possibly involving the functions max and min.

5.2 Guiding examples

Let us consider some examples and the way they fit into the formal setting of the previous section.

Example 1. The generating functions of section 3

$$GF(G, \mathcal{E}) =_{def} \sum \{w(E') : \langle V, E' \rangle \in \mathcal{E} \text{ and } E' \subseteq E\}$$

with $\mathcal{E}$ $MSOL$-definable by $\psi(E')$ can be written as a Monadic Second Order Term

$$\sum_{\psi(E') \wedge E' \subseteq E} \prod_{e \in E'} w(e)$$

Hence they are $\text{MSO}_{\mathbb{R}}$-evaluation problems.

Example 2 (4-$FEAS(A)$:). In order to formalize the 4-$FEAS(A)$ problem as an $\text{MSO}_{\mathbb{R}}$ extended decision problem, we use the representation of degree 4 polynomials as given in [GM96]: let $V = \{0, 1, \dots, n\}, E := V^4$ and $C : E \rightarrow \mathbb{R}$ be a weight function giving the coefficients of f in the following sense: for $(i, j, k, l) \in E$ the value $C(i, j, k, l)$ is the coefficient of the monomial $x_i \cdot x_j \cdot x_k \cdot x_l$ in f. Note that here we assume f to be homogeneous of degree 4. Later on f is dehomogenized by adding the condition $x_0 := 1$.

Let $A := \{s_1, \dots, s_m\}$; for every $1 \leq i \leq m$ define a function $w_i : V \rightarrow A$ such that $\forall x \in V \; w_i(x) := s_i$. We are looking for disjunct subsets $U_1, \dots, U_m$ of V such that the following holds: $\bigcup_{i=1}^{m} U_i = V$ and if we assign to every $x \in U_i$ the value $w_i(x) = s_i$ then this assignment of variables gives a zero of f.

For any quadruple $\lambda = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ where $\epsilon_i \in \{1, \ldots, m\}$ define the set $E_\lambda := E \cap (U_{\epsilon_1} \times U_{\epsilon_2} \times U_{\epsilon_3} \times U_{\epsilon_4})$ (i.e. a point $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in V^4$ belongs to E_λ iff every component α_i lies in U_{ϵ_i}).

For $\alpha \in E_\lambda$ the corresponding monomial $C(\alpha) \cdot x^\alpha$ gives the value $C(\alpha) \cdot s_{\epsilon_1} \cdot s_{\epsilon_2} \cdot s_{\epsilon_3} \cdot s_{\epsilon_4}$ under the above assignment. The $MSOL(\tau)$ formula for 4-$FEAS(A)$ simply is

$$\psi \equiv \exists U_1, \ldots, U_m \subseteq V \text{ such that } \forall i, j \in \{1, \ldots, m\} U_i \cap U_j = \emptyset \wedge U_1 \cup \ldots \cup U_m = V$$

(note that m is independent of the structure).

The real number part of the logical description is $\Phi \equiv \sum_{\lambda \in \{1,\ldots,m\}^4} T(E_\lambda) = 0$, where $T(E_\lambda) := \sum_{\alpha \in E_\lambda} := C(\alpha) \cdot s_{\epsilon_1} \cdot s_{\epsilon_2} \cdot s_{\epsilon_3} \cdot s_{\epsilon_4}$ is a summation/product term. Again note that m is independent of the structure; hence, the same holds true for the size of the first sum above. For dehomogenization one can introduce a further subset $U_0 \subseteq V$ which only consists of the element 0 and put $s_0 := 1$. It follows that f has a zero in A if and only if $(V, E, C) \models \psi \wedge \Phi$.

In a completely similar fashion the feasibility of polynomial inequality systems 2-$PIS(A)_\mathbb{R}$ fits into the framework of $\mathrm{MSO}_\mathbb{R}$ extended decision problems and $(d, \infty)-POS(A)_\mathbb{F}$ can be coded as an optimization problem. If the ring (field) is finite the coding yields an $\mathrm{MSO}_\mathbb{R}$ decision problem.

5.3 Hypergraphs of bounded tree width

Our main result can be stated as follows:

Theorem 11. *For structures of tree width at most k*

(i) $\mathrm{MSO}_\mathbb{R}$*-decision, optimization and evaluation problems can be solved in linear time (in the size of the structure).*

(ii) $\mathrm{MSO}_\mathbb{R}$*-extended decision problems can be solved in linear time provided the possible values of the subterms in Φ is bounded by $O(\log n)$ where n is the size of the structure.*

The proof is a tedious adaptation of the techniques developed in [ALS91] and [CMR00]. Some details were given in [Mak99] and will be given in the full paper.

6 Conclusions and further research

We have shown how the concept of tree width of multivariate polynomial systems can be used in finding polynomial time algorithms provided the tree width is bounded by a constant. The method has further extensions to linear and quadratic programming, previously analyzed in [Mee94].

Our proofs use a detour through logic. An alternative route would be through automata theory, as in [ALS91]. But automata theory and Monadic Second Order Logic are just two faces of the same definability phenomenon, cf. [Cou97,Cou90,Tho90].

It remains a challenging problem to find direct algebraic proofs and to overcome the limitations imposed by our coding technique.

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