Constructing Asymmetric Covering Codes by Tabu Search*

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Abstract

A set $C \subseteq \mathbb{F}_2^n$ is said to be an asymmetric covering code with radius R if every word $x \in \mathbb{F}_2^n$ can be obtained by replacing 1 by 0 in at most R coordinates of a word in C. In this paper, tabu search is employed in the search for good asymmetric covering codes of small length. Fifteen new upper bounds on the minimum size of such codes are obtained in the range $n \leq 13$.

1 Introduction

A *covering code* [2] has the property that any word in the space where it resides can be obtained by changing at most R of its coordinate values, for a prescribed parameter R called *covering radius*. We here only consider binary codes, that is, codes in $\mathbb{F}_2^n$, where $\mathbb{F}_2 = \{0, 1\}$. In certain applications—see [3]—it makes sense to allow only changes (w.l.o.g.) $1 \to 0$. This restriction leads to the concept of *asymmetric covering codes*.

Asymmetric covering codes were introduced by Cooper, Ellis, and Kahng [3]—who called them directed covering codes—and later studied by Applegate, Rains, and Sloane [1]. The size of the smallest asymmetric covering code of length n and radius R is denoted by $D(n, R)$. Bounds on $D(n, R)$ are tabulated for $n \leq 13$ in [3], and for $n \leq 11$, $R = 1$ in [1].

In the current paper, tabu search is used to find record-breaking asymmetric covering codes. The choice of the algorithm was inspired by the promising results of using tabu search in the search for covering codes [9].

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The set of words obtained by making at most R changes $1 \to 0$ of the coordinate values to a given word is called a *downward directed ball*. Then a code is an asymmetric covering code if and only if, for a given value of R, the union of the downward directed balls around the codewords is $\mathbb{F}_2^n$. (Generally, the words in a downward directed ball of an arbitrary code are said to be *covered*.) The packing version of this covering problem, when the downward directed balls *must not* intersect, has been thoroughly studied in the framework of *asymmetric error-correcting codes*; see [4, 5, 7, 10] and their references.

The paper is outlined as follows. Application of tabu search to the current problem is discussed in Section 2. Particular emphasis is put on handling instances where good codes have a relatively large size. In this manner, fifteen improved upper bounds on $D(n, R)$ are obtained. An updated table of upper bounds on $D(n, R)$ for $n \leq 13$ is presented in Section 3, and codes attaining the new bounds are listed.

2 The Algorithm

Tabu search [8] is a *local search method*. In local search, the current (feasible) solution is replaced by another solution from a set of solutions that in some sense are close—this set is called the *neighborhood*. The search is guided by a *fitness function* or *cost function*, defined for all solutions. Without loss of generality, we only consider minimization problems here.

Tabu search has evolved from the *steepest descent method*. In the steepest descent method the entire neighborhood is evaluated in each step and a solution with the smallest fitness value is selected. The search ends if there are no better solutions in the neighborhood. Similarly to the steepest descent method, in tabu search the entire neighborhood is examined. However, in tabu search the best solution is always chosen even if it is worse than the previous one. If there are several equally good solutions, then the selection is done randomly out of those. To prevent the algorithm from looping, a tabu list is used. The tabu list prevents the algorithm from changing the current solution into one that was recently traversed. For more details about tabu search, the reader is referred to [8].

We shall now have a closer look at the details of applying tabu search to finding good asymmetric covering codes.

2.1 Parameters of Tabu Search

A (feasible) solution is here simply defined as an M-element subset of $\mathbb{F}_2^n$, for a fixed M. In practice, M is chosen to be slightly smaller than the previous record for the instance considered. The obvious fitness function

$f(C)$ for a solution C is the number of uncovered words. When $f(C) = 0$, the code C is a covering code.

The main idea behind our choice of neighborhood is that the next solution should cover a particular word that is not covered currently. This idea is introduced for covering codes in [9], but we shall here arrive at a different definition of neighborhood.

Firstly, we require that exactly one codeword be altered, and that for such an alteration $c \to c'$ the Hamming distance between the words be 1, $d_H(c, c') = 1$. Secondly, we require that at least one currently uncovered word be covered after the alteration. (The second requirement implies that the size of the neighborhood decreases when the search proceeds.)

It turns out that the algorithm performs better for the described neighborhood than with a neighborhood definition that is analogous to the one in [9] (the new definition of neighborhood might therefore prove useful for covering codes in general).

For small codes, a random initial solution is chosen. For large codes, however, a search starting from a random initial solution often converges to a solution whose fitness value is relatively far from the value of a global optimum. We consider the search for large codes in more detail in Section 2.2.

Note that an asymmetric covering code will always contain the all-one codeword.

In [9] the tabu list is defined to contain the recently altered codewords, which must not be altered. It appears that with the neighborhood used here a more specific definition works better: if an alteration $c \to c'$ is carried out, then this alteration and its inverse $c' \to c$ are banned. Interestingly, experiments show that the length of the tabu list (telling the number of steps a move is banned) has little impact on the search, which makes tuning the algorithm easy. One may also define a so-called *aspiration criterion* which allows moves that are tabu under some particular conditions—we allow tabu moves that lead to a covering, $f(C) = 0$.

2.2 Searching for Large Codes

The basic tabu search approach is very effective for small codes, but the performance deteriorates rapidly when the size of the codes increases. There are several possible ways of combating this trend.

For large codes, instead of starting from a random initial solution one may start from a solution (code) with a small fitness value. There are several ways of obtaining such codes, for example, by removing codewords from a code attaining the best known bound, or by using a construction that gives a near-covering. One may as well start from (not so good) codes obtained through the direct sum construction and corresponding to the bounds $D(n+1, R+1) \leq D(n,R)$ and $D(n+1,R) \leq 2D(n,R)$, and remove codewords from these.

It is also possible to fix a part of the code and only alter some of the codewords; in particular, this idea is related to the following observation.

Theorem 2.1 *Puncture an asymmetric covering code C of length n and radius R to get the code $C' = \{x : (1,x) \in C\}$. Then C' is an asymmetric covering code of length $n-1$ and radius R.*

Proof. Let $C' = \{x : (1,x) \in C\}$ and $C'' = \{x : (0,x) \in C\}$. Then $C = C' \cup C''$. Any word $(1,y) \in \mathbb{F}_2^n$ must be covered (with radius R) by a word $(1,x) \in C$, so any word $y \in \mathbb{F}_2^{n-1}$ must be covered (with radius R) by a word $x \in C'$. □

To search for codes in $\mathbb{F}_2^n$, we may go in the opposite direction: take a good code in $\mathbb{F}_2^{n-1}$, append a 1 to all codewords, and then search for the rest of the words. Since one of the coordinates of these words is fixed to 0, the search actually takes place in $\mathbb{F}_2^{n-1}$. This search may be repeated for different, inequivalent codes in the part that is fixed. We may also take a good near-covering, fix the words with a nonzero weight in one or several coordinates, and continue the search by altering the words with zero weight in those coordinates (whereby the search is carried out in $\mathbb{F}_2^{n-k}$, where k is the number of fixed coordinates). Some of the codes were found by alternating different search strategies. The best known asymmetric covering code with given parameters may also be used as the starting point for a search, after removing one codeword.

Except for codes with very few codewords, it appears that the best asymmetric covering codes found often have no nontrivial automorphisms; see also [1, Proof of Theorem 2.1]. Therefore, the approach of prescribing automorphisms in the search for large codes does not seem promising.

3 The Results

The best known upper bounds on $D(n,R)$ for $3 \leq n \leq 13$ are shown in Table 1 (the cases $n = 1, 2$ are covered by $D(n, n-1) = 2$ and $D(n,R) = 1$ for $R \geq n$).

The bounds obtained in the current work are starred; the other bounds are from [1, 3, 6]. The exact values in the table are indicated by a period.

The new codes obtained in this study are here listed in decimal from. These codes can also be obtained in electronic form directly from the authors or at <URL:`http://www.hut.fi/~eseurane/papers/acc/`>.

$R\backslash n$	3	4	5	6	7	8	9	10	11	12	13
1	3.	6.	10.	18.	31.	58.	106	196	352	668	1253
2	2.	3.	5.	8.	14	23	40*	70	121*	218*	421*
3	1.	2.	3.	4.	7	12	19	31*	51*	92*	165*
4	1.	1.	2.	3.	4.	6.	10	15*	25	42	71*
5	1.	1.	1.	2.	3.	4.	6.	8.*	13*	21	35*
6	1.	1.	1.	1.	2.	3.	4.	5.	8	12*	18*
7	1.	1.	1.	1.	1.	2.	3.	4.	5.	7.	11
8	1.	1.	1.	1.	1.	1.	2.	3.	4.	5.	7.
9	1.	1.	1.	1.	1.	1.	1.	2.	3.	4.	5.
10	1.	1.	1.	1.	1.	1.	1.	1.	2.	3.	4.
11	1.	1.	1.	1.	1.	1.	1.	1.	1.	2.	3.
12	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	2.
13	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.

Table 1: Upper bounds on $D(n, R)$

$D(9,2) \leq 40$: 10, 31, 50, 57, 67, 163, 206, 213, 216, 231, 234, 241, 247, 252, 254, 255, 263, 284, 302, 309, 315, 319, 347, 356, 365, 374, 388, 411, 429, 438, 457, 463, 466, 477, 484, 491, 503, 505, 510, 511.

$D(10,3) \leq 31$: 67, 123, 220, 271, 402, 438, 445, 471, 485, 494, 507, 556, 637, 639, 689, 715, 767, 791, 857, 866, 895, 926, 935, 938, 957, 966, 975, 1010, 1011, 1020, 1023.

$D(10,4) \leq 15$: 158, 188, 247, 319, 374, 567, 638, 833, 950, 969, 989, 1003, 1007, 1019, 1023.

$D(10,5) \leq 8$: 222, 415, 702, 865, 879, 1013, 1019, 1023.

$D(11,2) \leq 121$: 54, 62, 93, 111, 115, 133, 216, 267, 322, 412, 430, 434, 461, 465, 471, 483, 495, 505, 528, 609, 639, 647, 681, 693, 699, 701, 730, 734, 740, 758, 780, 795, 799, 805, 823, 842, 854, 878, 888, 958, 971, 989, 997, 1011, 1015, 1022, 1023, 1064, 1124, 1151, 1179, 1183, 1189, 1207, 1226, 1238, 1262, 1272, 1291, 1297, 1303, 1327, 1337, 1350, 1384, 1396, 1402, 1404, 1440, 1470, 1471, 1485, 1499, 1507, 1519, 1525, 1531, 1564, 1582, 1586, 1613, 1617, 1627, 1635, 1653, 1663, 1666, 1687, 1711, 1721, 1759, 1786, 1788, 1789, 1851, 1853, 1863, 1886, 1887, 1897, 1910, 1917, 1932, 1936, 1946, 1954, 1967, 1972, 1979, 1985, 1990, 1991, 2001, 2013, 2024, 2025, 2027, 2038, 2039, 2046, 2047.

$D(11,3) \leq 51$: 57, 244, 362, 375, 423, 443, 463, 582, 635, 651, 735, 758, 789, 951, 984, 997, 1017, 1021, 1022, 1054, 1151, 1215, 1235, 1262, 1357, 1408, 1437, 1503, 1507, 1532, 1633, 1660, 1708, 1757, 1779, 1807, 1837, 1842, 1855, 1886, 1907, 1966, 1969, 1982, 2002, 2023, 2027, 2031, 2039, 2043, 2047.

$D(11,5) \leq 13$: 858, 895, 1189, 1207, 1263, 1453, 1525, 1725, 1765, 1959, 2011, 2046, 2047.

$D(12,2) \leq 218$: 10, 29, 91, 143, 176, 198, 215, 222, 291, 356, 376, 382, 430, 443, 489, 503, 557, 573, 577, 615, 634, 682, 695, 751, 757, 790, 829, 859, 863, 912, 915, 959, 961, 974, 980, 1023, 1078, 1147, 1151, 1176, 1203, 1249, 1262, 1268, 1284, 1293, 1309, 1351, 1370, 1418, 1431, 1468, 1487, 1493, 1533, 1539, 1604, 1624, 1630, 1678, 1691, 1709, 1737, 1751, 1787, 1824, 1829, 1831, 1854, 1890, 1893, 1895, 1903, 1911, 1912, 1913, 1933, 1962, 1963, 1968, 1973, 1983, 2014, 2023, 2028, 2034, 2043, 2070, 2125, 2135, 2136, 2143, 2177, 2195, 2206, 2250, 2255, 2260, 2265, 2271, 2303, 2358, 2363, 2414, 2415, 2417, 2468, 2493, 2527, 2531, 2548, 2554, 2607, 2609, 2658, 2669, 2679, 2717, 2725, 2747, 2783, 2786, 2808, 2812, 2824, 2827, 2892, 2942, 2958, 2968, 2974, 2981, 3007, 3015, 3026, 3033, 3051, 3061, 3067, 3112, 3115, 3180, 3246, 3256, 3262, 3295, 3303, 3314, 3321, 3327, 3343, 3345, 3388, 3394, 3405, 3415, 3451, 3461, 3483, 3497, 3505, 3511, 3522, 3544, 3548, 3566, 3567, 3575, 3581, 3606, 3611, 3637, 3662, 3663, 3665, 3709, 3716, 3720, 3741, 3759, 3779, 3796, 3802, 3823, 3830, 3831, 3837, 3861, 3871, 3880, 3885, 3890, 3891, 3935, 3937, 3943, 3946, 3956, 3966, 3967, 4006, 4024, 4025, 4026, 4027, 4028, 4045, 4051, 4062, 4063, 4064, 4065, 4071, 4077, 4085, 4088, 4094, 4095.

$D(12,3) \leq 92$: 167, 254, 455, 496, 573, 612, 926, 937, 943, 989, 1023, 1034, 1116, 1183, 1263, 1283, 1340, 1531, 1547, 1643, 1655, 1734, 1777, 1855, 1874, 1883, 1893, 1902, 1978, 2007, 2041, 2046, 2131, 2206, 2367, 2397, 2410, 2422, 2543, 2704, 2734, 2759, 2761, 2799, 2828, 2867, 2933, 2937, 2942, 3034, 3039, 3043, 3052, 3167, 3173, 3240, 3255, 3320, 3325, 3371, 3455, 3473, 3494, 3513, 3518, 3539, 3572, 3581, 3634, 3676, 3739, 3772, 3775, 3809, 3827, 3830, 3839, 3863, 3933, 3938, 3947, 3981, 4021, 4023, 4031, 4040, 4046, 4047, 4077, 4087, 4090, 4095.

$D(12,6) \leq 12$: 767, 1690, 1786, 1951, 2719, 3034, 3429, 3583, 3965, 4026, 4071, 4095.

$D(13,2) \leq 421$: 53, 90, 136, 236, 250, 303, 323, 351, 415, 422, 431, 469, 471, 476, 496, 543, 577, 601, 627, 659, 697, 719, 736, 751, 808, 831, 885, 890, 950, 985, 1004, 1022, 1023, 1100, 1103, 1151, 1169, 1180, 1249, 1269, 1279, 1302, 1305, 1312, 1335, 1400, 1423, 1454, 1490, 1507, 1526, 1529, 1583, 1588, 1630, 1637, 1642, 1674, 1683, 1695, 1737, 1779, 1788, 1820, 1841, 1858, 1871, 1877, 1891, 1892, 1914, 1925, 1961, 1979, 1981, 1994, 2007, 2011, 2023, 2029, 2046, 2054, 2063, 2111, 2131, 2166, 2265, 2271, 2332, 2380, 2410, 2437, 2473, 2483, 2492, 2502, 2527, 2528, 2549, 2554, 2608, 2630, 2683, 2723, 2732, 2773, 2778, 2799, 2831, 2879, 2889, 2903, 2909, 2918, 2921, 2926, 2948, 2954, 2959, 2998, 3001, 3024, 3070, 3071, 3113, 3151, 3157, 3187, 3196, 3199, 3202, 3238, 3247, 3327, 3338, 3387, 3419, 3455, 3503, 3504, 3523, 3532, 3557, 3562, 3574, 3577, 3583, 3589, 3610, 3691, 3711, 3743, 3749, 3754, 3795, 3804, 3824, 3827, 3836, 3862, 3865, 3870, 3879, 3885, 3919, 3956, 3957, 3962, 4027, 4032, 4038, 4041, 4055, 4077, 4094, 4154, 4181, 4207, 4239, 4277, 4287, 4322, 4368, 4442, 4457, 4463, 4484, 4559, 4579, 4588, 4599, 4603, 4642, 4684, 4703, 4728, 4745, 4783, 4819, 4820, 4837, 4842, 4855, 4861, 4879, 4882, 4887, 4897, 4902, 4921, 4987, 4989, 4995, 5045, 5050, 5064, 5078, 5081, 5091, 5118, 5151, 5155, 5180, 5238, 5241, 5247, 5315, 5318, 5327, 5365, 5368, 5370, 5407, 5426, 5445, 5491, 5500, 5512, 5525, 5530, 5554, 5565, 5598, 5599, 5611, 5614, 5623, 5636, 5679, 5690, 5713, 5750, 5753, 5766, 5788, 5797, 5800, 5815, 5822, 5844, 5851, 5870, 5871, 5885, 5887, 5950, 5951, 5958, 5961, 5981, 5982, 5991, 5996, 6011, 6031, 6032, 6035, 6044, 6054, 6057, 6075, 6085, 6090, 6103, 6125, 6127, 6129, 6141, 6143, 6153, 6182, 6201, 6245, 6255, 6287, 6294, 6296, 6335, 6374, 6377, 6419, 6452, 6464, 6485, 6490, 6492, 6511, 6627, 6636, 6647, 6655, 6666, 6677, 6707, 6716, 6751, 6850, 6867, 6876, 6885, 6890, 6909, 6927, 6966, 6969, 7024, 7035, 7046, 7049, 7055, 7071, 7072, 7087, 7093, 7098, 7119, 7126, 7129, 7139, 7148, 7152, 7166, 7199, 7242, 7248, 7286, 7289, 7295, 7305, 7318, 7321, 7347, 7356, 7375, 7413, 7418, 7427, 7436, 7455, 7461, 7466, 7539, 7548, 7573, 7578, 7613, 7632, 7646, 7659, 7678, 7727, 7733, 7738, 7747, 7756, 7776, 7798, 7801, 7856, 7863, 7877, 7882, 7899, 7918, 7919, 7933, 7935, 7998, 7999, 8006, 8009, 8029, 8031, 8032, 8039, 8055, 8059, 8079, 8083, 8092, 8095, 8102, 8105, 8112, 8119, 8123, 8133, 8138, 8144, 8151, 8155, 8160, 8173, 8175, 8177, 8178, 8179, 8180, 8184, 8189, 8191.

$D(13,3) \leq 165$: 46, 351, 435, 629, 779, 827, 967, 984, 1022, 1120, 1229, 1342, 1467, 1517, 1670, 1721, 1743, 1783, 1845, 1966, 2011, 2079, 2175, 2195, 2271, 2477, 2504, 2543, 2644, 2704, 2727, 2909, 2922, 2935, 3039, 3057, 3071, 3283, 3326, 3376, 3414, 3425, 3445, 3455, 3479, 3570, 3626, 3661, 3691, 3771, 3772, 3797, 3838, 3843, 3887, 3992, 4029, 4045, 4071, 4079, 4088, 4091, 4255, 4330, 4357, 4454, 4542, 4717, 4731, 4766, 4769, 4854, 5039, 5111, 5113, 5145, 5244, 5341, 5347, 5479, 5499, 5515, 5524, 5614, 5621, 5695, 5714, 5722, 5783, 5871, 5928, 5965, 6002, 6013, 6047, 6057, 6108, 6112, 6142, 6143, 6201, 6211, 6388, 6399, 6495, 6507, 6513, 6570, 6589, 6613, 6648, 6770, 6862, 6887, 6889, 6924, 6931, 6974, 7062, 7069, 7130, 7151, 7157, 7163, 7332, 7349, 7387, 7399, 7453, 7470, 7512, 7603, 7615, 7626, 7631, 7670, 7677, 7687, 7775, 7789, 7819, 7868, 7873, 7893, 7903, 7927, 7929, 7962, 7991, 8022, 8036, 8043, 8046, 8049, 8062, 8063, 8102, 8122, 8125, 8131, 8147, 8151, 8171, 8190, 8191.

$D(13,4) \leq 71$: 435, 1118, 1763, 1786, 1934, 1979, 2276, 2777, 2989, 3059, 3197, 3337, 3514, 3575, 3687, 3831, 3863, 3892, 3934, 4030, 4031, 4069, 4351, 4626, 4933, 4989, 5084, 5277, 5496, 5609, 5627, 5693, 5800, 5831, 5935, 5974, 5979, 6095, 6101, 6127, 6141, 6191, 6491, 6607, 6653, 6738, 6782, 6907, 7014, 7094, 7145, 7166, 7350, 7420, 7423, 7541, 7618, 7659, 7662, 7819, 7837, 7917, 7967, 8031, 8042, 8063, 8119, 8159, 8178, 8186, 8191.

$D(13,5) \leq 35$: 574, 989, 1022, 1137, 2743, 2900, 2927, 3007, 3483, 3560, 4095, 4338, 5767, 5990, 5997, 6070, 6077, 6103, 6475, 6585, 7167, 7291, 7403, 7423, 7646, 7669, 7709, 7854, 7902, 7925, 7983, 8059, 8148, 8171, 8191.

$D(13,6) \leq 18$: 887, 1478, 1719, 3071, 5582, 6135, 6713, 6719, 7622, 7643, 7657, 7676, 7807, 8127, 8141, 8152, 8170, 8191.

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