A CONTINUOUS METHOD FOR EXTREME EIGENVALUE PROBLEMS*

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Abstract. In this paper, a continuous method is introduced to compute both the extreme eigenvalues and their corresponding eigenvectors for a real symmetric matrix. The main idea is to convert the extreme eigenvalue problem into an optimization problem. Then a continuous method which includes both a merit function and an ordinary differential equation (ode) is introduced for the resulting optimization problem. The convergence of the ode solution is proved for any starting point. The limit of the ode solution for any starting point is fully studied. Both the extreme eigenvalues and their corresponding eigenvectors can be easily obtained under a very mild condition. Promising numerical results are also presented.

Key words. continuous method, extreme eigenvalue, extreme eigenvector.

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1. Introduction. Let $A \in R^{n\times n}$ be a symmetric matrix. From the Real Schur Theorem, we know that all eigenvalues of A are real and there exists an orthonormal matrix $U = (u_1, u_2, \ldots, u_n)$ and a diagonal matrix Λ such that

$$A = U\Lambda U^T, \tag{1.1}$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_1 = \cdots = \lambda_s < \lambda_{s+1} \le \cdots \le \lambda_n, \tag{1.2}$$

and $1 \le s \le n$. As a result, we denote

$$S = \{x \in R^n | Ax = \lambda_1 x,\ x^T x = 1\}. \tag{1.3}$$

Obviously, the columns of U form an orthonormal basis in R^n and S is the subset containing all the eigenvectors with l_2-norm one corresponding to the smallest eigenvalue of A. Since the minimum eigenvalue of $-A$ is $-\lambda_n$, so we only focus on finding λ_1 and an $x \in S$ in the rest of the paper.

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The eigenvalue problem is a classical but very important problem (see [12]). Besides the conventional methods in numerical analysis for the extreme eigenvalue problem (see [12] and the references therein), some continuous methods have been discussed in [1, 2, 3]. In [1, 2], various ode systems are introduced for many numerical analysis problems. Sparked by Hopfield's neural network approach [6, 7, 8], Cichocki and Unbehauen [3] introduced a neural network model for computing the minimum eigenvalue and the corresponding eigenvector. The idea in [3] is to convert the minimum eigenvalue problem into a constrained optimization problem. Then a neural network model was introduced to solve this constrained problem by using either the penalty method or the Lagrange multiplier method. However, the optimization problems formulated in [3] are not easy to solve. Therefore, the application of their methods is quite limited.

In this paper, we also convert the minimum eigenvalue problem into an optimization problem (Section 2). However, our optimization problem is to minimize a strictly concave function over a unit ball. Therefore, our optimization problem is very easy to solve. For the resulting optimization problem, a continuous method which consists of a merit function and an ordinary differential equation (ode) is introduced. The convergence of the ode solution is proved for any starting point (Section 3). Some promising numerical results are reported (Section 4). Finally, some conclusions are drawn (Section 5).

2. An equivalent optimization problem. To formulate the extreme eigenvalue problem into an optimization problem, we consider

$$\begin{aligned} \min_{x \in R^n} \quad & x^T A x \\ s.t. \quad & x^T x = 1. \end{aligned} \tag{2.1}$$

For any $x \in R^n$ with $x^T x = 1$, there exist $\alpha_i, \ i = 1, \ldots, n$ such that

$$x = \sum_{i=1}^{n} \alpha_i u_i, \tag{2.2}$$

where u_i's are the column vectors of U. Then problem (2.1) becomes

$$\begin{aligned} \min \quad & \sum_{i}^{n} \alpha_i^2 \lambda_i \\ s.t. \quad & \sum_{i=1}^{n} \alpha_i^2 = 1. \end{aligned} \tag{2.3}$$

LEMMA 2.1. *(i) Every local minimizer of (2.1) is also a global minimizer of (2.1). (ii) x is a global minimizer of (2.1) $\iff$ $x \in S$.*

Proof. (i) From (2.3), it is easy to see that

$$\sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq \lambda_1. \tag{2.4}$$

Then the result in (i) can be easily established by manipulating α_i's.

(ii) From the definition of S and (2.4), the result in (ii) is trivial. ◻

From the proof of Lemma 2.1, it is easy to see that the minimum objective function value of (2.1) is λ_1 and any optimal solution is an eigenvector corresponding to λ_1.

Problem (2.1) is to minimize a quadratic function on the surface of a ball. The difficulty for problem (2.1) is its constraint where the feasible region is not a convex set. Now we further convert problem (2.1) into another optimization problem which is much easier to solve. First, let's select a constant c such that

$$c \geq \lambda_n + 1. \tag{2.5}$$

Since

$$\max_{1\leq i\leq n} |\lambda_i| = \|A\|_2 \leq \|A\|_F, \tag{2.6}$$

we can always choose $c = \|A\|_F + 1$. Then we can establish the following problem:

$$\begin{aligned} \min_{x\in R^n} \quad & x^T Ax - cx^T x \\ s.t. \quad & x^T x \leq 1. \end{aligned} \tag{2.7}$$

Problem (2.7) differs from problem (2.1) in that the objective function is quadratic and strictly concave but the constraint is a simple ball constraint. The feasible region for (2.7) is a closed convex set. Therefore, it is much easier to solve (2.7) than (2.1).

LEMMA 2.2. *(i) Every local minimizer of (2.7) is also a global minimizer of (2.7). (ii) x is a global minimizer of (2.7) $\iff$ $x \in S$.*

Proof. For any $x \in R^n$, there exist $\alpha_i,\ i = 1, \ldots, n$ such that

$$x = \sum_{i=1}^{n} \alpha_i u_i, \tag{2.8}$$

where u_i's are the column vectors of U. Then problem (2.7) becomes

$$\begin{aligned} \min \quad & \sum_{i}^{n} \alpha_i^2(\lambda_i - c) \\ s.t. \quad & \sum_{i=1}^{n} \alpha_i^2 \leq 1. \end{aligned} \tag{2.9}$$

Since

$$0 > \lambda_i - c \geq \lambda_1 - c, \quad i = 1, \ldots, n, \tag{2.10}$$

the results in (i) and (ii) can be easily established. □

From Lemma 2.2, we can easily see that the minimum value of (2.7) is $\lambda_1 - c$ and any optimal solution is an eigenvector corresponding to λ_1.

3. A continuous method. Now we focus on problem (2.7). Generally speaking, a continuous method for an optimization problem consists of two components: a merit function (bounded below) and a dynamical system. In addition, the merit function must be monotonically nonincreasing along the solution of the dynamical system. Following the model developed in [9], we have our continuous method for problem (2.7):

Merit function:

$$f(x) = x^T A x - c x^T x. \tag{3.1}$$

Dynamical system:

$$\frac{dx(t)}{dt} = -[x - P_\Omega(x - \nabla f(x))], \tag{3.2}$$

where $\Omega = \{x \in R^n | x^T x \leq 1\}$ and $P_\Omega(\cdot)$ is the projection onto Ω. To ease the following discussion, we define

$$e(x) = x - P_\Omega(x - \nabla f(x)). \tag{3.3}$$

First, let's reveal an important property for $e(x)$.

LEMMA 3.1. *$e(x) = 0$ with $x \neq 0$ $\Longleftrightarrow$ x is an eigenvector of A with $\|x\| = 1$.*

Proof. "$\Longleftarrow$" This is straightforward.

"$\Longrightarrow$" $e(x) = 0$ implies

$$x = \begin{cases} [x - \nabla f(x)]/\|x - \nabla f(x)\|, & \text{if } \|x - \nabla f(x)\| > 1, \\ x - \nabla f(x), & \text{if } \|x - \nabla f(x)\| \leq 1. \end{cases} \tag{3.4}$$

If $\|x - \nabla f(x)\| \leq 1$, from (3.4), we have

$$\nabla f(x) = 2(A - cI_n)x = 0. \tag{3.5}$$

But $(cI_n - A)$ is a positive definite matrix, (3.5) implies $x = 0$ which contradicts with $x \neq 0$. Therefore, it must be true that $\|x - \nabla f(x)\| > 1$. Let $\gamma = \|x - \nabla f(x)\| - 1 > 0$, then from (3.4), we have

$$(A - cI_n)x = -\frac{\gamma}{2} x. \tag{3.6}$$

(3.6) indicates that x is an eigenvector of A.

From Theorem 1 in [5], $e(x) = 0$ implies

$$x \in \Omega, \quad (y - x)^T \nabla f(x) \geq 0, \quad \forall y \in \Omega. \tag{3.7}$$

If $\|x\| < 1$, then there exists a $\beta > 0$ such that $x \pm \beta x \in \Omega$. Then from (3.7), we have

$$\pm 2\beta x^T (A - cI_n) x \geq 0.$$

This implies $x^T(A - cI_n)x = 0$. But $(cI_n - A)$ is a positive definite matrix, therefore we must have $x = 0$ which is impossible. Thus, it must be true that $\|x\| = 1$. This completes our proof. □

Now we are ready to analyze the convergence properties for the solution of (3.2). These results will be summarized in the following theorems.

THEOREM 3.2. *For any $x_0 \in R^n$, there exists a unique solution $x(t)$ of the dynamical system (3.2) with $x(t = t_0) = x_0$ in $[t_0, +\infty)$.*

Proof. Since the right-hand-side of (3.2) is continuous in R^n, the Cauchy-Peano theorem ensures that there exists a solution $x(t)$ of the dynamical system (3.2) with $x(t = t_0) = x_0$. For this solution $x(t)$, we define

$$E(x(t)) = \|x(t) - P_\Omega(x(t))\|^2. \tag{3.8}$$

Obviously, $E(x(t))$ is the square of the distance of $x(t)$ to set Ω. Then we have from (3.2) and (3.3) that

$$\frac{dE(x(t))}{dt} = \begin{cases} -2(1 - \frac{1}{\|x\|})x^T e(x), & \text{if } \|x\| > 1, \\ 0, & \text{if } \|x\| \leq 1. \end{cases} \tag{3.9}$$

From (3.3), we have

$$x^T e(x) = \begin{cases} x^T x - \frac{(2c+1)x^T x - 2x^T Ax}{\|(2c+1)x - 2Ax\|}, & \text{if } \|(2c+1)x - 2Ax\| > 1, \\ 2x^T Ax - 2cx^T x, & \text{if } \|(2c+1)x - 2Ax\| \leq 1. \end{cases} \tag{3.10}$$

From the requirement on c in (2.5), we have

$$\|(2c+1)x - 2Ax\| \geq \|2cx - 2Ax\| - \|x\| \geq \|2x\| - \|x\| = \|x\|. \tag{3.11}$$

(3.10) and (3.11) indicate

$$x^T e(x) = x^T x - \frac{(2c+1)x^T x - 2x^T Ax}{\|(2c+1)x - 2Ax\|} > 0 \quad \text{if } \|x\| > 1. \tag{3.12}$$

(3.9) and (3.12) indicate that $E(x(t))$ is monotonically nonincreasing in t. Therefore, we have

$$\|e(x)\| \leq \|x - P_\Omega(x)\| + \|P_\Omega(x) - P_\Omega(x - \nabla f(x))\| \leq \|x(t_0)\| + 3. \tag{3.13}$$

(3.13) indicates that the right-hand-side of (3.2) is bounded for any given x_0. Again the Cauchy-Peano theorem ensures that the solution $x(t)$ exists in $[t_0, +\infty)$.

Since Ω is a closed convex set, from the nonexpansive property of the projection operator, we have

$$\|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|, \quad \forall u, v \in R^n.$$

Therefore,

$$\begin{aligned} \|e(x) - e(y)\| &= \|x - P_\Omega(x - \nabla f(x)) - y + P_\Omega(y - \nabla f(y))\| \\ &\leq \|x - y\| + \|P_\Omega(x - \nabla f(x)) - P_\Omega(y - \nabla f(y))\| \\ &\leq \|x - y\| + \|x - y\| + \|\nabla f(x) - \nabla f(y)\| \\ &\leq (2 + 2\|A\| + 2c)\|x - y\|, \quad \forall x, y \in R^n. \end{aligned} \tag{3.14}$$

(3.14) implies that $e(x)$ in (3.3) is Lipschitz continuous in R^n. From the Picard-Lindelöf theorem, the proof is completed. □

The result of Theorem 3.2 indicates that our dynamical system (3.2) is well defined. In the proof of Theorem 3.2, we can see that if $x_0 \notin \Omega$, then the solution $x(t)$ of (3.2) will move towards the feasible region, and if $x_0 \in \Omega$, then the solution $x(t)$ of (3.2) will stay in Ω from then on. Before we prove the convergence of the solution of (3.2), we need to observe the following properties. First, from (2.2), we can define

$$x(t) = \sum_{i=1}^{n} \alpha_i(t) u_i = U\alpha(t), \tag{3.15}$$

where $x(t)$ is the solution of (3.2), u_i's are the column vectors of U defined in (2.1), and $\alpha(t) = (\alpha_1(t), \cdots, \alpha_n(t))^T$. Then from (3.1), we have

$$g(\alpha) \equiv f(x) = \sum_{i=1}^{n} \alpha_i^2(t)(\lambda_i - c). \tag{3.16}$$

Therefore, it is straightforward to see that (3.2) is equivalent to

$$\frac{d\alpha(t)}{dt} = -[\alpha - P_\Omega(\alpha - \nabla g(\alpha))]. \tag{3.17}$$

In order to prove the convergence of the ode solution $x(t)$, we need the following lemma.

LEMMA 3.3. *Suppose scalar function $h(t)$ is differentiable on $[t_0, T]$ with $h(t_0) = 0$. If there exists an $M > 0$ such that $|\frac{dh}{dt}| \leq M|h(t)|,\ t \in [t_0, T]$, then $h(t) = 0,\ t \in [t_0, T]$.*

Proof. See Exercise 26, p. 119, [10]. □

It is easy to see that T can be extended to $+\infty$. Now we prove the following important convergence results for the solution of (3.2).

THEOREM 3.4. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of (3.2) with $x(t = t_0) = x_0$. Then (i) if $e(x_0) = 0$, $x(t) \equiv x_0$, $\forall t \geq t_0$; (ii) if $e(x_0) \neq 0$, then $\lim_{t\to+\infty} e(x(t)) = 0$.*

Proof. (i) From (3.17), we have for $i = 1, \cdots, n$

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} -\alpha_i(t) + \frac{(2c+1-2\lambda_i)\alpha_i(t)}{\|\alpha(t)-\nabla g(\alpha)\|}, & \text{if } \|\alpha(t) - \nabla g(\alpha)\| > 1, \\ 2(c-\lambda_i)\alpha_i(t), & \text{if } \|\alpha(t) - \nabla g(\alpha)\| \leq 1. \end{cases} \tag{3.18}$$

In both cases, we have for $i = 1, \cdots, n$

$$\left|\frac{d\alpha_i(t)}{dt}\right| \leq 2(c - \lambda_1 + 1)|\alpha_i(t)|, \quad \forall t \geq t_0. \tag{3.19}$$

If $e(x_0) = 0$, then from Lemma 3.1, we know either $x_0 = 0$ or x_0 is an eigenvector of A with $\|x_0\| = 1$. If $x_0 = 0$, from (3.15) we have $\alpha(t_0) = 0$. Therefore, (3.19) and Lemma 3.3 imply $\alpha(t) \equiv 0$, $\forall t \geq t_0$. Thus, $x(t) \equiv 0$, $\forall t \geq t_0$. If x_0 is an eigenvector of A with $\|x_0\| = 1$, let

$$Ax_0 = \lambda_k x_0.$$

Then from (3.15), $\alpha_i(t_0) = 0$ if $\lambda_i \neq \lambda_k$. Then from (3.19) and Lemma 3.3, we have

$$\alpha_i(t) \equiv 0, \ \forall t \geq t_0, \ \forall i \text{ with } \lambda_i \neq \lambda_k.$$

Therefore, $\|\alpha(t) - \nabla g(\alpha)\| = (2c+1-2\lambda_k)\|\alpha(t)\|$. Since $\|\alpha(t_0)\| = 1$ and $2c+1-2\lambda_k > 1$, (3.18) indicates that $\exists$ a $\bar{t} > t_0$ such that

$$\frac{d\alpha_j(t)}{dt} \equiv 0, \ \forall t \in [t_0, \bar{t}], \ \forall j \text{ with } \lambda_j = \lambda_k.$$

Therefore, $\alpha(t) \equiv \alpha(t_0)$ and $\|\alpha(t)\| = 1$, $\forall t \in [t_0, \bar{t}]$. This process can be repeated until $\bar{t} \to +\infty$. Therefore, $\alpha(t) \equiv \alpha(t_0)$, $\forall t \geq t_0$. Thus, $x(t) \equiv x_0$, $\forall t \geq t_0$.

(ii) Since $x_0 \in \Omega$, Theorem 3.2 ensures that $x(t) \in \Omega$, $\forall t \geq t_0$.

Since Ω is a closed convex set, from inequality (4) in [5], we have

$$[y - P_\Omega(y)]^T[x - P_\Omega(y)] \leq 0, \quad \forall x \in \Omega, \ \forall y \in R^n. \tag{3.20}$$

Taking $y = x - \nabla f(x)$ in (3.20), we have

$$[e(x) - \nabla f(x)]^T e(x) \leq 0. \tag{3.21}$$

From (3.1)-(3.3) and (3.21), we have

$$\frac{df(x)}{dt} = -[\nabla f(x)]^T e(x) \leq -\|e(x)\|^2 \leq 0, \quad \forall t \geq t_0. \tag{3.22}$$

From the LaSalle invariant set theorem (Theorem 3.4 in [11]) and (3.22), we know

$$\lim_{t\to+\infty} e(x(t)) = 0.$$

This completes the proof. □

Theorem 3.4 is a little bit short of proving the convergence of $x(t)$ which is much more desirable. This result is summarized in the following theorem.

THEOREM 3.5. *For any $x_0 \in \Omega$, let $x(t)$ be the solution of (3.2) with $x(t = t_0) = x_0$. Then $x(t)$ is convergent, i.e. there exists an $x^* \in \Omega$ such that $\lim_{t\to+\infty} x(t) = x^*$. In addition, if $x_0 \neq 0$, $\lim_{t\to+\infty} x(t)^T Ax(t) = \lambda_k$, where $k = \min\{i \mid x_0^T u_i \neq 0,\ i = 1, \cdots, n\}$.*

Proof. Obviously, if $x_0 = 0$, then $x(t) \equiv 0, \forall t \geq t_0$ from the proof of Theorem 3.4. Therefore $x(t)$ is convergent. So we assume $x_0 \neq 0$ in the rest of proof.

From the proof of Theorem 3.2, we know $x(t) \in \Omega$ for all t's. The boundedness of Ω implies that there exists at least one limit point for $\{x(t)\}$. Let $\bar{x}$ be any limit point of $\{x(t)\}$. Then $e(\bar{x}) = 0$ from (ii) of Theorem 3.4 and $\bar{x} \neq 0$ from $x_0 \neq 0$. From Lemma 3.1, we know that $\bar{x}$ is an eigenvector of A with $\|\bar{x}\| = 1$. Therefore, we have $\lim_{t\to\infty} \|x(t)\| = 1$.

From $x \in \Omega \iff \alpha \in \Omega$ and $\|x(t)\| = \|\alpha(t)\|$, there exists a $t^* > t_0$ such that if $t > t^*$, $\|\alpha(t)\| > \frac{1}{2}$. Thus if $t > t^*$, we have

$$\|\alpha(t) - \nabla g(\alpha)\|^2 = \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2 \geq 9\|\alpha(t)\|^2 > 1.$$

This and (3.15)-(3.17) imply that for any $i = 1, \cdots, n$, if $t > t^*$, then

$$\begin{aligned} \frac{d\alpha_i(t)}{dt} &= -\alpha_i(t) + \frac{\alpha_i(t) - 2\alpha_i(t)(\lambda_i - c)}{\|\alpha(t) - \nabla g(\alpha)\|} \\ &= \frac{\alpha_i(t)[1 + 2(c - \lambda_i) - \|\alpha(t) - \nabla g(\alpha(t))\|]}{\|\alpha(t) - \nabla g(\alpha(t))\|}. \end{aligned} \tag{3.23}$$

From $x \in \Omega \iff \alpha \in \Omega$, we have

$$\begin{aligned} \|\alpha(t) - \nabla g(\alpha(t))\|^2 &= \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2 \\ &\leq \sum_{i=1}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_1)]^2 \\ &\leq [1 + 2(c - \lambda_1)]^2. \end{aligned} \tag{3.24}$$

Obviously, from (3.17), we have

$$\frac{d\alpha_i(t)}{dt} = 2\alpha_i(c - \lambda_i), \quad \text{if } \|\alpha(t) - \nabla g(\alpha)\| \leq 1. \tag{3.25}$$

(3.23), (3.24) and (3.25) indicate that there exists an $M > 0$ such that

$$\left|\frac{d\alpha_i(t)}{dt}\right| \leq M|\alpha_i(t)|, \quad \forall t \geq t_0. \tag{3.26}$$

From the definition of k and Lemma 3.3, we know $x_i(t) = \alpha_i(t) = 0,\ i = 1, \cdots, k-1, \quad \forall t \geq t_0$. Therefore, (3.24) becomes

$$\|\alpha(t) - \nabla g(\alpha(t))\|^2 = \sum_{i=k}^{n} \alpha_i^2(t)[1 + 2(c - \lambda_i)]^2 \leq [1 + 2(c - \lambda_k)]^2. \tag{3.27}$$

On the other hand, (3.23), (3.27) and our assumption indicate that if $t > t^*$, we have

$$\frac{d\alpha_k(t)}{dt} \begin{cases} \geq 0, & \text{if } \alpha_k(t) > 0, \\ \leq 0, & \text{if } \alpha_k(t) < 0. \end{cases} \tag{3.28}$$

(3.28) is very important. Basically, it tells that when $t > t^*$

- if $\alpha_k(t_0) > 0$, $\alpha_k(t)$ will be monotonically nondecreasing in t but always stays in the interval $[\alpha_k(t_0), 1]$;
- if $\alpha_k(t_0) < 0$, $\alpha_k(t)$ will be monotonically nonincreasing in t but always stays in the interval $[-1, \alpha_k(t_0)]$;

Therefore, $\lim_{t\to+\infty} \alpha_k(t)$ exists and is nonzero since $\alpha_k(t_0) \neq 0$. Let $\alpha_k^* = \lim_{t\to+\infty} \alpha_k(t) \neq 0$. Similarly we can prove that $\alpha_i(t)$ is convergent if $\lambda_i = \lambda_k$, for any $i \geq k+1$. If $\lambda_n = \lambda_k$, then our proof is finished. Otherwise, we prove

$$\lim_{t\to+\infty} \alpha_i(t) = 0, \quad i = j, \cdots, n, \tag{3.29}$$

where $j = \min\{i \mid \lambda_k < \lambda_i,\ i = k+1, \cdots, n\}$.

Since j and n are finite and $\|\alpha(t)\| \leq 1$, then there exists a sequence of t_l with $t_l \to +\infty$ as $l \to +\infty$ such that $\lim_{l\to+\infty} \alpha_i(t_l)$, $i = j, \cdots, n$ exist. Let $\alpha_i^* = \lim_{l\to+\infty} \alpha_i(t_l)$, $i = j, \cdots, n$.

Now we show $\alpha_i^* = 0,\ i = j, \cdots, n$. Suppose not, then there must exist an $\alpha_i^* \neq 0$ for some $i = j, \cdots, n$, without loss of generality, let $\alpha_j^* \neq 0$. Then $\alpha(t_l)$ and $x(t_l)$ are convergent as $l \to +\infty$. But from our earlier discussion, the limit of $\{x(t_l)\}$, say x^* is an eigenvector of A with $\|x^*\| = 1$. Let λ be the corresponding eigenvalue. Then we have

$$x^* = \sum_{i=1}^{n} \alpha_i^* u_i \quad \text{and} \quad Ax^* = \lambda x^*, \tag{3.30}$$

where $\alpha_k^* \neq 0$ and $\alpha_j^* \neq 0$.

From (3.30), we have

$$\sum_{i=1}^{n} \alpha_i^* \lambda_i u_i = \sum_{i=1}^{n} \alpha_i^* \lambda u_i.$$

This implies $\lambda = \lambda_k = \lambda_j$ which contradicts with $\lambda_k < \lambda_j$. Therefore, (3.29) holds. Thus $x(t)$ is convergent as $t \to +\infty$ and $\lim_{t\to+\infty} x(t)^T Ax(t) = \lambda_k$. □

Even though we have proved that for any starting point, the ode solution would converge to an eigenvector of the matrix, yet this eigenvector will not correspond to the minimum eigenvalue if the projection of the initial point in the eigenspace corresponding to the minimum eigenvalue is zero. In other words, we can't say that for any starting point, the limit of the ode solution is the eigenvector corresponding to the minimum eigenvalue. From the optimality conditions for problem (2.7) and Lemma 3.1, we know that the followings are equivalent:

- $e(x) = 0$ but $x \neq 0$.
- x is an eigenvector of A with $\|x\|_2 = 1$.
- x satisfies the first-order necessary conditions for problem (2.7).

The last result indicates that it would be quite difficult to move away from x if x is an eigenvector of A corresponding to some λ_i with $i > s$. In this case, one remedy is to move away from x along a direction $d \neq 0$ satisfying $x^T d = 0$. Then, we can re-solve the dynamical system (3.2) with this new starting point.

4. Numerical results. In this section, we test our continuous method on two small and one large-scale examples. Our simulation will stop whenever the following condition is satisfied:

$$\|e(x(t))\|_\infty \leq \delta,$$

where δ is a preset value.

In our test for the first two small examples, we use Matlab **ode23s** to solve the ode (3.2).

Example 1:

We construct Example 1 in the following steps:

1. Select $\Lambda = diag(-1e-4, 0, 0, 1e-4, 1, 1, 1, 1, 1, 1) \in R^{10\times 10}$.
2. Let $B = rand(10, 10)$ and $[Q, R] = qr(B)$.
3. Define $A = Q^T \Lambda Q$.

Obviously, the minimum eigenvalue of A is $-1.e-4$. Our simulation results are summarized in the following Table 4.1 with $\delta = 10^{-8}$.

Table 4.1 Numerical results for Example 1

initial x_0	$\lambda_1 + 1.e-4$	$\|Ax - \lambda_1 x\|_\infty$
$(1, \cdots, 1)^T$	4.44e-11	3.66e-8
$-(1, \cdots, 1)^T$	-3.52e-12	8.70e-9
$(1, -1, \cdots, 1, -1)^T$	1.31e-12	2.62e-8

Example 2:

Example 2 is similar to Example 1 except $\Lambda = diag(0, 0, 1e-4, 1, \cdots, 1, 10^6) \in R^{20\times 20}$ and the dimension is increased to 20. Obviously, the minimum eigenvalue of A is zero. Our simulation results are summarized in the following Table 4.2 with $\delta = 10^{-8}$.

Table 4.2 Numerical results for Example 2

initial x_0	$\lambda_1 - 0$	$\|Ax - \lambda_1 x\|_\infty$
$(1, \cdots, 1)^T$	2.14e-5	4.63e-5
$-(1, \cdots, 1)^T$	1.42e-5	2.58e-5
$(1, -1, \cdots, 1, -1)^T$	3.81e-5	4.28e-5

Example 3:

Example 3 is a large-scale problem where $n = 5,000$. The matrix A is dense and generated from

$$[A]_{i,j} = \begin{cases} 2 \times U(0,1) - 1 & \text{if } i \geq j, \\ [A]_{i,j} & \text{if } i < j, \end{cases}$$

where $U(0,1)$ represents a uniform random number between 0 and 1. An off-shelf iterative ode solver is used to solve the dynamical system (3.2). The initial value used is $\bar{x}_0 = P_\Omega[(1/n, 2/n, 3/n, \cdots, n/n)^T]$ and $\hat{x}_0 = P_\Omega[rand]$, where $[rand]_i$ is a random number in $(-1, 1)$, $i = 1, \cdots, n$. Our simulation was conducted on a Sun Ultra Workstation, the simulation results are summarized in the following Table 4.3 with various δ values.

Table 4.3 Numerical results for Example 3

	initial value $\bar{x}_0$		initial value $\hat{x}_0$	
δ	λ_1	CPU (sec.)	λ_1	CPU (sec.)
$\delta = 10^{-4}$	-74.26	1,032	-76.07	1,112
$\delta = 10^{-6}$	-81.576	5,739	-81.577	5,933
$\delta = 10^{-8}$	-81.582	12,545	-81.582	11,727

5. Conclusions. In this paper, a new continuous method is proposed for symmetric extreme eigenvalue problems. Our approach is different from the existing ones in that a continuous path (or trajectory) of the minimum eigenvalue is achieved. This is represented by a dynamical system (or ode). Strong convergence results of our continuous method are obtained. Our limited simulation results clearly indicate that our new method is very effective.

Our final remark is on the dynamical system (3.2) since the success of our method relies on the solution of (3.2). Notice that (3.2) is an autonomous system and its right-hand-side is relatively simple. Therefore, it is anticipated that (3.2) could be solved easily even for large-scale system.

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