Decomposition of Images by the Anisotropic Rudin-Osher-Fatemi Model

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Abstract

The total variation based image de-noising model of Rudin, Osher, and Fatemi can be generalized in a natural way to privilege certain edge directions. We consider the resulting anisotropic energies and study properties of their minimizers.

1 Introduction

We introduce and study anisotropic versions of the total variation based noise-removal model developed by Rudin, Osher, and Fatemi (ROF) in [4]. Recall that the goal of the original ROF model is to remove noise from a corrupted digital image without blurring object boundaries (i.e. "edges"). If the corrupted image is denoted $f(x)$, one tries to recover the clean image as the minimizer of the following energy:

$$E(u) := \int_D |\nabla u| + \lambda \int_D (f-u)^2 \, dx \tag{1}$$

Our goal in this paper is to study the energy

$$E_\phi(u) := \int_D \phi(\nabla u) + \lambda \int_D (f-u)^2 \, dx \tag{2}$$

where ϕ is an anisotropic function with suitable properties explained in the next section. In particular, we will generalize to minimizers of (2) some of the interesting results that Y. Meyer shows in [3] for minimizers of (1).

From an applied point of view, our main results are of interest for restoring characteristic functions of convex regions having desired shapes. Standard total variation model (1) prefers convex shapes with smooth boundaries. Anisotropic, or Wulff, total variation model (2) prefers shapes which are compatible, in a sense explained below, with the Wulff shape (see e.g. [6]) associated with $\phi(x)$. For example, if $\phi(\nabla u) = |\nabla u|$, the characteristic function of an N-sphere is admissable as a minimizer, but not the characteristic function of an N-cube. On the other hand, if $\phi(\nabla u) = \sum_{i=1}^{N} |u_{x_i}|$, the situation is reversed: the characteristic function of an N-cube is admissable, while that of an N-sphere is not.

Note that for any ϕ as defined below, we have

$$c_1 E(u) \leq E_\phi(u) \leq c_2 E(u)$$

where $0 < c_1, c_2$ depend on ϕ but not on u. For instance, if $\phi(\nabla u) = \sum_{i=1}^{N} |u_{x_i}|$ then $c_1 = 1$ and $c_2 = \sqrt{N}$. This means that the "equivalent" variational models give quite different results. Moreover, one can tailor the image restoration (or other applied variational problem) to obtain the desired result, using the appropriate choice among an infinitude of equivalent convex variational problems.

2 Notation and Definitions

Let $\phi(x) : \mathbf{R}^N \to \mathbf{R}$ be a convex, positively 1-homogeneous function such that $\phi(x) > 0$ for $x \neq 0$.

Definition: The *Wulff shape* W_ϕ associated with $\phi(x)$ is defined to be the set:

$$W_\phi := \left\{ y \in \mathbf{R}^N : y \cdot x \leq \phi(x) \text{ for all } x \in \mathbf{R}^N \right\}. \qquad (3)$$

W_ϕ thus defined is a closed, bounded, and convex set that contains the origin in its interior. If $\phi(x)$ is an even function, as it in many applications is, then W_ϕ is centrally symmetric, i.e.

$$x \in W_\phi \Rightarrow -x \in W_\phi$$

The convex function $\phi(x)$ can be recovered from its associated Wulff shape W_ϕ according to the following formula:

$$\phi(x) = \sup_{y \in W_\phi} y \cdot x \qquad (4)$$

which, in case $\phi(x)$ is not convex, yields instead the convexification of $\phi(x)$. Let us note that by compactness of W_ϕ, the supremum in (4) is attained at some (possibly more than one) $y \in \partial W_\phi$.

It is also useful to introduce the following notation: we define the function $w_\phi : \mathbf{R}^N \setminus \{0\} \to \mathbf{R}^+$ as

$$w_\phi(x) := \inf_{\{y:\, x\cdot y>0\}} \frac{\phi(y)}{x \cdot y}$$

This function can be used to characterize the set W_ϕ as follows: For $\alpha > 0$ and $x \neq 0$, we have $\alpha x \in W_\phi$ if and only if $\alpha \leq w_\phi(x)$. In other words, $x \in W_\phi$ if and only if $w_\phi(x) \geq 1$.

Definition: Given $p, v \in \mathbf{R}^N$, let $H(p, v)$ denote the closed half space

$$H(p, v) = \big\{x \in \mathbf{R}^N : (x - p) \cdot v \leq 0\big\}.$$

Definition: Given a convex domain $\Omega \subset \mathbf{R}^N$ and a point $p \in \partial\Omega$, we define the collection of outer normals to Ω at p as

$$N_\Omega(p) = \Big\{v \in \mathbf{R}^N : \Omega \subset H(p, v)\Big\}.$$

See *Figure 1* for an illustration. Since Ω is convex, the set $N_\Omega(p)$ is non-empty for each $p \in \partial\Omega$. If $\partial\Omega$ happens to be differentiable at p, then $N_\Omega(p)$ contains a single direction. Using this notation, we can now state the following relation between $\phi(x)$ and the normals to its associated Wulff shape W_ϕ:

Lemma 1 *If $x \in \mathbf{R}^N$, $x \neq 0$, and $y \in W_\phi$, then $y \cdot x = \phi(x)$ if and only if $y \in \partial W_\phi$ and $x \in N_{W_\phi}(y)$.*

Proof: Let $x \in \mathbf{R}^N$, $x \neq 0$. If $y \in \partial W_\phi$ and $x \in N_{W_\phi}(y)$, then by definition of N_{W_ϕ}, we have $W_\phi \subset H(y, x)$. The definition of $H(y, x)$ in return reads

$$(\xi - y) \cdot x \leq 0 \text{ for all } \xi \in W_\phi.$$

so that

$$\phi(x) = \sup_{\xi \in W_\phi} \xi \cdot x \leq y \cdot x$$

which implies $y \cdot x = \phi(x)$. Conversely, if $y \in W_\phi$ and $y \cdot x = \phi(x) = \sup_{\xi \in W_\phi} \xi \cdot x$, then owing to the fact that $x \neq 0$ and $\xi \to \xi \cdot x$ is a linear function, $y \in \partial W_\phi$. Moreover,

$$(\xi - y) \cdot x \leq 0 \text{ for all } \xi \in W_\phi \Longrightarrow x \in N_{W_\phi}(y)$$

which proves the lemma. □

The dual characterization of $\phi(x)$ in terms of its Wulff shape given in (4) motivates the following definition of *anisotropic total variation energy*:

Definition: For a domain $\Omega \subset \mathbf{R}^N$ with Lipschitz boundary, we define

$$\int_\Omega \phi(\nabla u) := \sup_{\substack{g(x)\in C_c^1(\Omega;\mathbf{R}^N) \\ g(x)\in W_\phi \forall x\in\Omega}} -\int_\Omega u(x)\mathrm{div} g(x)\,dx. \tag{5}$$

This definition differs from that of standard (isotropic) total variation only in that the test vector fields $g(x)$ take their values in the set W_ϕ, instead of the unit ball $\{x : |x| \leq 1\}$.

When $\phi(x)$ is even, (5) defines a *semi-norm* on $L^1_{loc}(\Omega)$, which we will denote $\|\cdot\|_{\mathbf{BV}_\phi}$. If in addition $\Omega = \mathbf{R}^N$, then in fact $\|\cdot\|_{\mathbf{BV}_\phi}$ is a *norm*.

Definition: When $\phi(x)$ is even, we define the Banach space $\mathbf{BV}_\phi$ as

$$\mathbf{BV}_\phi := \left\{ u(x) \in L^1_{loc}(\mathbf{R}^N) : \int_{\mathbf{R}^N} \phi(\nabla u) \leq \infty \right\} \tag{6}$$

and equip it with the norm $\|\cdot\|_{\mathbf{BV}_\phi}$. □

We repeat that the spaces $\mathbf{BV}_\phi$ *are all equivalent*; in other words, there exist constants $C \geq c > 0$ such that

$$c\|u\|_{\mathbf{BV}} \leq \|u\|_{\mathbf{BV}_\phi} \leq C\|u\|_{\mathbf{BV}} \text{ for all } u \in L^1_{loc}(\mathbf{R}^N).$$

As $\mathbf{BV}_\phi$ is a Banach space, it naturally has a dual, whose norm, following Y. Meyer, will be denoted $\|\cdot\|_*$. Recall that by the Sobolev inequality for functions of bounded variation, the standard total variation norm controls the $L^{\frac{N}{N-1}}$-norm on $\mathbf{R}^N$. By our previous remark concerning the equivalence of norms, so does $\|\cdot\|_{\mathbf{BV}_\phi}$. It follows (from an application of Holder inequality) that any function $g \in L^N(\mathbf{R}^N)$ defines a bounded linear functional on $\mathbf{BV}_\phi$, under the standard L^2 inner product; its dual norm is then given by

$$\|g\|_* := \sup\left\{ \left| \int_{\mathbf{R}^N} g(x)u(x)\,dx \right| : u \in L^1_{loc}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} \phi(\nabla u) \leq 1 \right\}$$

In keeping with the spirit of Meyer's work, in this paper we will assume that $\phi(x)$ is even so that (5) defines a norm. However, let us point out that most of our results can be rephrased, and their proofs easily adapted, for general $\phi(x)$.

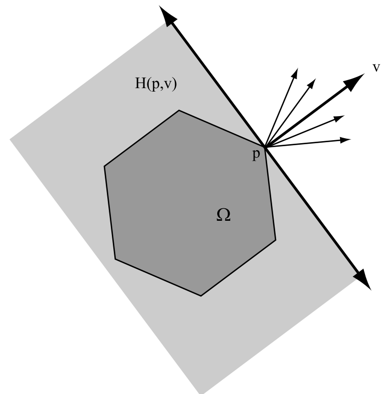

Figure 1: Illustration of some of the definitions in *Section 2*. Ω is a convex domain, and p is a point on its boundary $\partial\Omega$. The half space $H(p,v)$ contains Ω and "touches" $\partial\Omega$ at p; therefore, v belongs to $N_\Omega(p)$, the set of outward normal directions to Ω at p. When $\partial\Omega$ has a corner at p, as in the illustration, then $N_\Omega(p)$ can contain more than one direction, as indicated with the arrows.

3 Basic facts

In this section we state some fundamental facts that follow from (5).

Claim 1 *Let $u(x) \in C_c^1(\mathbf{R}^N)$. Then the anisotropic total variation energy of $u(x)$ as defined in (5) agrees with the natural sense of the integral in the left hand side of that formula.*

Claim 2 *Let Ω be an open set in $\mathbf{R}^N$, with Lipschitz boundary $\partial\Omega$. Let n_Ω denote the inward unit normal to Ω (which exists H^{N-1}-a.e. on $\partial\Omega$). Then,*

$$\int \phi(\nabla \mathbf{1}_\Omega(x)) = \int_{\partial\Omega} \phi(n_\Omega(x))\, dH^{N-1}(x)$$

The following proposition is the anisotropic analogue of one of Y. Meyer's results (his Proposition 5 on page 38 in AMS Lecture Notes [3]); we state a restricted version so as to minimize technical details, and include its proof for the sake of completeness:

Proposition 1 *Let $f(x) \in L^2(\mathbf{R}^N)$ be the given original image. Then $u(x) \in \mathbf{BV}_\phi(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ is the solution of the minimization problem*

$$\min_{w(x)\in \mathbf{BV}_\phi(\mathbf{R}^N)} E_\phi(u) \tag{7}$$

where E_ϕ is defined as in (2), if there exists a vector field $z(x) \in L^\infty(\mathbf{R}^N;\mathbf{R}^N)$ such that

1. $z(x) \in W_\phi$ *for all almost every* $x \in \mathbf{R}^N$*,*
2. $\operatorname{div} z(x) \in L^2(\mathbf{R}^N)$*,*
3. $\displaystyle\int_{\mathbf{R}^N} u(x) \operatorname{div} z(x)\, dx = -\int_{\mathbf{R}^N} \phi(\nabla u)$*,*
4. $\operatorname{div} z(x) = 2\lambda(u - f)$*.*

Proof: Let $u(x)$ and $z(x)$ satisfy the conditions of the claim. We will show that $E_\phi(u(x)) \le E_\phi(u(x) + h(x))$ for any $h(x) \in \mathbf{BV}_\phi(\mathbf{R}^N)$. Note that since $f(x) \in L^2(\mathbf{R}^N)$ by hypothesis, we can restrict attention to $h(x) \in \mathbf{BV}_\phi(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$.

Let $\xi(x) : \mathbf{R}^N \to \mathbf{R}$ be a smooth cut-off function such that $\xi(x) = 1$ in $\{x : |x| < 1\}$, and $\xi(x) = 0$ in $\{x : |x| > 2\}$. Let $\eta(x)$ be a compactly supported, radially symmetric, smooth, positive function of unit mass. Define the vector fields

$$z_j(x) := j^N \Big(\xi(\frac{x}{j}) z(x)\Big) * \eta(jx)$$

Then $z_j(x) \in C_c^\infty(\mathbf{R}^N;\mathbf{R}^N)$, and $\operatorname{div} z_j \to \operatorname{div} z$ in $L^2(\mathbf{R}^N)$ as $j \to \infty$. Also, by *Condition 1* placed on $z(x)$ in the hypothesis of the proposition and the fact that W_ϕ is a *convex* set, we have $z_j(x) \in W_\phi$ for all $x \in \mathbf{R}^N$ and all $j = 1, 2, \ldots$.

It follows from definition (5) and properties of $z_j(x)$ just mentioned that

$$\int_{\mathbf{R}^N} \phi\left(\nabla(u(x) + h(x))\right) \ge -\int_{\mathbf{R}^N} (u(x) + h(x)) \operatorname{div} z_j(x)\, dx$$

for every j. Therefore,

$$E_\phi(u(x)+h(x)) \ge \limsup_{j\to\infty} \bigg\{ -\int_{\mathbf{R}^N} u(x) \operatorname{div} z_j(x)\, dx - \int_{\mathbf{R}^N} h(x) \operatorname{div} z_j(x)\, dx + \lambda \int_{\mathbf{R}^N} (u + h - f)^2\, dx \bigg\}$$

By *Condition 3* of the claim we have

$$\lim_{j\to\infty} \int_{\mathbf{R}^N} u(x) \operatorname{div} z_j(x)\, dx = \int_{\mathbf{R}^N} u(x) \operatorname{div} z(x)\, dx = -\int_{\mathbf{R}^N} \phi(\nabla u),$$

and by *Condition 4* we have

$$\lim_{j\to\infty}\int_{\mathbf{R}^N} h(x)\,\mathrm{div}\,z_j(x)\,dx = \int_{\mathbf{R}^N} h(x)\,\mathrm{div}\,z(x)\,dx = \int_{\mathbf{R}^N} 2\lambda h(u-f)\,dx.$$

The last thee formulae give

$$\begin{aligned} E_\phi(u(x)+h(x)) &\geq \int_{\mathbf{R}^N}\phi(\nabla u) + \lambda\int_{\mathbf{R}^N} 2h(f-u)+(u+h-f)^2\,dx \\ &= E_\phi(u(x)) + \lambda\int_{\mathbf{R}^N} h^2(x)\,dx \geq E_\phi(u(x)) \end{aligned}$$

which proves the claim. □

Another of Meyer's important propositions can also be adapted to the anisotropic case. The following characterizes the resulting image decomposition via the anisotropic model (2) in terms of the dual norm of the original image f.

Proposition 2 *The minimizer u of the anisotropic energy (2) satisfies $u \equiv 0$ iff $\|f\|_* \leq \frac{1}{2\lambda}$. Moreover, if $u \not\equiv 0$, then $v \equiv f-u$ satisfies $\|v\|_* = \frac{1}{2\lambda}$.*

Proof: The proof follows word for word the one given by Meyer in [3] (Lemma 4 and Theorem 3 on page 32) for the isotropic case. □

4 Properties of the decomposition

In subsection 1, we exhibit some exact solutions. In subsection 2, we investigate regions whose characteristic functions can arise as the minimizer (i.e. as the u-part of the decomposition).

4.1 Exact solutions

Theorem 1 *Let $f(x) = \mathbf{1}_{W_\phi}(x)$. Then, for every large enough λ, the minimizer $u(x)$ of the variational problem (7) has the form*

$$u(x) = c\mathbf{1}_{W_\phi}(x)$$

for some $c > 0$.

Proof: We construct a vector field $z(x)$, associated with the proposed minimizer $u(x)$, that satisfies the conditions of *Proposition 1*. It is a slight modification of Meyer's choice in the isotropic case:

$$z(x) := \begin{cases} -x & \text{if } x \in W_\phi, \\ -x\big(w_\phi(x)\big)^N & \text{if } x \in W_\phi^c. \end{cases}$$

We first check that $z(x) \in W_\phi$ for all $x \in \mathbf{R}^N$. When $x \in W_\phi$, this is immediate from the definition of $z(x)$ and central symmetry of W_ϕ. Recall that if $x \neq 0$, then $w_\phi(x) < 1$ for $x \notin W_\phi$, and that $\alpha x \in W_\phi$ only if $\alpha \leq w_\phi(x)$, where $\alpha > 0$. These show that when $x \notin W_\phi$ we have $(w_\phi(x))^N \leq w_\phi(x)$, and thus $x(w_\phi(x))^N \in W_\phi$. Therefore, $z(x) \in W_\phi$ for all $x \in \mathbf{R}^N$. That verifies *Condition 1* of *Proposition 1*.

Next, let us compute $\operatorname{div} z(x)$. When $x \in W_\phi$, we have

$$\operatorname{div} z(x) = -\operatorname{div} x = -N$$

On the other hand, when $x \in W_\phi^c$,

$$\begin{aligned} \operatorname{div} z(x) &= -\operatorname{div}\left(x\big(w_\phi(x)\big)^N\right) = -\operatorname{div}\left(\frac{x}{|x|^N} w_\phi^N\left(\frac{x}{|x|}\right)\right) \\ &= -w_\phi^N\left(\frac{x}{|x|}\right)\operatorname{div}\left(\frac{x}{|x|^N}\right) - \frac{x}{|x|^N}\cdot\nabla\left(w_\phi^N\left(\frac{x}{|x|}\right)\right) \end{aligned}$$

Both terms on the right hand side vanish; the first because $x/|x|^N$ is divergence free, and the second because $w_\phi(x/|x|)$ is constant in the x-direction. Hence, $\operatorname{div} z(x) = 0$ for $x \in W_\phi^c$. Noting that $z(x)$ is globally Lipschitz, we get in particular that $\operatorname{div} z(x) \in L^2$, which verifies *Condition 2* of *Proposition 1*.

When $x \in \partial W_\phi$, we have $z(x) = -x$. Let $n(x)$ be the outward unit normal to ∂W_ϕ, which is well defined at H^{N-1}-a.e. point of ∂W_ϕ. For every such x, there is the following equality:

$$z(x) \cdot n(x) = -x \cdot n(x) = -\phi(n(x))$$

which holds by virtue of Lemma 1. Recalling *Claim 2*, it follows that

$$\int_{\mathbf{R}^N} \phi(\nabla \mathbf{1}_{W_\phi}(x)) = \int_{\partial W_\phi} \phi(n(x))\, dH^{N-1} = -\int_{\partial W_\phi} z(x)\cdot n(x)\, dH^{N-1}$$

where we once more used the fact that W_ϕ, being convex, is the closure of a Lipschitz domain. Divergence theorem is valid for such domains, and when applied to the last formula it gives

$$\int_{\mathbf{R}^N} \phi(\nabla \mathbf{1}_{W_\phi}(x)) = -\int_{\mathbf{R}^N} \mathbf{1}_{W_\phi}(x)\,\mathrm{div}\, z(x)\,dx$$

which verifies *Condition 3* of *Proposition 1*.

Finally, remembering that $\mathrm{div}\, z(x) = -N$ when $x \in W_\phi$ and $\mathrm{div}\, z(x) = 0$ when $x \notin W_\phi$, we see that whenever $\lambda \geq \frac{N}{2}$ we can verify *Condition 4* of *Proposition 1* by choosing $c = 1 - \frac{N}{2\lambda}$. That concludes the proof of the present theorem. ☐

Remark: *Theorem 1* of course generalizes to given images of the form $f(x) = c_1 \mathbf{1}_{W_\phi}(c_2 x + c_3)$, where $c_1 \in \mathbf{R}$, $c_2 > 0$, and $c_3 \in \mathbf{R}^N$ are constants.

For a given ϕ, the conclusion of *Theorem 1* may be true for regions other than those identified in the theorem and the remark that follows it; in other words there can be regions Ω that are distinct from scaled and translated versions of the corresponding Wulff shape W_ϕ such that when the original image is given by $f = \mathbf{1}_\Omega(x)$, the minimizer of (2) turns out to be a constant multiple of f (and so in particular has the same set of "edges" as the original image). Indeed, in the isotropic case, Bellettini, Caselles, and Novaga exhibited in [1] regions other than the ball that have this property. We would expect the same to be true in the anisotropic case. To illustrate this point, we have the following simple example: (We let $x = (x, y)$ in $\mathbf{R}^2$).

Claim 3 *Let $\phi(x, y) = |x| + |y|$. Let $f(x, y) = \mathbf{1}_R(x, y)$ where $R \in \mathbf{R}^2$ is a rectangle whose sides are parallel to the (x, y)-axis. Then for every $\lambda > \frac{a+b}{2ab}$, the minimizer of (2) is given by $u(x, y) = c\mathbf{1}_R(x, y)$ where $c = 1 - \frac{a+b}{2\lambda ab}$.*

Proof: Without loss of generality we may assume that $R = (-a, a) \times (-b, b)$ where $a, b > 0$. Define the vector field $z(x, y)$ as

$$z(x, y) := -\Big(\eta(\frac{x}{a})\mathbf{1}_{(-b,b)}(y), \eta(\frac{y}{b})\mathbf{1}_{(-a,a)}(x)\Big)$$

where the function $\eta(\xi) : \mathbf{R} \to \mathbf{R}$ is defined as

$$\eta(\xi) := \begin{cases} -1 & \text{if } \xi < -1, \\ \xi & \text{if } -1 \leq \xi \leq 1, \\ 1 & \text{if } \xi > 1. \end{cases}$$

Then $z(x)$ satisfies all the requirements of *Proposition 1*, provided that $u(x)$ is defined as claimed. Then, *Proposition 1* implies the desired conclusion. ☐

4.2 Properties of minimizers

In this section, we discuss properties of domains whose characteristic functions can arise as the minimizer of the anisotropic total variation model (2). Our goal is to obtain anisotropic analogues of the basic results of Meyer in [3], where he shows that the characteristic function of any smooth, bounded region can arise as the u-component of the standard ROF model, but not that of a domain with a corner (such as a square). When the corresponding Wulff shape W_ϕ of a given anisotropic energy density $\phi(x)$ is smooth and strictly convex, these facts have rather obvious analogues in the anisotropic setting. It is when W_ϕ either has corners or is non-strictly convex that we get qualitative differences. Therefore, we subsequently concentrate on the special case of polygonal Wulff shapes, which have both of these characteristics, in order to bring out the differences.

4.2.1 General considerations

In this section we write down a simple condition that makes it easier to identify certain domains for which a vector field $z(x)$ can be constructed that satisfies the hypothesis of *Proposition 1*

Lemma 2 *Let $\Omega \subset \mathbf{R}^N$ be a non-empty, closed, convex set. Then the projection map $\pi_\Omega : \mathbf{R}^N \to \Omega$ defined uniquely through the condition*

$$|x - \pi_\Omega(x)| = \min_{y\in\Omega} |x - y| \text{ for all } x \in \mathbf{R}^N$$

is globally Lipschitz.

Proof: If $p, q \in \Omega$, we simply have $|\pi_\Omega(p) - \pi_\Omega(q)| = |p - q|$. So assume that $p \notin \Omega$. Then, the convexity of Ω implies that

$$\Omega \subset H := H\big(\pi_\Omega(p), p - \pi_\Omega(p)\big).$$

Therefore, $\pi_\Omega(q) \in H$. Since $|q - \pi_\Omega(q)| \leq |q - \pi_\Omega(p)|$, we have

$$\pi_\Omega(q) \in \big\{x \in H : |x - q| \leq |q - \pi_\Omega(p)|\big\}.$$

Noting that

$$\begin{aligned} |\pi_\Omega(p) - q| &\leq |\pi_H(q) - q| + |\pi_H(q) - \pi_\Omega(p)| \\ &\leq |\pi_H(q) - q| + |p - q| \end{aligned}$$

we obtain

$$\pi_\Omega(q) \in \left\{ x \in H : |\pi_H(q) - q| \leq |x - q| \leq |\pi_H(q) - q| + |p - q| \right\}$$

The diameter of this set can be easily estimated from above by $3|p-q|$, and contains both $\pi_\Omega(p)$ and $\pi_\Omega(q)$. See *Figure 2* for an illustration. □

The real utility of the following statement is that when $f(x)$ is the characteristic function of a domain Ω with reasonable boundary, it reduces the conditions in *Proposition 1*, which involve constructing a vector field $z(x)$ on $\mathbf{R}^N$, to a condition that involves constructing a vector field on only $\partial\Omega$. It shows that such a vector field can then be suitably extended to $\mathbf{R}^N$. This seems to be very closely related to the notion of a *Lipschitz ϕ-regular set* introduced and studied in [2].

Lemma 3 *Let Ω be a bounded domain with piecewise smooth boundary in $\mathbf{R}^N$. Assume that there exists a Lipschitz map $\psi(x) : \partial\Omega \to \partial W_\phi$ such that*

> *The outer unit normal $n_\Omega(x) \in N_{W_\phi}(\psi(x))$ for all $x \in \partial\Omega$ at which it is defined.*

Then $\mathbf{1}_\Omega(x)$ can arise as the minimizer of the anisotropic ROF model for a suitable choice of a compactly supported given image $f(x)$.

Remark: The condition placed on the normals by the hypothesis of the claim above can imply additional smoothness on the boundary of Ω. For example, if ∂W_ϕ is itself smooth so that in particular $N_{W_\phi}(x)$ contains a single direction at every $x \in \partial W_\phi$, then $\partial\Omega$ has to be $C^{1,1}$ in order to satisfy the hypothesis.

Proof: Since ψ is Lipschitz from $\partial\Omega$ to ∂W_ϕ by hypothesis, by a standard result it can be extended to a globally Lipschitz function $\tilde{\psi} : \mathbf{R}^N \to \mathbf{R}^N$. Let $\pi_{W_\phi}(x) : \mathbf{R}^N \to W_\phi$ be the projection map onto the convex set W_ϕ, as defined in *Lemma 2*. By the conclusion of that lemma, $\pi_{W_\phi}(x)$ is Lipschitz. Let $\eta(x) : \mathbf{R}^N \to \mathbf{R}$ be a C^∞ cut-off function, of the following type: η is compactly supported, $\eta(x) \in [0,1]$ for all $x \in \mathbf{R}^N$, and $\eta(x) = 1$ for all $x \in \Omega$. Our proposed vector field is:

$$z(x) = -\eta(x)\pi_{W_\phi}\left(\tilde{\psi}(x)\right)$$

It satisfies the conditions of *Proposition 1*. Indeed, $z(x)$ is compactly supported and Lipschitz, so that $\operatorname{div} z(x) \in L^\infty$. Also, since $\eta(x) \in [0,1]$ and W_ϕ is convex, $z(x) \in W_\phi(x)$ for all x. Moreover, $z(x) = -\psi(x) \in \partial W_\phi$ for $x \in \partial\Omega$, so

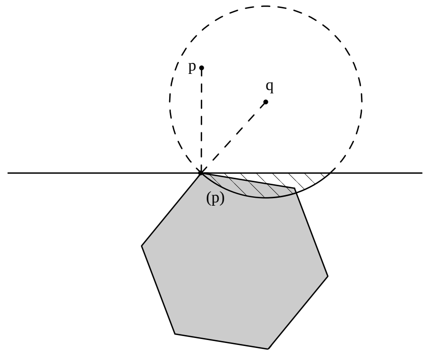

Figure 2: Setup used in the proof of Lemma 2. The projections $\pi_\Omega(p)$ and $\pi_\Omega(q)$ of the two points p and q, respectively, have to lie in the hatched area, which is the intersection of the half space $H(\pi_\Omega(p), p - \pi_\Omega(p))$ with the ball of radius $|q - \pi_\Omega(p)|$ and center q. The diameter of the hatched area is easily seen to be bounded from above by $3|p-q|$.

that if $\partial\Omega$ is smooth at a point x and $n_\Omega(x)$ is the outward normal there, then by hypothesis and the central symmetry of W_ϕ, we have $-n_\Omega(x) \in N_{W_\phi}(z(x))$. By *Lemma 1*, that means

$$z(x) \cdot n_\Omega(x) = -\phi(n_\Omega(x))$$

Therefore,

$$\int_{\mathbf{R}^N} \phi(\nabla \mathbf{1}_\Omega(x)) = -\int_{\partial\Omega} z(x) \cdot n_\Omega(x)\, dH^{N-1} = -\int_{\mathbf{R}^N} \mathbf{1}_\Omega(x) \operatorname{div} z(x)\, dx.$$

It remains to verify the final condition of *Proposition 1*. To that end, we can simply set $f(x) = \mathbf{1}_\Omega(x) - \frac{1}{2\lambda} \operatorname{div} z(x)$. □

4.2.2 Polygonal Wulff Shapes

In this section, we consider two dimensional anisotropic energies whose corresponding Wulff shapes are polygons. Let P be a closed, convex, centrally symmetric k-gon in $\mathbf{R}^2$ that contains the origin in its interior. Let $\{v_1, v_2, \ldots, v_k\}$ be its vertices in clockwise order. P is of course given by the convex hull, $\text{co}(\{v_1, \ldots, v_k\})$, of its vertices. Define the function $\phi : \mathbf{R}^2 \to \mathbf{R}^+$ as

$$\phi(x) := \sup_{y \in \{v_1, \ldots, v_k\}} x \cdot y$$

With ϕ defined as such, we have that $W_\phi = P$. Moreover, ϕ has all the usual properties we require.

Let n_j be the outward unit normal to ∂P along the segment $[v_j, v_{j+1}]$, and let $\theta_j := \text{arg}(n_j)$ denote the angle it makes with the positive x-axis. The requirements for the characteristic function of a piecewise smooth domain to be a possible minimizer of the anisotropic ROF model with $\phi(x)$ as given above can be expressed completely in terms of n_j.

The boundary ∂P of P consists of the vertices v_j and the open line segments (v_j, v_{j+1}) that connect them. The normal directions to ∂P at each one of its points can be listed as follows:

- For $j = 1, \ldots, k$ we have $N_P(v_j) = \big\{ n \in \mathbf{R}^2 : \theta_{j-1} \leq \text{arg}(n) \leq \theta_j \big\}$.
- For any $x \in (v_j, v_{j+1})$ we have $N_P(x) = \big\{ \alpha n_j : \alpha > 0 \big\}$

Our theorem in this section will apply to domains that, roughly speaking, satisfy the following condition:

> *The boundary of Ω is made up of piecewise smooth arcs whose normals remain in one of the "fans" $N_P(v_j)$. Two such piecewise smooth arcs whose normals belong to neighboring fans, say $N_P(v_j)$ and $N_P(v_{j\pm 1})$, can be connected with a line segment parallel to the side $(v_j, v_{j\pm 1})$ of the polygon P.*

A more precise description of these domains can be given as follows: Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. In other words,

$$\partial\Omega = \bigcup_{i=1}^{m} \bar{C}_i$$

where C_i are disjoint arcs of the form

$$C_i := \big\{ p \in \mathbf{R}^2 : p = h_i(t) \text{ for some } t \in (0,1) \big\}$$

where each $h_i(t)$ smoothly imbeds the unit interval $[0,1]$ into $\mathbf{R}^2$. The $\bar{C}_i$ are disjoint except for consequetive ones, which may touch at their corresponding endpoints. We let n_Ω denote the outer unit normal to Ω wherever it is defined. The arcs C_i are assumed to satisfy the following additional conditions:

Condition 1: Each C_i is one of the following two types:

Type A: There exists $j \in \{1, \ldots, k\}$ such that

$$\{n_\Omega(x) : x \in C_i\} \subset (N_P(v_j))^o = \{n \in \mathbf{R}^2 : \theta_{j-1} < \text{arg}(n) < \theta_j\}$$

Type B: There exists $j \in \{1, \ldots, k\}$ such that C_i is a line segment parallel to $\overline{(v_j, v_{j+1})}$, i.e. $\text{arg}(n_\Omega(x)) = \theta_j$ for all $x \in C_i$.

Condition 2: For each point $p \in \partial\Omega$, there exists a $j \in \{1, \ldots, k\}$ and a neighborhood G of p in $\mathbf{R}^2$ such that

$$\{n_\Omega(x) : x \in G \cap \partial\Omega\} \subset N_P(v_j)$$

Remark: Condition 2 says that if two Type A arcs, say C_{i_1} and C_{i_2} are connected, then their normals belong to the same "fan", i.e.

$$\{n_\Omega(x) : x \in C_{i_1} \cup C_{i_2}\} \subset (N_P(v_j))^o \text{ for some } j.$$

This means that any two Type A arcs on the boundary that belong to different fans have a line segment, also lying on the boundary, between them.

Figure 3 gives an example of these conditions in the special example of $\phi(x, y) = |x| + |y|$, whose corresponding Wulff shape is the square $P = [-1, 1] \times [-1, 1]$. The octagon shown on the left in the figure satisfies the conditions listed above, and hence by our Theorem 2 below, can arise as the u-component of the solution to the anisotropic ROF model. The triangle shown to the right, on the other hand, fails to satisfy the conditions listed above. In fact, the triangle (or even a disk) can never be the u-component of our decomposition for any reasonable original image $f(x)$. This fact follows from the more general statement of our *Theorem 3*.

Theorem 2 *Any shape of the form described above can be the u component (i.e. the minimizer) of anisotropic ROF model with the corresponding polygonal Wulff shape for an appropriate choice of the original image $f(x)$ and fidelity constant λ.*

Proof:

We will define a vector field $\psi(x) : \partial\Omega \to \partial P$ that satisfies the conditions of Lemma 3. We do so in two steps.

Step 1: We first define the vector field $\psi(x)$ along Type A arcs. By definition, for each Type A arc C_i there exists a j such that $\{n_\Omega(x) : x \in C_i\} \subset (N_P(v_j))^o$. We let

$$\psi(x) := v_j \text{ for all } x \in \bar{C}_i.$$

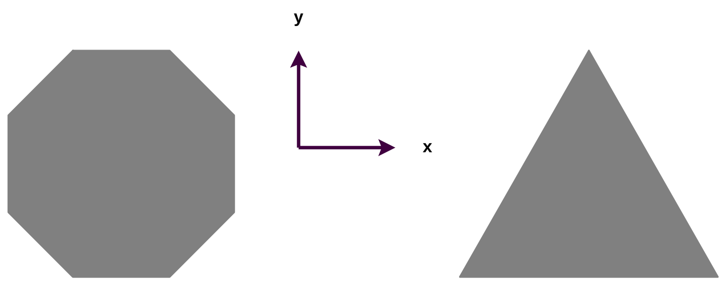

Figure 3: The octagon on the left can arise as the u component (i.e. as the minimizer) of anisotropic ROF model with $\phi(x,y) = |x| + |y|$ while the triangle on the right cannot (for any reasonable original image). These follow from our *Theorems 2* and *3*.

Step 2: Next, we define the vector field on Type B arcs. By definition, each Type B arc C_i is a line segment of the form (p,q) where $p,q \in \mathbf{R}^2$ such that (p,q) is parallel to one of the sides, say (v_{j_*}, v_{j_*+1}), of the polygon P. Therefore, $n_\Omega(x) = n_{j_*}$ along C_i. By Condition 2, there exist $j(p)$ and $j(q)$ and two neighborhoods G_p and G_q of the two end points p and q of C_i, respectively, such that

$$\left\{n_\Omega(x) : x \in G_p \cap \partial\Omega\right\} \subset N_P(v_{j(p)}) \text{ and } \left\{n_\Omega(x) : x \in G_q \cap \partial\Omega\right\} \subset N_P(v_{j(q)})$$

It follows that $n_{j_*} \in N_P(v_{j(p)}) \cap N_P(v_{j(q)})$. Therefore, $j(p), j(q) \in \{j_*, j_* + 1\}$, so that $v_{j(p)}$ and $v_{j(q)}$ are each either v_{j_*} or v_{j_*+1}. To define $\psi(x)$ on $\bar{C}_i$, first divide the segment $[p,q]$ into three equal subsegments along its length. On the subsegment that contains p, let $\psi(x) = v_{j(p)}$. And on the subsegment that contains q, let $\psi(x) = v_{j(q)}$. On the middle subinterval, we can therefore define $\psi(x)$ by smoothly interpolating between $\psi(p)$ and $\psi(q)$ so that for all $x \in C_i$ the resulting vector field $\psi(x)$ takes its values on the edge $[v_{j_*}, v_{j_*+1}]$ of the polygon P.

See *Figure 4* for an illustration of how the construction described above proceeds.

This completes the construction of the vector field $\psi(x)$ on each of the smooth arcs $\bar{C}_i$ of $\partial\Omega$. It remains to verify that $\psi(x)$ is well-defined and satisfies the conditions of Claim 3. By construction, and by Condition 2 that Ω satisfies, it can be seen that $\psi(x)$ is constant in a neighborhood of every point p of the boundary at which two distinct arcs C_i meet. In particular, the construction defines $\psi(x)$ unambiguously on all of $\partial\Omega$. Also by construction, $\psi(x)$ is Lipschitz (in fact smooth) on each C_i. It follows that $\psi(x)$ is Lipschitz on $\partial\Omega$.

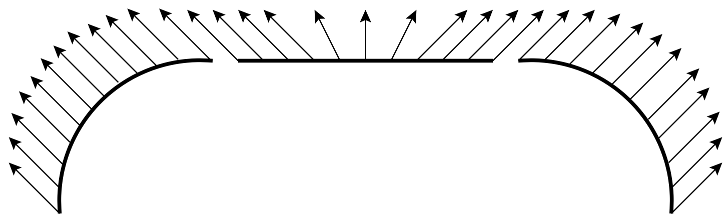

Figure 4: Illustration of how the construction of Theorem 2 proceeds, in the special case when $\phi(x,y) = |x| + |y|$. Horizontal and vertical intervals on the boundary can be used to interpolate between the values $(1,1),(1,-1),(-1,1),(-1,-1)$ that the vector field ψ might take on other parts (Type A arcs, in our terminology) of the boundary. It then becomes possible to extend such a vector field to a neighborhood of $\partial\Omega$ appropriately.

Moreover, $\psi(x)$ takes its values in $W_\phi = P$, and

$$n_\Omega(x) \in N_P(\psi(x)) \text{ for all } x \in C_i \text{ and } i = 1, \ldots, m.$$

Hence $\psi(x)$ satisfies the hypothesis of Claim 3. The conclusion of the present theorem now follows from this claim. □

In *Theorem 2*, roughly speaking, we required the boundary of the domain to contain a line segment whenever its tangent becomes parallel to one of the sides of the polygonal Wulff shape. Our next theorem shows that such a condition is in fact necessary, even for domains with smooth boundaries. To be more precise, we show that at any point $p \in \partial\Omega$ at which $\partial\Omega$ is *locally* either *strictly convex* or *strictly concave*, a plane parallel to one of the sides of W_ϕ cannot be a tangent hyperplane. Any domain Ω whose characteristic function $\mathbf{1}_\Omega(x)$ appears as the minimizer of energy (2) for some original image $f(x) \in L^\infty(\mathbf{R}^2)$ has to satisfy this condition. See *Figure 3* for an example.

Theorem 3 *Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary $\partial\Omega$. Assume there exists a point $p \in \partial\Omega$, an $r > 0$, and a $j \in \{1, 2, \ldots, k\}$ such that*

1. $\partial H(p, n_j) \cap \partial\Omega \cap B_r(p) = \{p\}$*, and*
2. *Either* $\Omega \cap B_r(p) \subset H(p, n_j)$ *or* $\Omega^c \cap B_r(p) \subset H(p, -n_j)$.

Then $u(x) = \mathbf{1}_\Omega(x)$ cannot arise as the minimizer of the anisotropic ROF model (2) for any choice of original image $f(x) \in L^\infty(\mathbf{R}^2)$.

We now give a couple of lemmas that will help us prove this statement.

Lemma 4 *Let $q = \frac{1}{2}(v_j + v_{j+1})$. Then, there exists a constant $\gamma > 0$ such that*

$$\phi(n) - q \cdot n \geq \gamma |n \cdot n_j^{\perp}|$$

for all $n \in \mathbf{R}^2$ with $|n| = 1$. Here, $n_j^{\perp}$ denotes one of the unit perpendicular directions to n_j.

Proof: Recall Lemma 1: For $q \in \partial W_\phi$ and $n \in \mathbf{R}^2$ with $|n| = 1$ we have $\phi(n) = q \cdot n$ only when $n \in N_{W_\phi}(q)$. Due to our choice of q as the midpoint of one of the sides of the polygon W_ϕ, this condition holds only if $n = n_j$. Based on this observation, and formula (4), it follows that $\phi(n) \geq q \cdot n$ for all unit vectors n, with equality holding only if $n = n_j$. Let G be a small enough neighborhood of n_j such that

$$\text{If } n \in G \text{ then } n \in N_{W_\phi}(v_j) \cup N_{W_\phi}(v_{j+1}).$$

(Note: here we used the fact that W_ϕ has *corners* at v_j and v_{j+1}.) By the continuity of ϕ, and our remarks above, we have

$$\min_{|n|=1, n \in G^c} \phi(n) - q \cdot n > 0. \tag{8}$$

On the other hand, if $n \in G$, then

$$\phi(n) - q \cdot n \in \big\{ (v_j - q) \cdot n, (v_{j+1} - q) \cdot n \big\}.$$

But $(v_j - q), (v_{j+1} - q) \perp n_j$. Therefore, there exists a $\tilde{\gamma} > 0$ such that

$$|\phi(n) - q \cdot n| \geq \tilde{\gamma} |n \cdot n^{\perp}| \text{ for all } n \in G. \tag{9}$$

Combining (8) with (9), we conclude it is possible to choose a $\gamma > 0$ that satisfies the conditions of the lemma. □

We also need the following rather obvious lemma.

Lemma 5 *Let E be a bounded set of finite perimeter in $\mathbf{R}^2$. Assume there exists $p, v \in \mathbf{R}^2$, and $d \geq 0$ such that $|v| = 1$ and*

$$E \subset H(p, v) \cap H^c(p - dv, v)$$

Then,

$$|E| \leq d \int_{\mathbf{R}^2} |D\mathbf{1}_E(x) \cdot v|$$

Proof: We may assume that $v = (1,0)$ and $p = (d,0)$. There exists a sequence $\{u_j\} \subset C_c^\infty(\mathbf{R}^2)$ such that $u_j \to \mathbf{1}_E(x)$ in $L^1(\mathbf{R}^2)$, and $\|Du_j\|(\mathbf{R}^2) \to \|D\mathbf{1}_E(x)\|(\mathbf{R}^2)$. Then, L^1 convergence of u_j implies

$$\lim_{j\to\infty} \int_{\mathbf{R}} \int_0^d |u_j| \, dx \, dy = |E|$$

We write

$$u_j(x,y) = \int_{-\infty}^x \partial_x u_j(\xi, y) \, d\xi$$

Then,

$$\begin{aligned} \int_{\mathbf{R}} \int_0^d |u_j| \, dx \, dy &= \int_0^d \int_{\mathbf{R}} \left| \int_{-\infty}^x \partial_x u_j(\xi, y) \, d\xi \right| \, dy \, dx \\ &\leq d \int_{\mathbf{R}} \int_{\mathbf{R}} |\partial_x u_j(x,y)| \, dx \, dy. \end{aligned}$$

Since $\|Du_j\|(\mathbf{R}^2) \to \|D\mathbf{1}_E(x)\|(\mathbf{R}^2)$, we have

$$\lim_{j\to\infty} \|\partial_x u_j\|(\mathbf{R}^2) = \int_{\mathbf{R}^2} |D\mathbf{1}_E(x) \cdot (1,0)|.$$

Consequently,

$$|E| \leq d \int_{\mathbf{R}^2} |D\mathbf{1}_E(x) \cdot (1,0)|$$

which proves the lemma. □

Proof of Theorem 3: We treat the case where the hypothesis $\Omega \cap B_r(p) \subset H(p, n_j)$ holds (i.e. Ω is locally strictly convex at p); the proof is very much the same for the other case (i.e. when Ω is locally strictly concave at p). Furthermore, to simplify the notation we will assume that $r > 0$ in the hypothesis is large enough so that $\Omega \subset B_r(p)$; it is then easy to see how to localize the argument presented below in order to cover the general case.

For $\varepsilon > 0$, let $H_\varepsilon := H(p - \varepsilon n_j, n_j)$. See *Figure 5* for an illustration. The idea of the proof is to compare $E_\phi(\mathbf{1}_\Omega(x))$ with $E_\phi(\mathbf{1}_{\Omega \cap H_\varepsilon}(x))$, and show that the latter is less than the former for small enough $\varepsilon > 0$.

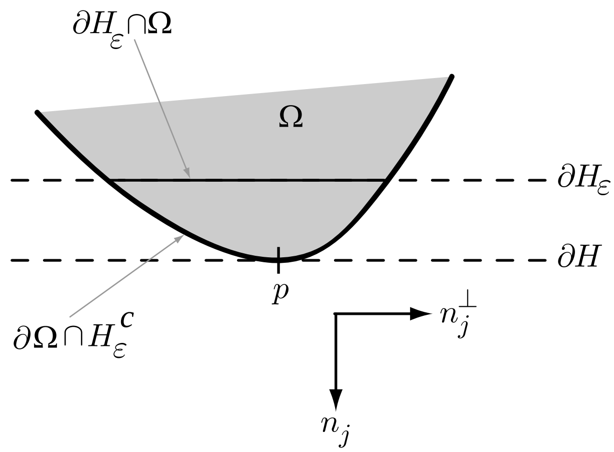

Figure 5: The set up used in the proof of Theorem 3, which is based on a simple cut and paste argument.

Define $d(\varepsilon) := \mathrm{diam}(\Omega \cap H_\varepsilon^c)$. Then, by hypothesis on Ω, we have $d(\varepsilon) > 0$ for all $\varepsilon > 0$, and

$$\lim_{\varepsilon \to 0^+} d(\varepsilon) = 0. \tag{10}$$

Let $n(x) := n_\Omega(x)$ denote the outward unit normal to $\partial\Omega$. For a.e. $\varepsilon > 0$ we have

$$\begin{aligned} \int_{\partial(\Omega\cap H_\varepsilon)} \phi(n_{\Omega\cap H_\varepsilon}(x))\, d\sigma &= \int_{\partial H_\varepsilon \cap \Omega} \phi(n_j)\, d\sigma + \int_{\partial\Omega\cap H_\varepsilon} \phi(n(x))\, d\sigma \\ \text{and } \int_{\partial\Omega} \phi(n(x))\, d\sigma &= \int_{\partial\Omega\cap H_\varepsilon^c} \phi(n(x))\, d\sigma + \int_{\partial\Omega\cap H_\varepsilon} \phi(n(x))\, d\sigma. \end{aligned} \tag{11}$$

Set $q = \frac{1}{2}(v_j + v_{j+1})$. Recall Lemma 4:

$$\phi(n) \geq q \cdot n + \gamma |n \cdot n_j^\perp|, \text{ with } \gamma > 0.$$

Using this inequality, we have

$$\int_{\partial\Omega\cap H_\varepsilon^c} \phi(n(x))\, d\sigma \geq \int_{\partial\Omega\cap H_\varepsilon^c} n(x)\cdot q\, d\sigma + \gamma \int_{\partial\Omega\cap H_\varepsilon^c} |n(x)\cdot n_j^\perp|\, d\sigma \tag{12}$$

Now,

$$\begin{aligned}\int_{\partial\Omega\cap H_\varepsilon^c} n(x)\cdot q\,d\sigma = q\cdot\int_{\partial\Omega\cap H_\varepsilon^c} n(x)\,d\sigma = q\cdot\int_{\partial H_\varepsilon\cap\Omega} n_j\,d\sigma \\ = \int_{\partial H_\varepsilon\cap\Omega}\phi(n_j)\,d\sigma.\end{aligned} \tag{13}$$

Combining (12) with (13) gives

$$\int_{\partial\Omega\cap H_\varepsilon^c}\phi(n(x))\,d\sigma \geq \int_{\partial H_\varepsilon\cap\Omega}\phi(n_j)\,d\sigma + \gamma\int_{\partial\Omega\cap H_\varepsilon^c}|n(x)\cdot n_j^\perp|\,d\sigma$$

This last formula, along with (11) implies

$$\int_{\partial\Omega}\phi(n(x))\,d\sigma \geq \int_{\partial(\Omega\cap H_\varepsilon)}\phi(n_{\Omega\cap H_\varepsilon}(x))\,d\sigma + \gamma\int_{\partial\Omega\cap H_\varepsilon^c}|n(x)\cdot n_j^\perp|\,d\sigma. \tag{14}$$

On the other hand, we can apply *Lemma 5* with $E = \Omega\cap H_\varepsilon^c$, $d = d(\varepsilon)$, and $v = n_j^\perp$ to get

$$\begin{aligned}|\Omega\cap H_\varepsilon^c| &\leq d(\varepsilon)\int_{\partial(\Omega\cap H_\varepsilon^c)}|n_j^\perp\cdot n_{\Omega\cap H_\varepsilon^c}(x)|\,d\sigma \\ &= d(\varepsilon)\int_{\partial\Omega\cap H_\varepsilon^c}|n_j^\perp\cdot n(x)|\,d\sigma\end{aligned} \tag{15}$$

Finally, (14) and (15) imply

$$E_\phi(\mathbf{1}_\Omega(x)) \geq E_\phi(\mathbf{1}_{\Omega\cap H_\varepsilon}(x)) + \left(\frac{\gamma}{d(\varepsilon)} - C\lambda\right)\left|\Omega\cap H_\varepsilon^c\right| \tag{16}$$

where $C = 1 + 2\|f\|_{L^\infty}$. For small enough $\varepsilon > 0$, by (10) we have that $(\frac{\gamma}{d(\varepsilon)} - C\lambda) > 0$. Therefore, (16) shows that $\mathbf{1}_\Omega(x)$ cannot be the minimizer. □

5 Conclusion

We generalized the total variation based image de-noising model of Rudin, Osher, and Fatemi to favor certain edge directions. We studied the resulting anisotropic energies by investigating properties of their possible minimizers. Our results characterize the sets whose indicator functions can arise as solutions to the anisotropic models. They also exhibit some exact solutions.

This line of research can be continued in several ways. Based on the results of [1] in the isotropic case, we would expect it to be possible to identify more general conditions under which the solution to the anisotropic model turns out to be a constant multiple of the original image, when the latter is binary. Also, the improvements on the standard (isotropic) Rudin, Osher, Fatemi model introduced by Meyer in [3] and studied further in [7, 5] can be adapted in a very natural way to the anisotropic setting.

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