On the Number of Hamiltonian Cycles in a Tournament

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Abstract

Let $P(n)$ and $C(n)$ denote, respectively, the maximum possible numbers of Hamiltonian paths and Hamiltonian cycles in a tournament on n vertices.

The study of $P(n)$ was suggested by Szele [14], who showed in an early application of probabilistic method that $P(n) \geq n!2^{-n+1}$, and conjectured that

$$\lim(P(n)/n!)^{1/n} = 1/2.$$

This was proved by Alon [2], who observed that the conjecture follows from a suitable bound on $C(n)$, and showed $C(n) < O(n^{3/2}(n-1)!2^{-n})$. Here we improve this to

$$C(n) < O(n^{3/2-\xi}(n-1)!2^{-n}),$$

with $\xi = 0.2507\ldots$.

Our approach is mainly based on entropy considerations.

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1 Introduction

A tournament is a complete directed graph. For a tournament T we denote by $C(T)$ the number of (directed) Hamiltonian cycles in T and by $P(T)$ the number of Hamiltonian paths. Let $C(n) = \max C(T)$, the maximum taken over tournaments with n vertices, and define $P(n)$ similarly.

The problem of estimating $P(n)$ seems to have been first suggested by Szele [14], whose proof that $P(n) \geq n!2^{-n+1}$ is considered the first combinatorial application of the probabilistic method (see e.g. [4]). The same argument—namely that $P(n)$ is at least the expected number of Hamiltonian paths in a *random* T—shows $C(n) \geq (n-1)!2^{-n}$. Szele also showed that $\lim(P(n)/n!)^{1/n}$ exists and conjectured that its value is 1/2. This was proved by Alon [2], who derived from the following theorem of Brégman (formerly the *Minc Conjecture*) the upper bound $C(n) < O(n^{1/2}n!2^{-n})$ (we will not worry about the constants in the "big Oh's"), and used this to show

$$P(n) < O(n^{3/2}n!2^{-n}). \tag{1}$$

In what follows we set $\Psi(n) = n^{1/2}n!2^{-n}$. This is the order of magnitude in Alon's bound on $C(n)$ and the benchmark against which we will measure our results.

Theorem 1.1 (Brégman's Theorem [5]) *For an $n \times n$ $\{0,1\}$-matrix A with row sums r_i,*

$$per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}$$

(where *per* means permanent). If one regards A in Theorem 1.1 as the adjacency matrix of a digraph T on $[n]$ ($A_{ij} = 1$ iff there is an arc from i to j), then the permanent of A is the number of *1-factors* of T—that is, spanning subdigraphs with all in- and outdegrees equal to 1—so in particular is a bound on the number of Hamiltonian cycles of T. Essentially, Alon shows that

$$4P(n) \leq C(n+1) \tag{2}$$

(his argument for this is repeated in the proof of Corollary 3.3 below), which, with the trivial

$$P(n) \geq nC(n), \tag{3}$$

says the two problems are not much different. Our main purpose here is to slightly improve the upper bounds:

Theorem 1.2 *For any* $\xi < 2(1 - \exp[\sqrt{3/4} - 1]) = 0.2507\ldots$,

$$C(n) < O(n^{1/2-\xi} n! 2^{-n}),$$

and (consequently)

$$P(n) < O(n^{3/2-\xi} n! 2^{-n}).$$

For convenience we will prove the theorem with

$$\xi = (1 + o(1))2(1 - \exp[\sqrt{1 - (.99)(.24)} - 1]), \tag{4}$$

but it is easy to see that the same argument can be used to prove the theorem as stated.

Our proof of this follows the beautiful entropy proof of Brégman's Theorem given by J. Radhakrishnan in [11] (itself inspired by Schrijver's proof [13], especially as presented in [4]). In retrospect, at least, the entropy approach is quite natural in such settings, where one wants to bound the size of some subset of a product of sets (as is the case here, the set of 1-factors of a digraph being a subset of $[n]^{[n]}$). See e.g. [12] for a discussion of other results of this type.

It turns out that a convenient way to explain the present argument is to review what Radhakrishnan does and see where there might be room for improvement in our special situation. This is done in Section 2; in a few words the plan is as follows. We treat 1-factors as functions in the obvious way: $f(x) = y$ means the arc (x, y) is part of the 1-factor. In the entropy approach we consider $\mathbf{f}$ chosen uniformly at random from the set of 1-factors and try to bound the entropy $H(\mathbf{f})$, which is simply the log of the number of 1-factors. (For entropy see Section 2.) This is done by taking the vertices in some order $x_1, \ldots, x_n$ and considering for $i = 1, \ldots, n$ the conditional entropy of $\mathbf{f}(x_i)$ given $(\mathbf{f}(x_j) : j < i)$, the idea being that some *a priori* possibilities for $\mathbf{f}(x_i)$ are ruled out because they have been chosen earlier in the sequence. The mild improvement in the present situation then derives from the fact that when we allow only Hamiltonian cycles, we have the additional restriction that $\mathbf{f}(x_i)$ cannot close a cycle of length less than n. Details of the implementation of this plan are contained in Sections 3 and 4.

The actual growth rate of $C(n)$ (or $P(n)$)—and some related questions suggested by it—seem quite interesting; see Section 5. We tend to think Szele's lower bound is close(r) to the truth. A small but ingenious improvement—multiplication by $e - o(1)$)—in Szele's bound was given by Alon et. al. in

[1]; and very recently Wormald [15], motivated by the question at the end of the present paper, improved this to 2.85584..., which he conjectures to be close to the truth. Both these improvements are based on analysis of suitable random *regular* tournaments.

Notation and conventions

We use $\mathcal{H}$ or $\mathcal{H}(T)$ for the set of Hamiltonian cycles of T (treated, as above, as functions) and often abbreviate "Hamiltonian cycle" to "cycle." We use $\Gamma^+(x)$ for the set of out-neighbors of x, $\Gamma^-(x)$ for the set of in-neighbors and $d(x) = |\Gamma^+(x)|$.

We write $[n]$ for $\{1, \ldots, n\}$, S_n for the symmetric group (here the set of orderings of the vertex set of our tournament), $\mathbb{E}$ for expectation, and log for $\log_2$. For simplicity we pretend throughout that all large numbers are integers.

2 Review and preview

Here we recall Radhakrishnan's proof of Brégman's Theorem and indicate what needs to be added for the proof of Theorem 1.2. The *entropy* of a discrete random variable $\mathbf{X}$ (meaning, simply, a random variable taking values in some countable set) is

$$H(\mathbf{X}) = \sum_x p(x) \log(1/p(x)),$$

where we write $p(x)$ for $\Pr(\mathbf{X} = x)$. The *conditional entropy* of $\mathbf{X}$ given $\mathbf{Y}$ is (with the obvious meanings for $p(y), p(x|y)$)

$$H(\mathbf{X}|\mathbf{Y}) = \mathbb{E}H(\mathbf{X}|\mathbf{Y} = y) = \sum_y p(y) \sum_x p(x|y) \log \frac{1}{p(x|y)} .$$

The only entropy facts we will need here are, first, that for a random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ (note this is also a random variable), one has

$$H(\mathbf{X}) = H(\mathbf{X}_1) + H(\mathbf{X}_2|\mathbf{X}_1) + \cdots + H(\mathbf{X}_n|\mathbf{X}_1, \ldots, \mathbf{X}_{n-1}) \tag{5}$$

(this is the "chain rule"), and, second, that for any $\mathbf{X}$,

$$H(\mathbf{X}) \leq \log |\text{range}(\mathbf{X})|. \tag{6}$$

(Note this is sharp if $\mathbf{X}$ is chosen uniformly from its range.) For more entropy background see [10] or [6].

We deal with Brégman's Theorem in its digraph formulation: We are given a digraph T on $[n]$ and $\mathbf{f}$ drawn from an (*arbitrary*) probability measure μ on $\mathcal{F}$, the set of 1-factors in T, and want to bound the entropy $H(\mathbf{f})$. (For Brégman's Theorem μ is uniform measure, which according to (6) is actually the worst case; but it will later be helpful to have this more general version.) Fix $\sigma \in S_n$. We think of sequentially exposing the values $\mathbf{f}(x)$ according to the order on vertices x given by σ. By (5) we have

$$\begin{aligned} H(\mathbf{f}) &= \sum_{i=1}^{n} H\left[\mathbf{f}(\sigma^{-1}(i)) \mid \mathbf{f}(\sigma^{-1}(1)), \ldots, \mathbf{f}(\sigma^{-1}(i-1))\right] \\ &= \sum_{x} H\left[\mathbf{f}(x) \mid (\mathbf{f}(y) : \sigma(y) < \sigma(x))\right]. \end{aligned}$$

Suppose now that

$$\mathbf{f} \text{ is chosen uniformly from some } \mathcal{M} \subseteq \mathcal{F} \text{ with } |\mathcal{M}| = m.$$

Then with f ranging over $\mathcal{M}$,

$$H(\mathbf{f}) = \frac{1}{m} \sum_{f} \sum_{x} H\left[\mathbf{f}(x) \mid \mathbf{f}(y) = f(y) \;\; \forall \sigma(y) < \sigma(x)\right]. \tag{7}$$

Set

$$k_f(x, \sigma) := |\Gamma^{+}(x) \setminus \{f(y) : \sigma(y) < \sigma(x)\}|. \tag{8}$$

This quantity, the number of possibilities for $\mathbf{f}(x)$ at the time it is exposed given agreement with f up to that point, will play a central role in this paper. By (6) the summand in (7) is at most

$$\log k_f(x, \sigma). \tag{9}$$

So if we now unfix σ, choosing it at random according to some probability measure on S_n (which in practice will just be uniform measure), then

$$\begin{aligned} H(\mathbf{f}) &\leq \mathbb{E}_{\sigma} \frac{1}{m} \sum_{f} \sum_{x} \log k_f(x, \sigma) \\ &= \frac{1}{m} \sum_{f} \sum_{x} \mathbb{E}_{\sigma} \log k_f(x, \sigma). \end{aligned}$$

If σ is chosen uniformly from S_n then the summand does not depend on f, and is equal to

$$\frac{1}{d(x)}\sum_{k=1}^{d(x)}\log k = \frac{1}{d(x)}\log(d(x)!),$$

since—this is perhaps the main point—for fixed f,

$$k_f(x,\sigma) = |\{y \in f^{-1}(\Gamma^+(x)) : \sigma(y) \geq \sigma(x)\}|$$

is uniform from $[d(x)]$. This gives Theorem 1.1.

□

To do better in the case of Hamiltonian cycles we intend to strengthen the bound (9) in some cases (with $\mathcal{M}$ now a subset of $\mathcal{H}(T)$). Given f, define for each x and σ (with $\sigma(x) < n$),

$$i(x,\sigma) = \min\{i \geq 1 : \sigma(f^{-i}(x)) > \sigma(x)\}$$

and

$$b(x,\sigma) = f^{-i(x,\sigma)+1}(x).$$

That is, if we follow f backwards from x, $b(x,\sigma)$ is the last vertex we see before we first encounter a vertex that follows x in σ. If $b(x,\sigma) \in \Gamma^+(x)$ then (given $\mathbf{f}(y) = f(y)\ \forall \sigma(y) < \sigma(x)$ as in (7)) we cannot have $\mathbf{f}(x) = b(x,\sigma)$, since this choice is incompatible with what's known at the time we choose $\mathbf{f}(x)$ (it would imply that f contains a cycle of length $i(x,\sigma) < n$). So, with

$$A(x,\sigma) := \{b(x,\sigma) \in \Gamma^+(x)\},$$

we may replace the bound (9) on $H[\mathbf{f}(x)|\mathbf{f}(y) = f(y)\ \forall \sigma(y) < \sigma(x)]$ in the proof of Theorem 1.1 by $\log(k_f(x,\sigma) - \mathbf{1}_{A(x,\sigma)})$. Setting

$$p_k(x) = \Pr_\sigma[b(x,\sigma) \in \Gamma^+(x) \mid k_f(x,\sigma) = k],$$

this gives

$$\begin{aligned} H(\mathbf{f}) &\leq \frac{1}{m}\sum_f\sum_x \frac{1}{d(x)}\sum_{k=1}^{d(x)}[\log k - p_k(x)\log\frac{k}{k-1}] \\ &= \sum_x \frac{1}{d(x)}\log(d(x)!) - \frac{1}{m}\sum_f\sum_x\sum_k \frac{1}{d(x)}p_k(x)\log\frac{k}{k-1}. \quad (10) \end{aligned}$$

So we would like to show that the subtracted expression is substantial. In practice this will go as follows. We first introduce the notion of a "good" Hamiltonian cycle, taking $\mathcal{M}$ to be the set of such cycles (and $m = |\mathcal{M}|$). We show (Lemma 3.4) that the number of bad cycles is negligible:

$$|\mathcal{H} \setminus \mathcal{M}| \leq (1 - 10^{-4} + o(1))^{.01n}\Psi(n). \tag{11}$$

This is essentially a consequence of Theorem 1.1, as is the observation that we may assume T is fairly regular (see (21)). The final and main point of the proof is showing that, given this regularity, the desired gain in (10) even holds "locally":

$$\text{for each } f \in \mathcal{M}, \quad \sum_x \sum_k \frac{1}{d(x)} p_k(x) \log \frac{k}{k-1} > \xi \log n \tag{12}$$

(ξ as in (4). Combining these with (10) gives Theorem 1.2.

3 Few bad cycles

Here we say what's meant by a bad cycle and prove (11). This requires a few preliminaries. We first observe that Theorem 1.1 gives

Corollary 3.1 *For a tournament T on n vertices with outdegrees $r_1, \ldots, r_n$, and r the geometric mean of the r_i's, the number of 1-factors in T is less than* $\exp[(1/4 + o(1)) \ln^2 n](\frac{r}{e})^n$.

Proof. This is a simple calculation using Theorem 1.1 and, for instance, the fact that the sum of any t of the r_i's is at least $\binom{t}{2}$. We omit the details.

□

Say an arc (j, i) in a tournament on $[n]$ is *oriented backwards* if $i < j$.

Lemma 3.2 *Suppose T is a tournament on $[n]$ in which at most $(1/4-\alpha)\binom{n}{2}$ arcs are oriented backwards. Then the number of 1-factors in T is less than $(1 - \alpha^2 + o(1))^n \Psi(n)$, provided $\alpha > (\ln n)/\sqrt{n}$.*

(In practice α will just be a positive constant.)

Proof. Suppose for simplicity that n is even, and again write $r_1, \ldots, r_n$ for the outdegrees. The arithmetic mean of those r_i's with $n/2 < i \leq n$ is less than

$(1/2-\alpha)n$ (it may be helpful to picture the adjacency matrix with at most $(1/4-\alpha)\binom{n}{2}$ 1's below the main diagonal); so since the arithmetic mean of *all* the r_i's is $(n-1)/2$, concavity of the logarithm implies that the *geometric* mean of (all) the r_i's is less than $n\sqrt{(1/2-\alpha)(1/2+\alpha)} < (1-2\alpha^2)n/2$. Inserting this in Corollary 3.1 we find that the number of 1-factors in T is less than

$$\exp[(1/4+o(1))\ln^2 n]\left((1-2\alpha^2)\frac{n}{2e}\right)^n,$$

which gives the statement in the lemma.

□

The reason for shifting our attention to $P(T)$ in the following corollary is that our proof of a bound on the number of "bad" cycles (Lemma 3.4) will involve breaking such cycles into paths.

Corollary 3.3 *For T, α as in Lemma 3.2, $P(T) < (1-\alpha^2+o(1))^n\Psi(n)$.*

Proof. We follow [2]: Let T' be the random tournament on $[n+1]$ obtained from T by choosing uniformly from the 2^n possible orientations of the edges joining $n+1$ to vertices from $[n]$. Then each Hamiltonian path of T extends to a Hamiltonian cycle of T' with probability 1/4, so that

$$P(T)/4 = \mathbb{E}C(T') < (1-\alpha^2+o(1))^n\Psi(n),$$

the inequality following from Lemma 3.2. This gives the stated bound. (Note the $o(1)$ term can be used to absorb minor factors such as $\Psi(n+1)/\Psi(n) \approx n$.)

□

We take the *length* of an interval I (of a cycle) to be the number of edges (arcs) in I, and the *length* of a chord to be the number of edges in the shorter interval connecting the ends of the chord. We then use *j-interval* (*j-chord*) to mean an interval (chord) of length j.

Let ω be a function of n tending slowly to infinity ($\omega < n^{o(1)}$ will be slow enough). ¿From now on k will always denote an integer satisfying

$$\omega < k < n/(2\omega). \tag{13}$$

For a cycle f say a chord (x, y) goes backwards if $x = f^i(y)$ for some $i < n/2$. Given $f \in \mathcal{H}$, we say an interval of f is *bad* if less than .24 of its chords (say of length at least 2, though this makes no significant difference) go backwards (and of course, here and elsewhere, "good" means "not bad"). A cycle is k-*bad* if more than .01 of its k-intervals are bad, and *bad* if it is k-bad for some k in our range. This is the notion of "bad" for which we prove (11):

Lemma 3.4 *In any tournament on* $[n]$ *the number of bad cycles is less than* $(1-10^{-4}+o(1))^{.01n}\Psi(n)$.

Proof. It is enough to show that the number of k-bad cycles is less than

$$(1-10^{-4}+o(1))^{.01n}\Psi(n). \tag{14}$$

Say a k-set K (of vertices) is bad if it has an ordering with respect to which the fraction of edges in K which are oriented backwards in T is less than .24. If a cycle H is k-bad, then it may be partitioned into n/k k-intervals at least .01 of which are bad, which in particular implies that their underlying k-sets are bad. So we may specify a bad cycle H as follows.

(i) Choose a cyclically ordered partition $I_1 \cup \cdots \cup I_{n/k}$ of $[n]$ into k-sets and some $.01n/k$ of these which are bad (requiring that H traverse the I_j's in the cyclic order).

(ii) For each j choose the Hamiltonian path which is the restriction of H to I_j.

The number of possibilities in (i) is at most

$$\frac{k}{n}\binom{n}{k,\ldots,k}\binom{n/k}{.01n/k}, \tag{15}$$

while (1) and Corollary 3.3 bound the number of possibilities in (ii) by

$$(O(k\Psi(k)))^{n/k}(1-10^{-4}+o(1))^{.01n}; \tag{16}$$

and the product of (15) and (16) is easily seen to be no more than (14). (We again bury minor factors—in this case mainly something like $k^{n/k}$—in the $o(1)$.)

□

4 Not too many good cycles

Here we prove Theorem 1.2; that is, for a tournament T on $[n]$,

$$C(T) < O(n^{1/2-\xi} n! 2^{-n}). \tag{17}$$

We need one preliminary observation. For a cycle f let $back_j(f)$ be the fraction of j-chords of f that go backwards. Notice that if f is k-good then

$$\sum_{j=2}^{k} back_j(f)(k+1-j) \geq \beta \sum_{j=2}^{k} (k+1-j), \tag{18}$$

where $\beta = (.99)(.24)$, since each j-chord belongs to exactly $k+1-j$ intervals of length k.

Lemma 4.1 *For any good cycle f, k satisfying* (13) *and* $q = 1 - \frac{2k}{n}$,

$$\sum_{j=2}^{\frac{n}{2k}} back_j(f) q^j \geq (\gamma - o(1)) \frac{n}{2k},$$

where $\gamma = 1 - \exp[\sqrt{1-\beta} - 1] = 0.119\ldots$

Proof. Set $k' = \frac{n}{2k}$. We have (18) for k'—i.e.

$$\sum_{j=2}^{k'} back_j(f)(k'+1-j) \geq \beta \sum_{j=2}^{k'} (k'+1-j) \tag{19}$$

—and should minimize $\sum_{j=2}^{k'} back_j(f) q^j$. But since $\frac{q^j}{k'+1-j}$ is increasing in j for $j \leq k'$, the minimum is achieved by taking $back_j = 1$ if $j \leq m$ and 0 otherwise, where m is the smallest value for which this assignment achieves (19). An easy calculation then gives $m = (1 - \sqrt{1-\beta})k'$, and

$$\sum_{j=2}^{\frac{n}{2k}} back_j(f) q^j = \sum_{j=2}^{m} q^j = \frac{q^2 - q^{m+1}}{1-q} = (1 - o(1))\gamma \frac{n}{2k}.$$

□

Remark. The value of γ can presumably be improved by exploiting (18) in general (rather than just for k'), but we will not try to optimize here.

We now turn to (17). To begin notice we may assume that r, the geometric mean of the outdegrees in T, satisfies

$$r > (1 - \frac{\ln^2 n}{n})n/2,$$

since when this is not the case Corollary 3.1 immediately gives (17).

This implies that T is fairly regular, as follows. Let $r_i = (1+\delta_i)\frac{n-1}{2}$. Then

$$\prod (1+\delta_i)^{1/n} = \frac{2r}{n-1} > 1 - \frac{\ln^2 n}{7n}.$$

On the other hand, letting $\delta = \frac{1}{n}\sum |\delta_i|$, and using $\sum \delta_i = 0$, the Taylor expansion of $\ln(1+x)$ and Cauchy-Schwarz we have

$$\frac{1}{n}\sum \ln(1+\delta_i) \leq \frac{1}{n}\sum \left(\delta_i - \frac{\delta_i^2}{2} + \frac{\delta_i^3}{3}\right) \leq \frac{1}{n}\sum \left(-\frac{\delta_i^2}{6}\right) \leq -\frac{\delta^2}{6}.$$

So

$$1 - \frac{\ln^2 n}{7n} < e^{-\delta^2/6} < 1 - \frac{\delta^2}{7}$$

implying $\delta < n^{-1/2}\ln n$. So if we set $\zeta = n^{-1/4}\sqrt{\ln n}$ (say) and call a vertex x *good* if

$$|d(x) - (n-1)/2| < \zeta n. \tag{20}$$

Then

$$|\{x : x \text{ bad}\}| < \zeta n = o(n). \tag{21}$$

As in (12), write $\mathcal{M}$ for the set of good cycles in T, and set $m = |\mathcal{M}|$. In view of Lemma 3.4 (= (11)) the proof of (17) will be complete if we can show (12) for T satisfying (21). So we consider a fixed $f \in \mathcal{M}$ and write $k(x,\sigma)$ for $k_f(x,\sigma)$ (see (8) for k_f). The main remaining point is

Lemma 4.2 *If x is good, k satisfies* (13) *and* $0 \leq j \leq n/(2k)$, *then*

$$\Pr\left[b(x,\sigma) = f^{-j}(x) \mid k(x,\sigma) = k\right] > (1 - o(1))\frac{2k}{n}\left(1 - \frac{2k}{n}\right)^j.$$

This easily gives (12): writing $back_j(x)$ for the indicator of $\{f^{-j}(x) \in \Gamma^+(x)\}$, Lemma 4.2 says that for good x,

$$p_k(x) > (1-o(1))\frac{2k}{n}\sum_{j=2}^{\frac{n}{2k}}\left(1-\frac{2k}{n}\right)^j back_j(x), \tag{22}$$

whence the sum in (12) (even restricted to the good vertices, of which there are more than $(1-\zeta)n$), is at least

$$(1-o(1))\frac{2}{n}\sum_k \frac{2k}{n}\sum_{j=2}^{\frac{n}{2k}}\left(1-\frac{2k}{n}\right)^j (back_j(f)-\zeta)n\log\frac{k}{k-1}$$

$$> (1-o(1))2\gamma\sum_k \log\frac{k}{k-1} = (2\gamma - o(1))\log n,$$

where k ranges over $(\omega, n/(2\omega))$, and we used Lemma 4.1 for the inequality and $\omega < n^{o(1)}$ for the $\log n$.

□

Proof of Lemma 4.2. The intuition behind the proof is as follows. Notice that "$b(x,\sigma) = f^{-j}(x)$" means that in the sequence

$$(f^{-i}(x) : i = 1, \ldots), \tag{23}$$

f^{-j} is the first vertex y for which

$$\sigma(f^{-1}(y)) > \sigma(x). \tag{24}$$

We are given k, the number of $y \in \Gamma^+(x)$ for which (24) holds, and will show—as one would expect given (20)—that the number of such y in $\Gamma^-(x)$ is typically also close to k. Thus (typically) *each* y (other than $f(x)$) satisfies (24) with probability approximately $2k/n$, so we may guess that the number of y's from (23) that need to be examined before finding one satisfying (24) is approximately geometric with mean $n/(2k)$, which is what the lemma says.

Some additional notation will be helpful. Let

$$S = f^{-1}(\Gamma^+(x)),$$

$$T = [n] \setminus S$$

(the f-preimages of x and its in-neighbors), $s = |S|$ $(= d(x))$ and $t = |T| = n - s$. Note that since x is good (see (20)), we have

$$\left|\frac{s}{t} - 1\right| < o(1) \tag{25}$$

(with $o(1) \approx 4\zeta$).

Let $A_\sigma = \{y : \sigma(y) \geq \sigma(x)\}$. Then

$$k(x,\sigma)(= |\Gamma^+(x) \setminus \{f(y) : \sigma(y) < \sigma(x)\}|) = |S \cap A_\sigma(x)|,$$

and we use ℓ for the analogous quantity for in-neighbors:

$$\ell(x,\sigma) = |T \cap A_\sigma(x)|.$$

We henceforth write Pr for $\Pr_\sigma$.

Our first task is to say something about the conditional dsitribution of $\ell(x,\sigma)$ given $k(x,\sigma)$. This is in fact an instance of "Pólya's urn" (see e.g. [9]): we may think of starting with an urn containing $b = k$ blue balls and $r = s + 1 - k$ red balls, and repeatedly drawing balls (uniformly) at random, following the rule that after each draw we replace the chosen ball together with an additional ball of the same color. Then the number of blue balls chosen in t draws is distributed as $\ell(x,\sigma)$ (given $k(x,\sigma) = k$), and we have (e.g. [9], p.240, Prob.30)

$$\mathbb{E}[\ell(x,\sigma) \mid k(x,\sigma) = k] = \frac{kt}{s+1} \sim k, \tag{26}$$

$$\mathrm{Var}[\ell(x,\sigma) \mid k(x,\sigma) = k] = \frac{tk(s-k+1)(s+t+1)}{(s+1)^2(s+2)} \sim 2k \tag{27}$$

(using (25) and (13)). This gives sufficient concentration for our purposes:

Lemma 4.3 *For any good x, k satisfying* (13), *and* $\lambda > 0$,

$$\Pr\left[|\ell(x,\sigma) - \frac{kt}{s+1}| > \lambda \mid k(x,\sigma) = k\right] < (1 + o(1))\lambda^{-2}2k,$$

and in particular

$$\Pr\left[|\ell(x,\sigma) - k| < o(k) \mid k(x,\sigma) = k\right] > 1 - o(1).$$

Proof. This follows from (26) and (27) via Chebyshev's inequality.

$\square$

Write $Q(k,\ell)$ for the event $\{k(x,\sigma)=k, \ell(x,\sigma)=\ell\}$. By Lemma 4.3, Lemma 4.2 will follow if we show

$$\Pr\left[b(x,\sigma)=f^{-j}(x) \mid Q(k,\ell)\right] > (1-o(1))\frac{2k}{n}\left(1-\frac{2k}{n}\right)^j \tag{28}$$

whenever

$$|\ell-k|<o(k). \tag{29}$$

Let

$$I=\{f^{-i}(x): 1\le i\le j\},$$

$a=|S\cap I|$ and $b=|T\cap I|=j-a$. Let $w=f^{-j-1}(x)$. We will assume that $w\in S$ since this is the only case that contributes to the sum in (22).

If we condition on $Q(k,\ell)$ then

$$K:=S\cap A_\sigma(x)\setminus\{x\} \text{ and } L:=T\cap A_\sigma(x)$$

are chosen uniformly (and independently) from $\binom{S\setminus\{x\}}{k-1}$ and $\binom{T}{\ell}$ respectively, and we have $b(x,\sigma)=f^{-j}(x)$ iff

$$w\in K \text{ and } K\cap I=\emptyset=L\cap I.$$

Thus, with $(m)_u:=m(m-1)\cdots(m-u+1)$,

$$\Pr\left[b(x,\sigma)=f^{-j}(x) \mid Q(k,\ell)\right]=\frac{\binom{s-a-2}{k-2}}{\binom{s-1}{k-1}}\frac{\binom{t-b}{\ell}}{\binom{t}{\ell}}=\frac{k-1}{s-1}\frac{(s-a-2)_{k-2}}{(s-2)_{k-2}}\frac{(t-b)_\ell}{(t)_\ell},$$

which for ℓ satisfying (29) (and x,k,j as in the lemma) is easily seen to be at least the right hand side of (28).

$\square$

5 Remarks

Recall we began with an $O(n^{3/2})$ gap between Szele's lower bound, $C(n) \geq (n-1)!2^{-n} := R(n)$ (the expected number of Hamiltonian cycles in a random tournament) and Alon's upper bound, $O(\Psi(n)) = O(n^{3/2}R(n))$. Theorem 1.2 replaces $n^{3/2}$ by $n^{3/2-\xi}$. The value of ξ given here is certainly not optimal (even without going beyond the present ideas), but of course the real question at this point is, what is the true behavior? We think it likely that $C(n)$ is no more than $O(n^{1/2}R(n))$, and would not rule out the perhaps surprising possibility that $C(n) = O(R(n))$, i.e. that *the number of cycles in a tournament cannot exceed the expected number of cycles in a random tournament by more than a multiplicative constant.* (These natural guesses have also been suggested by Noga Alon [3].)

A heuristic reason for the first guess—i.e. that one can reduce the bound one gets from Brégman's Theorem by at least a factor like n—is the feeling that the Hamiltonian cycles ought to account for no more than about a $1/n$ proportion of the 1-factors. (It seems reasonable to expect this for more general digraphs, but we have no concrete conjecture here.) A more "practical" reason is that one can see where the proof should be strengthened to give essentially this gain; namely, it should be the case that for typical f one has $back_j(f) \approx 1/2$ for most j (more precisely, the number of f violating this should be small compared to $R(n)$), which would allow us to replace the ξ in (12) by something close to 1 and our upper bound by about $n^{-1}\Psi(n)$. (This would also require somewhat expanding the range of j in Lemma 4.2, but such a change causes no difficulty.)

As to a further reduction, we do feel that $n^{-1}\Psi(n)$ is still too large. A vague reason for this is that the matrices associated with tournaments are (in various senses) quite unlike the block diagonal matrices which are the tight cases for Brégman's Theorem. Thus (another conjecture) the number of 1-factors in a tournament should be significantly less than $\Psi(n)$, say at most $O(n^{-\delta}\Psi(n))$ for some positive constant δ, suggesting, according to the preceding paragraph, that $C(n) < O(n^{-1-\delta}\Psi(n))$. But we have no particular reason (other than the appeal of a clean answer) to expect $\delta = 1/2$.

A related, also seemingly quite interesting question is, what can one say about *lower* bounds on $C(T)$ for *regular* T? There is of course no such lower bound for general T; but as far as we know, it could even be that one has $C(T) > \Omega(R(n))$ for all regular T. For a start, it seems plausible that using the van der Waerden Conjecture [8], one may show that $R(n)$ gives the right asymptotics for the logarithm: $C(T) > n^{(1-o(1))n}$ for regular T.

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