SUPERVISORY CONTROL OF FAMILIES OF LINEAR SET-POINT CONTROLLERS - PART 2: ROBUSTNESS

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Abstract— **A simply-structured high-level controller called a 'supervisor' has recently been proposed in [1] for the purpose of orchestrating the switching of a sequence of candidate set-point controllers into feedback with an imprecisely modeled siso process so as to cause the output of the process to approach and track a constant reference input. The process is assumed to be modeled by a siso linear system whose transfer function is in the union of a number of subclasses, each subclass being small enough so that one of the candidate controllers would solve the set-point tracking problem, were the process's transfer function to be one of the subclass's members. In [1] it is shown that in the absence of unmodelled process dynamics the proposed supervisor can successfully perform its function {i.e., achieve a zero steady state tracking error} even if process disturbances are present, provided they are constant. This paper proves that without any further modification, the same supervisor can also perform this function in the face of norm-bounded unmodelled dynamics and moreover that none of the signals within the overall system can grow without bound in response to bounded disturbance and noise inputs, be they constant or not.**

I. Introduction

A simply-structured high-level controller called a "supervisor," has recently been proposed in [1] for the purpose of orchestrating the switching of a sequence of candidate controllers into feedback with an imprecisely modeled process so as to stabilize the process and make its output track a constant reference input. In [1], the process is assumed to be modeled by a SISO linear system which is known only to the extent that its transfer function is within the union of a number of given subclasses, each subclass being small enough so that one of the candidate controllers would solve the problem, were the process's transfer function to be one of the subclass's members. Each subclass contains a "nominal process model transfer function" about which the subclass is centered. The supervisor's unique feature, distinguishing it from other logics which might be used for the same purpose, is that controller selection is made by (i) continuously comparing in real time suitably defined norm-squared output estimation errors or "performance signals" determined by the nominal models and (ii) by placing in the feedback-loop, from time to time, that candidate controller whose corresponding performance signal is the smallest. The paradigm is a manifestation of the idea of certainty equivalence from parameter adaptive control.

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In [1] it is shown that in the absence of unmodelled process dynamics the proposed supervisor can successfully perform its function {i.e., achieve a zero steady state tracking error} even if process disturbances are present, provided they are constant. The purpose of this paper is to prove that without any further modification, the same supervisor can also perform this function in the face of norm-bounded unmodelled dynamics and moreover that none of the signals within the overall system can grow without bound in response to bounded disturbance and noise inputs, be they constant or not.

Relevant background material from [1] is briefly summarized in §II. Included is a description of the class of admissible process model transfer functions, a statement of requisite properties of the candidate loop-controller transfer functions, a characterization of the system which is to be supervised, and a description of the estimator-based supervisor which carries out the controller selection process. The paper's main result, Theorem 1, is presented in §III. The theorem states, in essence, that the overall supervisory control system performs its function in the face of large nominal modeling errors as well as norm-bounded unmodelled dynamics and constant disturbances. An explicit upper bound for the allowable norm-bound ϵ_{p^*} on the unmodelled dynamics is given in subsection VII-A.

Theorem 1 is the main technical result of this paper. Its proof employs an output-injection based argument [2] which is to some extent similar to that used in [1]. The correctness proof in [1] depend crucially on the readily discernible fact that at least one of the supervisor's performance signals must be bounded. While this is also true when norm-bounded unmodelled dynamics and disturbances are present, it is no longer possible to conclude that this is so without first delving into the detailed analysis of the system. For this reason the proof of Theorem 1 is much longer than its counterpart in [1].

The analysis of the system is carried out in five sections. Section IV outlines the notation and explains the underlying concepts upon which the analysis is based. Key mathematical relationships are enclosed in boxes. In analyzing the system it is especially useful to distinguish between two different types of supervision: perfect and imperfect. Roughly speaking, the system is being {almost} perfectly supervised whenever the controller in the loop is close to the "correct" one. Otherwise the system is being imperfectly supervised. The concept of perfect supervision is made precise in §V. This section also contains a technical result, namely Proposition 1, which characterizes the behavior of a perfectly supervised system. The behavior of an imperfectly supervised system is correspondingly characterized by Proposition 2 which is stated and proved in §VI.

The hypotheses of Propositions 1 and 2 include constraints on ϵ_{p^*} of the forms

$$\epsilon_{p^*} < \frac{1}{\mathfrak{g}_{p^*}} \qquad \text{and} \qquad \epsilon_{p^*} < \frac{1}{\bar{\mathfrak{g}}_{p^*}}$$

respectively, where $\mathfrak{g}_{p^*}$ and $\bar{\mathfrak{g}}_{p^*}$ are positive system gains depending only on the family of candidate loop-controllers to be supervised, and on the estimator-based supervisor which carries out the controller selection process. Explicit formulas for $\mathfrak{g}_{p^*}$ and $\bar{\mathfrak{g}}_{p^*}$ are given respectively by equation (26) in §V and equation (39) in §VI. These gains are not optimized {but could be} with respect to the selections of the various quantities upon which they depend. In any event, conservative but computable upper bounds for both $\mathfrak{g}_{p^*}$ and $\bar{\mathfrak{g}}_{p^*}$ can easily be derived using standard techniques.

Theorem 1 is a more or less direct consequence of Propositions 1 and 2. A proof of Theorem 1 is given in subsection VII-C. The proof of Proposition 1 is straightforward and is given in the Appendix. All the difficulty in proving Proposition 2 is hidden in the proof of Lemma 5 which is the main technical result upon which the proposition's proof depends. The lemma's proof relies on several key inequalities implied by dwell-time switching. These inequalities are derived in a self-contained subsection of §VIII and are then used there to prove Lemma 5.

Preliminaries:

In the sequel prime denotes transpose. $\mathbb{R}^{n\times m}$ is the linear space of real $n\times m$ matrices. The norm of $M\in\mathbb{R}^{n\times m}$, written $|M|$ is the sum of the magnitudes of its entries. If $f:[0,\infty)\rightarrow\mathbb{R}^n$ and $g:[0,\infty)\rightarrow\mathbb{R}^n$ are piecewise-continuous time functions we sometimes write $f\rightarrow g$ if normed difference $|f(t)-g(t)|$ goes to zero as $t\rightarrow\infty$; if μ is a positive number and $|f(t)-g(t)|$ goes to zero as fast as $e^{-\mu t}$ we sometimes denote this by writing $f=g \bmod e^{-\mu t}$. Time functions such as f and g are *bounded* if they are bounded in the $\mathcal{L}^\infty[0,\infty)$ sense.

A square time-varying matrix A is *exponentially stable* if for some positive numbers a and μ, the state transition matrix of A satisfies $|\Phi(t,\tau)|\leq ae^{-\mu(t-\tau)}$, $\forall t\geq\tau\geq 0$. If, in addition, A depends on either a signal or parameter q in some given family $\mathcal{F}$, and a and μ are the same for all $q\in\mathcal{F}$, then A's exponential stability is *uniform* over $\mathcal{F}$.

Throughout this paper λ is a fixed positive number {later to be used in the definitions of performance signals} and $\mathbb{L}$ denotes the linear space of all real, rational, proper transfer functions in s, whose poles all lies to the left of the vertical line $s=-\lambda$ in the complex plane. For $\alpha\in\mathbb{L}$, $\left\|\alpha\right\|$ denotes the norm

$$\left\|\alpha\right\| = \sup_{\omega\in\mathbb{R}} |\alpha(j\omega-\lambda)|$$

For any stable, proper, transfer function β and any piecewise-continuous signal $u:[0,\infty)\rightarrow\mathbb{R}$, $\widehat{\beta}\circ u$ denotes the convolution product

$$\widehat{\beta}\circ u \triangleq \int_0^t \widehat{\beta}(t-\tau)u(\tau)d\tau$$

where $\widehat{\beta}$ is the inverse Laplace transform of β.

II. Background

The objective of this paper is to continue with the analysis of the estimator-based supervisory control system proposed in [1]. The process to be controlled is taken to be a linear system Σ_P with control input u and controlled output y_c; Σ_P's transfer function from u to y_c is presumed to be a member of a known class of admissible transfer functions of the form

$$\mathcal{C}_P = \bigcup_{p\in\mathcal{P}} \mathcal{C}(p)$$

where $\mathcal{P}$ is either a finite set of indices or a closed, bounded subset of a real, finite-dimensional, normed linear space. Here $\mathcal{C}(p)$ denotes the subclass

$$\mathcal{C}(p) = \{\nu_p(1+\delta^m)+\delta^a : (\delta^m,\delta^a)\in\mathbb{U}\}$$

where

$$\nu_p \triangleq \frac{\alpha_p}{\beta_p}$$

is a prespecified, strictly proper, *nominal transfer function*, $\mathbb{U}$ is a specified, bounded subset of $\mathbb{L}\oplus\mathbb{L}$, and δ^m and δ^a are transfer functions representing unmodelled dynamics of the multiplicative and additive types. It is assumed for each $p\in\mathcal{P}$, that β_p is monic and that α_p and β_p are coprime. Prompted by the requirements of set-point control, it is further assumed that the numerator of each transfer function in $\mathcal{C}_P$ is nonzero at $s=0$. For each $p\in\mathcal{P}$, $\frac{1}{s}\mathcal{C}(p)$ is required be at least small enough so that it can be robustly stabilized with a single, fixed-parameter, linear controller. Of course $\frac{1}{s}\mathcal{C}_P$ need not have this property. The specific model of the process to be controlled is depicted in Figure 1. Here y is the process's measured output, $\mathbf{n}$ is bounded measurement noise, $\mathbf{d}$ is a bounded disturbance, η_{p^*} is a polynomial of degree less than that of β_{p^*} and p^* is a fixed but unknown element of $\mathcal{P}$.

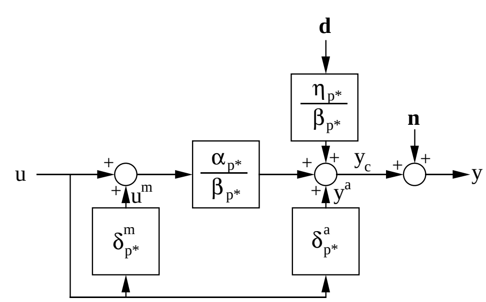

Fig. 1. Process Model

Presumed given is a family of candidate loop-controller transfer functions $\mathcal{K}\triangleq\{\kappa_p : p\in\mathcal{P}\}$ possessing at least the following property.

Stability Margin Property: For each $p\in\mathcal{P}$, the real parts of all of the closed-loop poles of the feedback interconnection shown in Figure 2 are less than $-\lambda$.

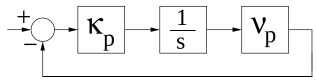

Fig. 2. Feedback Interconnection

In addition $\mathcal{K}$ must be defined so that the assignment $\nu_p \longmapsto \kappa_p$ is a well-defined function from the nominal process model transfer function class $\mathcal{N} \stackrel{\Delta}{=} \{\nu_p : p \in \mathcal{P}\}$ to $\mathcal{K}$. This is equivalent to the following.

Assumption 1: $\mathcal{K}$ and $\mathcal{N}$ have the property that $\kappa_p = \kappa_q$ whenever $p, q \in \mathcal{P}$ are such that $\nu_p = \nu_q$.

Also presumed given is a family of n_C-dimensional, stabilizable realizations of the form

$$\left\{ \begin{bmatrix} A_C & 0 \\ 0 & A_C \end{bmatrix} + \begin{bmatrix} b_C \\ 0 \end{bmatrix} f_p, \begin{bmatrix} g_p b_C \\ b_C \end{bmatrix}, f_p, g_p \right\}, \qquad (1)$$

one for each $\kappa_p \in \mathcal{K}$. Here $n_C \stackrel{\Delta}{=} 2n_\kappa$ where n_κ is an upper bound on the McMillan Degrees of the κ_p and (A_C, b_C) is a parameter-independent, n_κ-dimensional, siso, controllable pair with $\lambda I + A_C$ stable. All uncontrollable eigenvalue of each realization, are presumed to have real parts less than $-\lambda$. The reader is referred to [1] for an explanation of why realizations of this particular type are being considered.

Remark 1: In [1] it is explained how to construct such realizations so that $\kappa_p \longmapsto [\, f_p \quad g_p \,]$ is a bijective function from $\mathcal{K}$ to the space of such row matrices. This and Assumption 1 thus imply that $[\, f_p \quad g_p \,] = [\, f_q \quad g_q \,]$ whenever $\nu_p = \nu_q$. ♠

The sub-system to be supervised is of the form

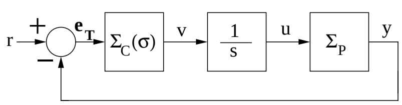

Fig. 3. Supervised Sub-System

where $\Sigma_C(\sigma)$ is the n_C-dimensional "state-shared" dynamical system

$$\dot{x}_C = \begin{bmatrix} A_C & 0 \\ 0 & A_C \end{bmatrix} x_C + \begin{bmatrix} b_C \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ b_C \end{bmatrix} \mathbf{e_T} \qquad (2)$$

$$v = f_\sigma x_C + g_\sigma \mathbf{e_T}, \qquad (3)$$

called a *multi-controller*, v is the input to the *integrator*

$$\dot{u} = v, \qquad (4)$$

$\mathbf{e_T}$ is the *tracking error*

$$\mathbf{e_T} \stackrel{\Delta}{=} r - y, \qquad (5)$$

and σ is a piecewise constant *switching signal* taking values in $\mathcal{P}$. The job of a supervisor is to generate σ so as to achieve 1. *global boundedness* {of all system signals} in the face of arbitrary but bounded noise and disturbance inputs, and 2. *set-point regulation* {i.e., $\mathbf{e_T} \rightarrow 0$} in the event that the disturbance signal is constant and the noise signal is zero. The supervisor considered in [1] which accomplishes this is an estimator-based algorithm of the hybrid type whose output is σ and whose inputs are v and y. Internally the supervisor consists of three subsystems: a *multi-estimator dynamic* Σ_E, a *performance-weight generator* Σ_W, and a *dwell-time switching logic* Σ_D. Σ_E is a n_E-dimensional linear dynamical system of the form

$$\dot{x}_E = \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} x_E + \begin{bmatrix} b_E \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b_E \end{bmatrix} v \qquad (6)$$

where $n_E \stackrel{\Delta}{=} 2(n_\nu + 1)$ and (A_E, b_E) is a parameter-independent, n_ν+1-dimensional siso, controllable pair with $\lambda I + A_E$ stable. Here n_ν is an upper bound on the McMillan Degrees of the ν_p, $p \in \mathcal{P}$. In [1] it is explained how to construct a function $p \longmapsto c_p$ so that for each $p \in \mathcal{P}$,

$$\left\{ \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} + \begin{bmatrix} b_E \\ 0 \end{bmatrix} c_p, \begin{bmatrix} 0 \\ b_E \end{bmatrix}, c_p \right\}$$

is a stabilizable realization of $\frac{1}{s}\nu_p$ whose uncontrollable eigenvalues have real parts less than $-\lambda$. The c_p are used in the definition of Σ_W which will be given in a moment. The c_p also enable us to define *output estimation errors*

$$e_p \stackrel{\Delta}{=} c_p x_E - y, \quad p \in \mathcal{P} \qquad (7)$$

While these error signals are not actually generated by the supervisor, they play an important role in explaining how the supervisor functions.

In the sequel we will make use of the following technical assumption.

Assumption 2: The functions $p \longmapsto c_p$ and $p \longmapsto [\, f_p \quad g_p \,]$ are continuous on $\mathcal{P}$.

This of course means that both the ν_p and the κ_p must vary continuously as p ranges over any connected component of $\mathcal{P}$.

The supervisor's second subsystem, Σ_W, is a causal dynamical system whose inputs are x_E and y and whose state and output W is a "weighting matrix" which takes values in a linear space $\mathcal{W}$. W together with a suitably defined *performance function* $\Pi : \mathcal{W} \times \mathcal{P} \rightarrow \mathbb{R}$ determine a scalar-valued *performance signal* of the form

$$\pi_p = \Pi(W, p) \qquad (8)$$

which is viewed by the supervisor as a measure of the expected performance of controller p. Σ_W and Π are defined by

$$\dot{W} = -2\lambda W + \begin{bmatrix} x_E \\ y \end{bmatrix} \begin{bmatrix} x_E \\ y \end{bmatrix}' \qquad (9)$$

and

$$\Pi(W, p) = [\, c_p \quad -1 \,] \, W \, [\, c_p \quad -1 \,]' \qquad (10)$$

respectively. The definitions of Σ_W and Π are prompted by the observation that if π_p are given by (8), then

$$\dot{\pi}_p = -2\lambda \pi_p + e_p^2, \quad p \in \mathcal{P} \qquad (11)$$

because of (7), (9) and (10).

The supervisor's third subsystem, called a *dwell-time switching logic* Σ_D, is a hybrid dynamical system whose input and output are W and σ respectively, and whose state is the ordered triple $\{X, \tau, \sigma\}$. Here X is a discrete-time matrix which takes on sampled values of W, and τ is a continuous-time variable called a *timing signal.* τ takes values in the closed interval $[0, \tau_D]$, where τ_D is a prespecified positive number called a *dwell time.* Also assumed prespecified is a *computation time* $\tau_C \leq \tau_D$ which bounds from above for any $X \in \mathcal{W}$, the time it would take a supervisor to compute a value $p = p_X \in \mathcal{P}$ which minimizes $\Pi(X, p)$. Between "event times," τ is generated by a reset integrator according to the rule $\dot{\tau} = 1$. Event times occur when the value of τ reaches either $\tau_D - \tau_C$ or τ_D; at such times τ is reset to either 0 or $\tau_D - \tau_C$ depending on the value of Σ_D's state. Σ_D's internal logic is defined by the computer diagram shown in Figure 4 where p_X denotes a value of $p \in \mathcal{P}$ which minimizes $\Pi(X, p)$.

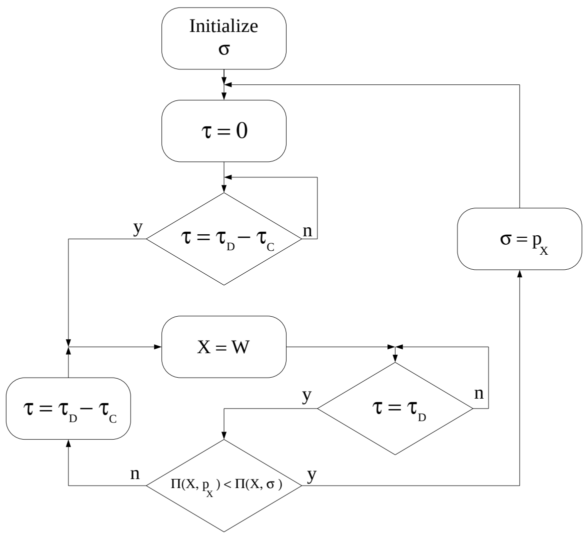

Fig. 4. Computer Diagram of Σ_D

In the sequel we call a piecewise-constant signal $\bar{\sigma} : [0, \infty) \to \mathcal{P}$ *admissible* if it either switches values at most once, or if it switches more than once and the set of time differences between each two successive switching times is bounded below by τ_D. We write $\mathbb{S}$ for the set of all admissible switching signals. Because of the definition of Σ_D, it is clear its output σ will be admissible. This means that switching cannot occur infinitely fast and thus that existence and uniqueness of solutions to the differential equations involved is not an issue. .

III. Main Result

The overall supervisory control system just defined, henceforth denoted by Σ, admits a block diagram description of the form

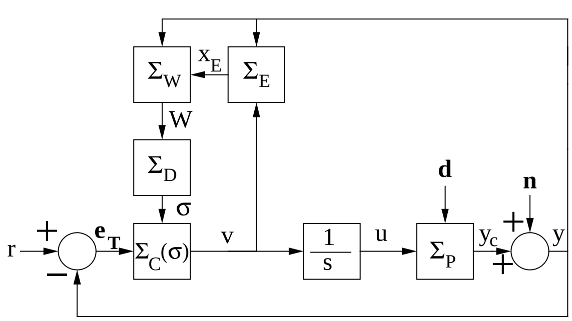

Fig. 5. Supervisory Control System Σ

Thus Σ consists of the Σ_P, the multi-controller Σ_C defined by (2) and (3), the integrator (4), the tracking error $\mathbf{e_T}$ defined by (5), the multi-estimator dynamic Σ_E defined by (6), the performance weight generator Σ_W defined by (9), the performance function Π given by (10), and the dwell-time switching logic Σ_D.

Theorem 1: Let ω_E be the characteristic polynomial of A_E. Let $\tau_C \geq 0$ be fixed. Let τ_D be any positive number no smaller than τ_C. There are positive numbers ϵ_l, $l \in \mathcal{P}$, for which the following statements are true. Suppose that for some $p^* \in \mathcal{P}$, Σ_P is as depicted in Figure 1 with transfer function of the form

$$\frac{\alpha_{p^*}}{\beta_{p^*}}(1 + \delta^m_{p^*}) + \delta^a_{p^*} \qquad (12)$$

where $\delta^a_{p^*}$ and $\delta^m_{p^*}$ are transfer functions in $\mathbb{L}$ satisfying

$$\left\| \frac{\beta_{p^*}}{\omega_E} \right\| \left\| \delta^a_{p^*} \right\| + \left\| \frac{\alpha_{p^*}}{\omega_E} \right\| \left\| \delta^m_{p^*} \right\| \leq \epsilon_{p^*} \qquad (13)$$

Then the following statements are true.
1. **Global Boundedness:** For each constant set-point value r, each pair of piecewise-continuous and bounded disturbance and noise inputs $\mathbf{d}$ and $\mathbf{n}$ respectively, and each initialization of Σ, y_c, u, x_C, x_E, W, and X are bounded responses.
2. **Tracking and Disturbance Rejection:** Suppose $\mathbf{n} = 0$. For each constant set-point value r, each constant disturbance input $\mathbf{d}$ and each initialization of Σ,

$$y_c \to r$$

and u, x_C, x_E, W, and X tend to finite limits, all as fast as fast as $e^{-\lambda t}$.

The implications of Theorem 1 are clear: The supervisory control system under consideration performs its function {i.e., controls Σ_P's set-point} is the face of large nominal modeling errors, norm-bounded unmodelled dynamics and constant disturbance. Moreover none of the signals within the overall system, namely y_c, u, x_C, x_E, W, and X, can grow without bound in response to bounded disturbance and noise inputs, be they constant or not. Apart from its ability to handle nominal modeling errors, the supervisory control system's capabilities are the same as those of a standard, non-self-adjusting linear set-point control system. A bound for ϵ_{p^*} which assures that the conclusions of Theorem 1 hold, is given in §VII-A.

IV. Basic Concepts

The aim of this section is to outline basic concepts and key equations which are central to proof of Theorem 1. We begin by defining in §IV-A, a composite subsystem composed of Σ_E and Σ_C. This system has an important property: it is detectable through output estimation error e_p if controller p is in the feedback-loop { §IV-B}. In §IV-C we define an exponentially weighted 2-norm and discuss its relationship to the performance signals defined earlier. In §IV-D we summarize several key equalities and inequalities upon which the analysis of the overall system depends.

A. Composite Subsystem

In the sequel we will analyze the closed-loop behavior of the system described previously assuming that the process model transfer function is in $\mathcal{C}_P$ and that r is an arbitrary but constant input. In carrying out the analysis it is useful to introduce the composite state

$$x = \begin{bmatrix} \bar{x}_E \\ x_C \end{bmatrix} \tag{14}$$

where $\bar{x}_E$ is the shifted state

$$\bar{x}_E \triangleq x_E + \begin{bmatrix} A_E^{-1} b_E \\ 0 \end{bmatrix} r \tag{15}$$

As noted in [1], these definitions imply that for all $l \in \mathcal{P}$,

$$\left.\begin{array}{rcl} \dot{x} & = & (A_l + bf_{\sigma l})x + (bg_\sigma + d)e_l \\ v & = & f_{\sigma l}x + g_\sigma e_l \\ e_p & = & c_{pl}x + e_l, \quad p \in \mathcal{P} \\ \mathbf{e_T} & = & e_l - [\,c_l \quad 0\,]\,x \end{array}\right\} \tag{16}$$

where, for each triple of points p, q, l in $\mathcal{P}$,

$$f_{ql} \triangleq [\,-g_q c_l \quad f_q\,], \qquad c_{pl} \triangleq [\,c_p - c_l \quad 0\,], \tag{17}$$

$$d \triangleq \begin{bmatrix} -b_E \\ 0 \\ 0 \\ b_C \end{bmatrix}, \quad A_l \triangleq \bar{A} - d\,[\,c_l \quad 0\,], \quad b \triangleq \begin{bmatrix} 0 \\ b_E \\ b_C \\ 0 \end{bmatrix},$$

and

$$\bar{A} \triangleq \begin{bmatrix} A_E & 0 & 0 & 0 \\ 0 & A_E & 0 & 0 \\ 0 & 0 & A_C & 0 \\ 0 & 0 & 0 & A_C \end{bmatrix}$$

Remark 2: Recall that for $p \in \mathcal{P}$, c_p has been defined so that

$$\left\{ \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} + \begin{bmatrix} b_E \\ 0 \end{bmatrix} c_p, \begin{bmatrix} 0 \\ b_E \end{bmatrix}, c_p \right\}$$

realizes $\frac{1}{s}\nu_p$. Using this and the above definitions, it is easy to verify that $\nu_p = \nu_l$ whenever p and l are such that $c_{pl}(sI - A_l)^{-1}b = 0$. ♠

B. Detectability

An important property of the matrices just defined is that for any fixed $p, l \in \mathcal{P}$, the matrix pair $(c_{pl}, \lambda I + A_l + bf_{pl})$ is detectable. This is a consequence of certainty equivalence, the Stability Margin Property, and the requirement that λ be smaller than the negative of the real part of each eigenvalue of A_E and A_C [1].

One implication of detectability, is that each $\lambda I + A_l + bf_{ll}$ must be a stability matrix. This is because $c_{ll} = 0$, $l \in \mathcal{P}$. Even more is true. Suppose q and l are such that $c_{ql}(sI - A_l)^{-1}b = 0$. Then $\nu_q = \nu_l$ because of Remark 2. Consequently $[\,f_q \quad g_q\,] = [\,f_l \quad g_l\,]$ because of Remark 1. Therefore $f_{ql} = f_{pl}$ because of the definition of f_{ql} in (17). In other words, if $c_{ql}(sI - A_l)^{-1}b = 0$, then $f_{ql} = f_{ll}$ and so $\lambda I + A_l + bf_{ql}$ and $\lambda I + A_l + bf_{ll}$ must be one and the same. Since $\lambda I + A_l + bf_{ll}$ is a stability matrix, it follows that, for such q and l, $\lambda I + A_l + bf_{ql}$ must be a stability matrix as well. On the other hand, for values of q and l such that $c_{ql}(sI - A_l)^{-1}b \neq 0$, one can state the following.

Lemma 1 (**Dwell-Time Switching**) Let l and q be elements of $\mathcal{P}$ such that $c_{ql}(sI - A_l)^{-1}b \neq 0$. There exist bounded, vector-valued functions $p \longmapsto \bar{k}_p$, $p \longmapsto \bar{h}_p$ on $\mathcal{P}$ which, for any admissible switching signal $\bar{\sigma} : [0, \infty) \to \mathcal{P}$, exponentially stabilizes the time-varying matrix

$$\lambda I + A_l + bf_{\bar{\sigma} l} + k_{\bar{\sigma}} c_{\bar{\sigma} l} + \bar{h}_{\bar{\sigma}} c_{ql}$$

Moreover, this is true uniformly over the class of all such $\bar{\sigma}$.

A more general version of this result is proved in [1]. The lemma will be used in the proof of Proposition 2 in §VI.

C. Norms and System Gains

It is especially useful to introduce the following. For any piecewise-continuous function $z : [0, \infty) \to \mathrm{IR}^n$, let $||z||_{\{t_1,t_2\}}$ denote the exponentially weighted 2-norm

$$||z||_{\{t_1,t_2\}} \triangleq \sqrt{\int_{t_1}^{t_2} e^{2\lambda t}|z(t)|^2 dt}$$

whenever $t_2 > t_1$ and the number zero otherwise. The utility of this norm stems from the identity

$$\boxed{e^{2\lambda t}\pi_p(t) = ||e_p||^2_{\{0,t\}} + \pi_p(0), \quad p \in \mathcal{P}} \tag{18}$$

which, in turn, is a direct consequence of (11).

We state without proof several easily derived facts. If z is bounded on $[0, \infty)$ {in the $\mathcal{L}^\infty$ sense}, then so is $e^{-\lambda t}||z||_{\{0,t\}}$. If $z \to 0$ as $t \to \infty$, then so does $e^{-\lambda t}||z||_{\{0,t\}}$.

To proceed we need the following notation. Let

$$\dot{x} = A(t)x + B(t)u \qquad y = C(t)x + D(t)u$$

be a linear system whose coefficient matrices A, B, C, and D are piecewise continuous and bounded on $[0, \infty)$. We denote by

$$\left\{\!\!\left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\}\!\!\right\}$$

the input-output system operator $u \longmapsto y_0$, where y_0 is the zero initial state, output response signal

$$t \longmapsto \int_0^t C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

on $[0, \infty)$, and $\Phi(t, \tau)$ is the state transition matrix of A . In the sequel we invariably write

$$\left\{\!\!\left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\}\!\!\right\} \circ u$$

for $y_0(t)$ and $[\![A]\!]_\tau$ for $\Phi(t, \tau)$.

The *gain* of the above system, is the induced 2-norm

$$\left\| \left\{\!\!\left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\}\!\!\right\} \right\| \triangleq \sup_{\mathcal{U}} \left\| \left\{\!\!\left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\}\!\!\right\} \circ u \right\|_{\{0,\infty\}}$$

where $\mathcal{U} \triangleq \{u : ||u||_{\{0,\infty\}} \leq 1\}$. For strictly causal systems, we will also make use of the "∞-gain"

$$\left| \left\{\!\!\left\{ \begin{array}{cc} \lambda I + A & B \\ C & 0 \end{array} \right\}\!\!\right\} \right| \triangleq \sup_{\mathcal{U}} \sup_{t \geq 0} \left| \left\{\!\!\left\{ \begin{array}{cc} \lambda I + A & B \\ C & 0 \end{array} \right\}\!\!\right\} \circ \{e^{\lambda t}u\} \right|$$

It is easy to prove that both of these gains are finite if $\lambda I + A$ is exponentially stable. It is also easily proved that if A, B, C, and D depend on some function or parameter q in some family $\mathcal{F}$, if A's exponential stability is uniform with respect to q, and if there are finite $\mathcal{L}^\infty$-bounds on B, C and D which are the same for all $q \in \mathcal{F}$, then the two gains just defined are uniformly bounded on $\mathcal{F}$.

D. Key Equations

So far we've assumed that the process model transfer function is in $\mathcal{C}_P$. On the other hand, we've not yet made use of the assumption that this transfer function is specifically as shown in (12) and consequently that the process is as depicted in Figure 1. It is shown in [1] that what this assumption implies is that

$$\boxed{e_{p^*} = \widehat{\delta}_{p^*} \circ v + \mathbf{b}} \tag{19}$$

where

$$\mathbf{b} = -\frac{\widehat{s\beta_{p^*}}}{\omega_E} \circ \mathbf{n} - \frac{\widehat{s\eta_{p^*}}}{\omega_E} \circ \mathbf{d} \bmod e^{-\lambda t}, \tag{20}$$

$$\delta_{p^*} = -\left(\frac{\beta_{p^*}}{\omega_E}\delta^a_{p^*} + \frac{\alpha_{p^*}}{\omega_E}\delta^m_{p^*}\right), \qquad (21)$$

and ω_E is the characteristic polynomial of A_E.

Remark 3: Note that the transfer functions appearing in (20) and (21) are all proper and δ_{p^*} is strictly proper. This is because the degrees of β_{p^*} and α_{p^*} are both less than $\nu_{p^*}+1$ which in turn is the degree of ω_E. Observe that the transfer functions in (20) and (21) are also all stable. This is a consequence of the definition of ω_E and the assumed stability of $\delta^a_{p^*}$ and $\delta^m_{p^*}$. It thus can be concluded that $\mathbf{b}$ will be bounded whenever $\mathbf{n}$ and $\mathbf{d}$ are. Notice also that because of the presence of the zero at zero in the numerators of the two transfer functions appearing in (20), $\mathbf{b}$ will tend to zero whenever $\mathbf{n}$ and $\mathbf{d}$ equal or tend to constants. In fact, if $\mathbf{n}$ and $\mathbf{d}$ are constant, the convergence of $\mathbf{b}$ to zero will be as fast as $e^{-\lambda t}$ because λ is {by design} smaller than the negative of the real part of each zero of ω_E. ♠

The expression for e_{p^*} in (19) implies that

$$||e_{p^*}||_{\{t_0,t\}} \leq ||\widehat{\delta}_{p^*} \circ v||_{\{t_0,t\}} + ||\mathbf{b}||_{\{t_0,t\}}, \qquad t \geq t_0 \geq 0$$

Since for such t_0 and t, $||\widehat{\delta}_{p^*} \circ v||_{\{t_0,t\}} \leq ||\widehat{\delta}_{p^*} \circ v||_{\{0,t\}}$, it follows that

$$||e_{p^*}||_{\{t_0,t\}} \leq ||\widehat{\delta}_{p^*} \circ v||_{\{0,t\}} + ||\mathbf{b}||_{\{t_0,t\}}, \qquad t \geq t_0 \geq 0 \quad (22)$$

This inequality can be simplified. For this first recall that δ_{p^*} is a proper rational function whose poles all lies to the left of the vertical line $s = -\lambda$ in the complex plane. It is well know that for any such rational function α, and any piecewise-continuous signal $u : [0,\infty) \to \mathbb{R}$,

$$||\widehat{\alpha} \circ u||_{\{0,t\}} \leq \Big|\alpha\Big| ||u||_{\{0,t\}}$$

where $\widehat{\alpha}$ is the inverse Laplace transform of α. Applying this to (22) thus yields

$$||e_{p^*}||_{\{t_0,t\}} \leq \Big|\delta_{p^*}\Big| ||v||_{\{0,t\}} + ||\mathbf{b}||_{\{t_0,t\}}, \qquad t \geq t_0 \geq 0$$

Note next that $\Big|\delta_{p^*}\Big| \leq \Big|\frac{\beta_{p^*}}{\omega_E}\Big|\Big|\delta^a_{p^*}\Big| + \Big|\frac{\alpha_{p^*}}{\omega_E}\Big|\Big|\delta^m_{p^*}\Big|$ because of (21) and the fact that $\Big|\cdot\Big|$ is submultiplicative. But the hypothesis of Theorem 1 stipulates that $\Big|\frac{\beta_{p^*}}{\omega_E}\Big|\Big|\delta^a_{p^*}\Big| + \Big|\frac{\alpha_{p^*}}{\omega_E}\Big|\Big|\delta^m_{p^*}\Big| \leq \epsilon_{p^*}$. What this clearly implies is that $\Big|\delta_{p^*}\Big| \leq \epsilon_{p^*}$ and thus that

$$||e_{p^*}||_{\{t_0,t\}} \leq \epsilon_{p^*}||v||_{\{0,t\}} + ||\mathbf{b}||_{\{t_0,t\}}, \qquad t \geq t_0 \geq 0$$

Therefore, for $t \geq t_0 \geq 0$,

$$\boxed{||e_{p^*}||_{\{t_0,t\}} \leq \epsilon_{p^*}\left\{||v||_{\{0,t_0\}} + ||v||_{\{t_0,t\}}\right\} + ||\mathbf{b}||_{\{t_0,t\}}} \qquad (23)$$

Finally, as we've already noted, the equations in (16) hold for all $l \in \mathcal{P}$. In the sequel, we will need these equations evaluated at $l \triangleq p^*$:

$$\begin{array}{rcl} \dot{x} &=& (A_{p^*} + bf_{\sigma p^*})x + (bg_\sigma + d)e_{p^*} \\ v &=& f_{\sigma p^*}x + g_\sigma e_{p^*} \\ e_p &=& c_{pp^*}x + e_{p^*}, \quad p \in \mathcal{P} \\ \mathbf{e_T} &=& e_{p^*} - [\, c_{p^*} \quad 0\,]\, x \end{array} \qquad (24)$$

V. Perfect Supervision

Let us consider for the moment the very special case when for some reason σ happens to take the value p^* for all time. In other words, what we want to briefly discuss is the hypothetical situation in which the supervisor has somehow figured out what the "correct" control index ought to be and set σ accordingly. If this were so, then $A_{p^*} + bf_{\sigma p^*}$ would be the constant matrix $A_{p^*} + b_{p^*p^*}$. But as we've already noted in §IV-B, $\lambda I + A_{p^*} + bf_{p^*p^*}$ is a stability matrix so $A_{p^*} + bf_{\sigma p^*}$ would have to be an exponentially stable. Starting with this, and then using (23) and (24), one could then easily derive a bound for ϵ_{p^*} which would insure x's boundedness and, if $\mathbf{n}$ and $\mathbf{d}$ were constant, even x's exponential convergence to zero. Our aim in this section is to carry out this analysis not for the case when σ takes on the single value p^* but rather for the case when σ's values for all time are such that $f_{\sigma p^*}$ is close enough to $f_{p^*p^*}$ so that $A_{p^*} + bf_{\sigma p^*}$ is an exponentially stable time varying matrix. In other words, what we intend to do next, is to focus on the special case when the supervisor has somehow figured out how to choose control indices in $\mathcal{P}$ corresponding controllers which are close to the correct one. We refer to this hypothetical situation as *perfect supervision.*

For each $l \in \mathcal{P}$, let us pick a number

$$\boxed{\mathfrak{y}_l \in (0,1)}$$

and then define

$$\boxed{\mathcal{Q}_l \triangleq \left\{p : |f_{pl} - f_{ll}| \left\| \left\{\!\!\left\{ \begin{array}{cc} A_l + bf_{ll} & b \\ I & 0 \end{array} \right\}\!\!\right\} \right\| < \mathfrak{y}_l,\ p \in \mathcal{P}\right\}} \qquad (25)$$

Since $l \in \mathcal{Q}_l$, each such subclass is nonempty. In the sequel we will say that the supervisor is performing perfectly whenever σ's value is in $\mathcal{Q}_{p^*}$. We can now state the following.

Lemma 2: The matrix $\lambda I + A_l + bf_{\bar{\sigma}l}$ is exponentially stable for every piecewise-constant signal $\bar{\sigma} : [0,\infty) \to \mathcal{Q}_l$. Moreover this is true uniformly over the class of all such $\bar{\sigma}$.

This standard perturbational result follows directly from the stability of $\lambda I + A_l + bf_{ll}$ and the smallness of $|f_{pl} - f_{ll}|$ for $p \in \mathcal{Q}_l$. A proof will not be given.

A. System Gain

For each $l \in \mathcal{P}$, define the *perfect supervision system gain*

$$\boxed{\mathfrak{g}_l \triangleq \sup_{\bar{\sigma}} \left\| \left\{\!\!\left\{ \begin{array}{cc} A_l + bf_{\bar{\sigma}l} & bg_{\bar{\sigma}} + d \\ f_{\bar{\sigma}l} & g_{\bar{\sigma}} \end{array} \right\}\!\!\right\} \right\|} \qquad (26)$$

and the constants

$$\begin{array}{rcl} \mathfrak{a}_l &\triangleq& \sup_{\bar{\sigma}} \left| \left\{\!\!\left\{ \begin{array}{cc} \lambda I + A_l + bf_{\bar{\sigma}l} & bg_{\bar{\sigma}} + d \\ I & 0 \end{array} \right\}\!\!\right\} \right| \\ \mathfrak{b}_l &\triangleq& \sup_{t_0 \geq 0} \sup_{\bar{\sigma}} ||f_{\bar{\sigma}l} [\![A + bf_{\bar{\sigma}l}]\!]_{t_0} ||_{\{t_0,\infty\}} \\ \mathfrak{c}_l &\triangleq& \sup_{t_0 \geq 0} \sup_{\bar{\sigma}} \sup_{t \geq t_0} | [\![\lambda I + A_l + bf_{\bar{\sigma}l}]\!]_{t_0} | \end{array}$$

where for each definition, the supremum with respect to $\bar{\sigma}$ is to be taken over the class of all piecewise-constant switching signals $\bar{\sigma}$ taking values only in $\mathcal{Q}_l$. In view of Lemma 2, $\mathfrak{g}_l$, $\mathfrak{a}_l$, $\mathfrak{b}_l$ and $\mathfrak{c}_l$ are finite numbers.

B. Behavior of x and v

The following proposition characterizes the closed-loop behavior of x and v for those time intervals, if any, on which there is perfect supervision.

Proposition 1: Suppose that
$$\epsilon_{p^*} < \frac{1}{\mathfrak{g}_{p^*}} \quad (27)$$
If $[t_a, t_b)$ is an interval on which σ taking values only in $\mathcal{Q}_{p^*}$, then for all $t \in [t_a, t_b)$
$$|x(t)| \leq e^{-\lambda t}\Big\{e^{\lambda t_a}\mathfrak{c}_{p^*}|x(t_a)| + \frac{\mathfrak{a}_{p^*}}{(1-\epsilon_{p^*}\mathfrak{g}_{p^*})}\Big(||\mathbf{b}||_{\{t_a,t\}} + \epsilon_{p^*}\mathfrak{b}_{p^*}|x(t_a)| + \epsilon_{p^*}||v||_{\{0,t_a\}}\Big)\Big\} \quad (28)$$
and
$$||v||_{\{0,t\}} \leq \frac{1}{(1-\epsilon_{p^*}\mathfrak{g}_{p^*})}\Big\{||v||_{\{0,t_a\}} + \mathfrak{b}_{p^*}|x(t_a)| + \mathfrak{g}_{p^*}||\mathbf{b}||_{\{t_a,t\}}\Big\} \quad (29)$$
This result does not depend on dwell-time switching and can be derived in a straight forward manner using well known results. To maintain continuity, we defer the proposition's proof to the appendix.

Let us note that along any system trajectory, σ can exhibit one of two possible types of behavior. Either there must be a finite time T beyond which σ takes values only in $\mathcal{Q}_{p^*}$, or no such time must exist. If the former is true, we say the system is *ultimately perfectly supervised*. Otherwise the system is said to be *imperfectly supervised*. It should be noted that imperfect supervision is more likely to be the rule than the exception, because of noise and disturbance signals. In other words, if $\mathbf{n}$ and/or $\mathbf{d}$ are not constant, it is unlikely that σ will eventually enter and remain in $\mathcal{Q}_{p^*}$ and even more unlikely that it will stop switching.

For an ultimately perfectly supervised system, the asymptotic behavior of x is completely characterized by Proposition 1. In particular, by applying the proposition to the interval $[T, \infty)$ one can readily deduce that x must be bounded on $[0, \infty)$ and moreover that x must tend to zero as $t \to \infty$ if $\mathbf{n}$ and $\mathbf{d}$ are constant {ie, if $e^{-\lambda t}||\mathbf{b}||_{0,t} \to 0$ as $t \to \infty$}. Proving that x must have these same properties, even if supervision is imperfect, is much more challenging.

VI. Imperfect Supervision

Our aim here is to characterize the behavior of x and v in the face of imperfect supervision. We will derive a technical result {Proposition 2} similar to Proposition 1 which norm bounds x anr v for those values of t, if any, at which $\sigma \notin \mathcal{Q}_{p^*}$.

To begin, let us note that the complement of $\mathcal{Q}_l$ in $\mathcal{P}$, written $\bar{\mathcal{Q}}_l$, is the subset
$$\boxed{\bar{\mathcal{Q}}_l \triangleq \left\{p : |f_{pl} - f_{ll}| \left\| \begin{Bmatrix} A_l + bf_{ll} & b \\ I & 0 \end{Bmatrix} \right\| \geq \mathfrak{y}_l,\ p \in \mathcal{P}\right\}} \quad (30)$$
Observe that $\bar{\mathcal{Q}}_{p^*}$ is a compact subset {finite subset if $\mathcal{P}$ is finite} because of Assumption 2 and the compactness of $\mathcal{P}$. Recall that $\mathbb{S}$ denotes the set of all admissible switching signals. We can now state the following.

Lemma 3: Fix $l \in \mathcal{P}$ and suppose that $\bar{\mathcal{Q}}_l$ is nonempty. There exist bounded functions $(p,q) \longmapsto k_{pql}$ and $(p,q) \longmapsto h_{pql}$ mapping $\mathcal{P} \times \bar{\mathcal{Q}}_l$ into $\mathbb{R}^{(n_E+n_C)}$ which exponentially stabilize the time varying matrix
$$\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}c_{\bar{\sigma}l} + h_{\bar{\sigma}ql}c_{ql}$$
for every $q \in \bar{\mathcal{Q}}_l$ and every admissible switching signal $\bar{\sigma}$. Moreover, this is true uniformly over $\bar{\mathcal{Q}}_l \times \mathbb{S}$.

Proof of Lemma 3: In view of Lemma 1 and the compactness of $\bar{\mathcal{Q}}_l$, it is enough to show that $c_{ql}(sI - A_l)^{-1}b \neq 0$ for all $q \in \bar{\mathcal{Q}}_l$. Now the definition $\bar{\mathcal{Q}}_l$ in (30), implies that $f_{ql} \neq f_{ll}$ for all $q \in \bar{\mathcal{Q}}_l$. Hence for all such q, $[f_q \quad g_q] \neq [f_l \quad g_l]$ because of (17). From this and Remark 1 it follows that $\nu_q \neq \nu_l$ for all $q \in \bar{\mathcal{Q}}_l$. Thus by Remark 2, $c_{ql}(sI - A_l)^{-1}b \neq 0$ for all $q \in \bar{\mathcal{Q}}_l$. ■

A. System Gains

The purpose of this subsection is to define an upper bound for ϵ_{p^*} which, if satisfied, guarantees bounds for x and v similar to those given in Proposition 1 but which hold whenever $\sigma(t) \in \bar{\mathcal{Q}}_{p^*}$. This will be done in several steps. In the sequel, we will assume that l is such that $\bar{\mathcal{Q}}_l$ is nonempty and that k_{pql} and h_{pql} are fixed functions which have been defined in accordance with Lemma 3.

To proceed, let us note because of Lemma 3, that
$$\sup_{q\in\bar{\mathcal{Q}}_l}\sup_{\bar{\sigma}\in\mathbb{S}} \left\| \begin{Bmatrix} A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql} + h_{\bar{\sigma}ql}c_{ql} & k_{\bar{\sigma}ql} \\ I & 0 \end{Bmatrix} \right\| < \infty$$
Moreover, $\inf_{q\in\bar{\mathcal{Q}}_l}|c_{ql}| > 0$; this is because $c_{ql}(sI - A_l)^{-1}b \neq 0$, $q \in \bar{\mathcal{Q}}_l$, as was just noted in the proof of Lemma 3. It it is thus possible to define a positive number $\mathfrak{d}_l$ which is sufficiently small so that
$$\boxed{\mathfrak{d}_l \leq \inf_{q\in\bar{\mathcal{Q}}_l}|c_{ql}|} \quad (31)$$
and
$$\boxed{\frac{1}{\mathfrak{d}_l} > \sup_{q\in\bar{\mathcal{Q}}_l}\sup_{\bar{\sigma}\in\mathbb{S}} \left\| \begin{Bmatrix} A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql} + h_{\bar{\sigma}ql}c_{ql} & k_{\bar{\sigma}ql} \\ I & 0 \end{Bmatrix} \right\|} \quad (32)$$
With $\mathfrak{d}_l$ so defined, we can then state the following.

Lemma 4: Let μ_1 and μ_2 be positive numbers. Under the conditions of Lemma 3, the matrix
$$\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}(c_{\bar{\sigma}l} - \bar{c} - \tilde{c}) + h_{\bar{\sigma}ql}(1-\psi)c_{ql}$$
is exponentially stable for every $q \in \bar{\mathcal{Q}}_l$, every $\bar{\sigma} \in \mathbb{S}$, and every triple of piecewise-continuous functions $\bar{c} : [0,\infty) \to \mathbb{R}^{1\times(n_E+n_C)}$, $\tilde{c} : [0,\infty) \to \mathbb{R}^{1\times(n_E+n_C)}$, and $\psi : [0,\infty) \to \mathbb{R}$ satisfying
$$\sup_{t\geq 0}|\bar{c}(t)| \leq \mathfrak{d}_l, \quad \int_0^\infty |\tilde{c}(s)|ds \leq \mu_1, \quad \text{and} \quad \int_0^\infty |\psi(s)|ds \leq \mu_2$$
respectively. Moreover this is true uniformly over the Cartesian product of $\bar{\mathcal{Q}}_l \times \mathbb{S}$ with the set of all such triples.

This lemma can be proved in two steps. First it can be shown that
$$[\{\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}c_{\bar{\sigma}l} + h_{\bar{\sigma}ql}c_{ql}\} - k_{\bar{\sigma}ql}\bar{c}]$$
must be uniformly exponentially stable because of (i) the uniform exponential stability of
$$\{\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}c_{\bar{\sigma}l} + h_{\bar{\sigma}ql}c_{ql}\}$$
noted in Lemma 3, (ii) the smallness of $|\bar{c}(t)|$ and (iii) the smallness of $\mathfrak{d}_l$ required by the inequality in (32). The uniform exponential stability of
$$[\{\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}(c_{\bar{\sigma}l} + h_{\bar{\sigma}ql}\} - k_{\bar{\sigma}ql}\bar{c}] - \{k_{\bar{\sigma}ql}\tilde{c} + h_{\bar{\sigma}ql}\psi c_{ql}\}$$
can then be deduced directly from this and the uniform $\mathcal{L}^1[0,\infty)$ boundedness of all such $\tilde{c}$ and ψ. Since these

claims can be easily justified using well-known perturbational results, a proof of the lemma will not be given.

What we intend to do next is to derive a result similar to Proposition 1 for those times, if any, at which σ takes values in $\bar{\mathcal{Q}}_{p^*}$. For this we need the following. Set

$$\mathfrak{v}_l \triangleq n_E\left(1+\frac{\sup_{p\in\mathcal{P}}|c_{pl}|}{\mathfrak{d}_l}\right)^{n_E} \tag{33}$$

Next define $\mathbb{F}_l$ to be the family of all quadruples of the form $\{\bar{c},\tilde{c},\psi,\bar{g}\}$ where $\bar{c}:[0,\infty)\rightarrow \mathbb{R}^{1\times(n_C+n_E)}$, $\tilde{c}:[0,\infty)\rightarrow \mathbb{R}^{1\times(n_E+n_C)}$, $\psi:[0,\infty)\rightarrow[0,1]$ and $\bar{g}:[0,\infty)\rightarrow\mathbb{R}$ are piecewise-continuous functions satisfying

$$\sup_{t\geq 0}|\bar{c}(t)| \leq \mathfrak{d}_l \tag{34}$$

$$\int_0^\infty |\tilde{c}(s)|ds \leq (n_E)(\tau_D+\tau_C)\mathfrak{v}_l(\sup_{p\in\mathcal{P}}|c_{pl}|) \tag{35}$$

$$\int_0^\infty |\psi(s)|ds \leq \tau_D+\tau_C \tag{36}$$

$$\sup_{t\geq 0}|\bar{g}(t)| \leq \mathfrak{v}_l \tag{37}$$

Note that any functions $\bar{c}$, $\tilde{c}$ and ψ satisfying (34)-(36), also satisfy the conditions of Lemma 4. Thus $\lambda I + A_l + bf_{\bar{\sigma}l} + k_{\bar{\sigma}ql}(c_{\bar{\sigma}l}-\bar{c}-\tilde{c}) + h_{\bar{\sigma}ql}(1-\psi)c_{ql}$ must be exponentially stable, uniformly over $\bar{\mathcal{Q}}_l\times\mathbb{S}\times\mathbb{F}_l$. Note in addition, that ψ is bounded because its codomain is bounded.

Set

$$\left.\begin{array}{l}\bar{A}\triangleq A_l+bf_{\bar{\sigma}l}+k_{\bar{\sigma}ql}(c_{\bar{\sigma}l}-\bar{c}-\tilde{c})+h_{\bar{\sigma}ql}(1-\psi)c_{ql}\\ \bar{B}\triangleq[\,-\mathfrak{v}_l k_{\bar{\sigma}ql}\quad k_{\bar{\sigma}ql}\bar{g}+bg_{\bar{\sigma}}+d+h_{\bar{\sigma}ql}(1-\psi)\quad h_{\bar{\sigma}ql}\,]\\ \bar{G}\triangleq[\,0\quad g_{\bar{\sigma}}\quad 0\,]\end{array}\right\} \tag{38}$$

Define finally the *imperfect supervision system gain*

$$\bar{\mathfrak{g}}_l \triangleq 3\sup_{\bar{\mathcal{Q}}_l}\sup_{\mathbb{S}}\sup_{\mathbb{F}_l}\left\|\left\{\!\!\left\{\begin{array}{cc}\bar{A} & \bar{B}\\ f_{\bar{\sigma}l} & \bar{G}\end{array}\right\}\!\!\right\}\right\| \tag{39}$$

and the constants

$$\begin{aligned}\bar{\mathfrak{a}}_l &= 3\sup_{\bar{\mathcal{Q}}_l}\sup_{\mathbb{S}}\sup_{\mathbb{F}_l}\left|\left\{\!\!\left\{\begin{array}{cc}\lambda I+\bar{A} & \bar{B}\\ I & 0\end{array}\right\}\!\!\right\}\right|\\ \bar{\mathfrak{b}}_l &\triangleq \sup_{\bar{\mathcal{Q}}_l}\sup_{\mathbb{S}}\sup_{\mathbb{F}_l}||f_{\bar{\sigma}l}[\![\bar{A}]\!]_0||_{\{0,\infty\}}\\ \bar{\mathfrak{c}}_l &\triangleq \sup_{\bar{\mathcal{Q}}_l}\sup_{\mathbb{S}}\sup_{\mathbb{F}_l}\sup_{t\geq 0}|[\![\lambda I+\bar{A}]\!]_0|\end{aligned}$$

Note that $\bar{\mathfrak{g}}_l$, $\bar{\mathfrak{a}}_l$, $\bar{\mathfrak{b}}_l$ and $\bar{\mathfrak{c}}_l$ must all be finite because of Lemma 4 and the boundedness of ψ and $\bar{g}$.

B. Behavior of x and v

Let $\mathfrak{w}\triangleq\sqrt{2}\sup_{p\in\mathcal{P}}|[\,c_p\quad -1\,]|$. The following proposition characterizes closed-loop behavior of x and v for values of t, if any, at which $\sigma\in\bar{\mathcal{Q}}_{p^*}$.

Proposition 2: Suppose that $\bar{\mathcal{Q}}_{p^*}$ is nonempty and that

$$\epsilon_{p^*}<\frac{1}{\bar{\mathfrak{g}}_{p^*}} \tag{40}$$

Then

$$|x(t)| \leq e^{-\lambda t}\left\{\bar{\mathfrak{c}}_{p^*}|x(0)|+\frac{\bar{\mathfrak{a}}_{p^*}}{1-\epsilon_{p^*}\bar{\mathfrak{g}}_{p^*}}\left(\epsilon_{p^*}\bar{\mathfrak{b}}_{p^*}|x(0)| + ||\mathbf{b}||_{\{0,t\}}+\frac{2}{3}\mathfrak{w}\sqrt{|W(0)|}\right)\right\} \tag{41}$$

and

$$||v||_{\{0,t\}} \leq \frac{1}{1-\epsilon_{p^*}\bar{\mathfrak{g}}_{p^*}}\left\{\bar{\mathfrak{b}}_{p^*}|x(0)|+\bar{\mathfrak{g}}_{p^*}\left(||\mathbf{b}||_{\{0,t\}} + \frac{2}{3}\mathfrak{w}\sqrt{|W(0)|}\right)\right\} \tag{42}$$

at any time t at which the value of σ is in $\bar{\mathcal{Q}}_{p^*}$.

This proposition proves to be a simple consequence of the following lemma.

Lemma 5: Let T be any time at which $\sigma(T)\in\bar{\mathcal{Q}}_{p^*}$ and let $q\triangleq\sigma(T)$. There exists a quadruple of functions $\{\bar{c},\tilde{c},\psi,\bar{g}\}\in\mathbb{F}_{p^*}$, depending on T, such that

$$||(1-\psi)e_q||_{\{0,T\}}\leq||e_{p^*}||_{\{0,T\}}+\mathfrak{w}\sqrt{|W(0)|} \tag{43}$$

and

$$||\hat{e}||_{\{0,T\}}\leq||e_{p^*}||_{\{0,T\}}+\mathfrak{w}\sqrt{|W(0)|} \tag{44}$$

where

$$\hat{e}\triangleq\frac{1}{\mathfrak{v}_{p^*}}\{e_\sigma-(1-\bar{g})e_{p^*}-(\bar{c}+\tilde{c})x\},\quad t\geq 0 \tag{45}$$

The proof of this lemma involves a number of steps. For continuity, the proof will be deferred to §VIII.

Proof of Proposition 2: Let $\{\bar{c},\tilde{c},\psi,\bar{g}\}$ and $\hat{e}$ be as in Lemma 5 and define $\bar{e}\triangleq[\,\hat{e}\quad e_{p^*}\quad(\psi-1)e_q\,]'$. Then

$$||\bar{e}||_{\{0,T\}}\leq 3||e_{p^*}||_{\{0,T\}}+2\mathfrak{w}\sqrt{|W(0)|} \tag{46}$$

because of Lemma 5. Observe that the differential equation for x and the equation for v in (24) can be rewritten as

$$\dot{x}=\bar{A}x+\bar{B}\bar{e} \tag{47}$$

and

$$v=f_{\sigma p^*}x+\bar{G}\bar{e}$$

respectively, where $\bar{A},\bar{B}$ and $\bar{G}$ are as in (38) with $l\triangleq p^*$ and $\bar{\sigma}\triangleq\sigma$. Hence

$$v=\Gamma\circ\bar{e}+\Phi x(0)$$

where

$$\Gamma\triangleq\left\{\!\!\left\{\begin{array}{cc}\bar{A} & \bar{B}\\ f_{\sigma p^*} & \bar{G}\end{array}\right\}\!\!\right\}\qquad\text{and}\qquad\Phi\triangleq[\![\bar{A}]\!]_0$$

Therefore

$$\begin{aligned}||v||_{\{0,T\}} &\leq ||\Gamma\circ\bar{e}||_{\{0,T\}}+||\Phi x(0)||_{\{0,T\}}\\ &\leq ||\Gamma||\,||\bar{e}||_{\{0,T\}}+||\Phi||_{\{0,T\}}|x(0)|\\ &\leq \frac{\bar{\mathfrak{g}}_{p^*}}{3}||\bar{e}||_{\{0,T\}}+\bar{\mathfrak{b}}_{p^*}|x(0)|\\ &\leq \bar{\mathfrak{g}}_{p^*}\left(||e_{p^*}||_{\{0,T\}}+\frac{2}{3}\mathfrak{w}\sqrt{|W(0)}\right)\\ &\quad +\bar{\mathfrak{b}}_{p^*}|x(0)|\end{aligned} \tag{48}$$

Inequality (42) follows at once from this and key inequality (23). So does the inequality

$$||e_{p^*}||_{\{0,T\}} \leq \frac{1}{(1-\epsilon_{p^*}\bar{\mathfrak{g}}_{p^*})}\left\{\epsilon_{p^*}\left(\bar{\mathfrak{b}}_{p^*}|x(0)| + \bar{\mathfrak{g}}_{p^*}\frac{2}{3}\mathfrak{w}\sqrt{|W(0)|}\right)+||\mathbf{b}||_{\{0,T\}}\right\} \tag{49}$$

It is possible to rewrite (47) as

$$\frac{d}{dt}\{e^{\lambda t}x\}=(\lambda I+\bar{A})\{e^{\lambda t}x\}+\bar{B}\{e^{\lambda t}\bar{e}\}$$

Thus

$$\{e^{\lambda t}x\}=\Upsilon\circ\{e^{\lambda t}\bar{e}\}+\Theta x(0)$$

where

$$\Upsilon\triangleq\left\{\!\!\left\{\begin{array}{cc}\lambda I+\bar{A} & \bar{B}\\ I & 0\end{array}\right\}\!\!\right\}\qquad\text{and}\qquad\Theta\triangleq[\![\lambda I+\bar{A}]\!]_0$$

Therefore

$$\begin{aligned} e^{\lambda t}|x(t)| &\leq \left|\Upsilon \circ \{e^{\lambda t}\bar{e}\}\right| + |\Theta x(0)\}| \\ &\leq |\Upsilon|\,||\bar{e}||_{\{0,T\}} + |\Theta||x(0)| \\ &\leq \frac{\bar{\mathfrak{a}}_{p^*}}{3}||\bar{e}||_{\{0,T\}} + \bar{\mathfrak{c}}_{p^*}|x(0)| \\ &\leq \bar{\mathfrak{a}}_{p^*}\left(||e_{p^*}||_{\{0,T\}} + \frac{2}{3}\mathfrak{w}\sqrt{|W(0)|}\right) \\ &\quad + \bar{\mathfrak{c}}_{p^*}|x(0)| \end{aligned}$$

From this and (49) it follows that (41) is true. ■

VII. Perfect or Imperfect Supervision

The goal of this section is to prove Theorem 1. This will be done by first combining, in §VII-B, the characterizations of x and v given by Propositions 1 and 2, into a single pair of inequalities which are valid for all time. This will lead at once, in §VII-C, to a validation of the claims of Theorem 1. We begin by specifying a bound for ϵ_{p^*}.

A. A Bound for ϵ_{p^*}

Suppose that

$$\epsilon_{p^*} < \begin{cases} \frac{1}{\mathfrak{g}_{p^*}} & \text{if } \bar{\mathcal{Q}}_{p^*} \text{ is empty} \\ \min\left\{\frac{1}{\mathfrak{g}_{p^*}}, \frac{1}{\bar{\mathfrak{g}}_{p^*}}\right\} & \text{if } \bar{\mathcal{Q}}_{p^*} \text{ is nonempty} \end{cases} \tag{50}$$

where $\mathfrak{g}_{p^*}$ and $\bar{\mathfrak{g}}_{p^*}$ are as defined in (26) and (39) respectively. In the sequel it will be shown that if ϵ_{p^*} is so bounded, then the conclusions of Theorem 1 are correct.

B. Behavior of x and v

Whether $\bar{\mathcal{Q}}_{p^*}$ is empty or not, we claim that there are nonnegative constants $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_6$ such that for $t \geq 0$

$$\begin{aligned} |x(t)|e^{\lambda t} &\leq \mathbf{c}_1|x(0)| + \mathbf{c}_2||\mathbf{b}||_{\{0,t\}} + \mathbf{c}_3\sqrt{|W(0)|} \\ ||v||_{\{0,t\}} &\leq \mathbf{c}_4|x(0)| + \mathbf{c}_5||\mathbf{b}||_{\{0,t\}} + \mathbf{c}_6\sqrt{|W(0)|} \end{aligned} \tag{51}$$

In the event $\bar{\mathcal{Q}}_{p^*}$ is empty, Proposition 1 applies on $[t_a, t_b) = [0, \infty)$. Therefore in this case one can define the $\mathbf{c}_i$ in the obvious way, in terms of the constants $\epsilon_{p^*}, \mathfrak{g}_{p^*}, \mathfrak{a}_{p^*}, \mathfrak{b}_{p^*}$ and $\mathfrak{c}_{p^*}$ appearing in the inequalities in (28) and (29).

Suppose $\bar{\mathcal{Q}}_{p^*}$ is nonempty. Therefore the inequalities in Propositions 1 and 2 both hold at the times specified. Establishing the existence of nonnegative constants $\mathbf{c}_i$ for which the inequalities in (51) hold is the same as establishing the existence of 2-vectors $\mathbf{h}$, $\mathbf{k}$, and $\mathbf{l}$, each with nonnegative entries, such that the vector inequality

$$\begin{bmatrix} |x(t)|e^{\lambda t} \\ \\ ||v||_{\{0,t\}} \end{bmatrix} \leq \mathbf{h}|x(0)| + \mathbf{k}||\mathbf{b}||_{\{0,t\}} + \mathbf{l}\sqrt{|W(0)|}, \tag{52}$$

holds component-wise, for all $t \geq 0$. Note that the inequalities of Proposition 2 can be written in this form for suitably defined $\mathbf{h}_1$, $\mathbf{k}_1$, and $\mathbf{l}_1$ depending on $\epsilon_{p^*}, \bar{\mathfrak{g}}_{p^*}, \bar{\mathfrak{a}}_{p^*}, \bar{\mathfrak{b}}_{p^*}$ and $\bar{\mathfrak{c}}_{p^*}$. Meanwhile the inequalities in Proposition 1 can be written together as

$$\begin{bmatrix} |x(t)|e^{\lambda t} \\ \\ ||v||_{\{0,t\}} \end{bmatrix} \leq \mathbf{M}\begin{bmatrix} |x(t_a)|e^{\lambda t_a} \\ \\ ||v||_{\{0,t_a\}} \end{bmatrix} + \mathbf{k}_2||\mathbf{b}||_{\{t_a,t\}},\ t \in [t_a, t_b) \tag{53}$$

where $\mathbf{M} \triangleq [\mathbf{h_2}\ \ \mathbf{m}]_{2\times 2}$; here $\mathbf{h}_2$, $\mathbf{k}_2$, and $\mathbf{m}$ are 2-vectors with nonnegative components depending on $\epsilon_{p^*}, \mathfrak{g}_{p^*}, \mathfrak{a}_{p^*}, \mathfrak{b}_{p^*}$ and $\mathfrak{c}_{p^*}$.

Suppose that t_0, t_a and t_b are successive times such that $\sigma([t_0, t_a)) \subset \bar{\mathcal{Q}}_{p^*}$ and $\sigma([t_a, t_b)) \subset \mathcal{Q}_{p^*}$. Then (52) is valid for $t \in [t_0, t_a)$ if $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_1, \mathbf{k}_1, \mathbf{l}_1\}$ where as (53) is valid for $t \in [t_a, t_b)$. From this it follows that (52) holds for $t \in [t_a, t_b)$ provided $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{M}\mathbf{h}_1, \mathbf{M}\mathbf{k}_1 + \mathbf{k}_2, \mathbf{M}\mathbf{l}_1\}$. Therefore (52) must hold for $t \in [t_0, t_b)$ if provided $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_3, \mathbf{k}_3, \mathbf{l}_3\}$ where $\mathbf{h}_3 \triangleq \max\{\mathbf{h}_1, \mathbf{M}\mathbf{h}_1\}$, $\mathbf{k}_3 \triangleq \max\{\mathbf{k}_1, \mathbf{M}\mathbf{k}_1 + \mathbf{k}_2\}$ and $\mathbf{l}_3 \triangleq \max\{\mathbf{l}_1, \mathbf{M}\mathbf{l}_1\}$[1]. Since this is true for *every* such t_0, t_a and t_b, it must be true that if t_0 is any time at which $\sigma(t_0) \in \bar{\mathcal{Q}}_{p^*}$ and $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_3, \mathbf{k}_3, \mathbf{l}_3\}$, then (52) holds for *all* $t \in [t_0, \infty)$.

Let us now note that one of the following must be true: either $\sigma([0,\infty)) \subset \mathcal{Q}_{p^*}$ or there must be a least time $t_0 \geq 0$ such that $\sigma(t_0) \in \bar{\mathcal{Q}}_{p^*}$. If the former is true, then, because of (53) and the definition of $\mathbf{M}$, it must be that with $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_2, \mathbf{k}_2, 0\}$, (52) holds for all $t \geq 0$. On the other hand, if the latter is true then (52) must hold for $t \in [0, t_0)$ with $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_2, \mathbf{k}_2, 0\}$ and for $t \in [t_0, \infty)$ with $\{\mathbf{h}, \mathbf{k}, \mathbf{l}\} \triangleq \{\mathbf{h}_3, \mathbf{k}_3, \mathbf{l}_3\}$. It therefore follows that under any conditions, (52) must hold for all $t \geq 0$ provided $\mathbf{h} \triangleq \max\{\mathbf{h}_2, \mathbf{h}_3\}$, $\mathbf{k} \triangleq \max\{\mathbf{k}_2, \mathbf{k}_3\}$, and $\mathbf{l} \triangleq \mathbf{l}_3$. This therefore establishes the validity of the inequalities in (51) for the case when $\bar{\mathcal{Q}}_{p^*}$ is nonempty. ■

C. Proof of Theorem 1

Observe that the convolution product $\widehat{\delta}_{p^*} \circ v$ in key equation (19) can also be written as

$$\widehat{\delta}_{p^*} \circ v = e^{-\lambda t}\left(\left\{e^{\lambda t}\widehat{\delta}_{p^*}\right\} \circ \left\{e^{\lambda t}v\right\}\right)$$

From this and (19) it follows that

$$|e_{p^*}(t)| \leq \mathbf{c}e^{-\lambda t}||v||_{\{0,t\}} + |\mathbf{b}(t)|, \quad t \geq 0 \tag{54}$$

where

$$\mathbf{c} \triangleq \sup\left\{\sup_{t\geq 0}\left|\{e^{\lambda t}\widehat{\delta}_{p^*}\} \circ \{e^{\lambda t}v\}\right| : ||v||_{\{0,\infty\}} \leq 1\right\}$$

Note that $\mathbf{c} < \infty$ because the inverse Laplace transform of $e^{\lambda t}\widehat{\delta}_{p^*}$, namely $\delta_{p^*}(s - \lambda)$, is a stable, strictly proper transfer function {cf Remark 3}. From (54) and inequality for v in (51), it follows that

$$\begin{aligned} |e_{p^*}(t)| &\leq \mathbf{c}e^{-\lambda t}\left\{\mathbf{c}_4|x(0)| + \mathbf{c}_5||\mathbf{b}||_{\{0,t\}} + \mathbf{c}_6\sqrt{|W(0)|}\right\} \\ &\quad + |\mathbf{b}(t)|, \quad t \geq 0 \end{aligned} \tag{55}$$

As noted in Remark 3, $\mathbf{b}$ will be bounded if $\mathbf{n}$ and $\mathbf{d}$ are and will tend to zero as fast as $e^{-\lambda t}$ if $\mathbf{n}$ and $\mathbf{d}$ are both constant. In view of (51) and (55), x and e_{p^*} must also behave in the same manner. In other words, x and e_{p*} will

[1] Here the max operation is to be interpreted component-wise.

be bounded if $\mathbf{n}$ and $\mathbf{d}$ are and will tend to zero as fast as $e^{-\lambda t}$ if $\mathbf{n}$ and $\mathbf{d}$ are both constant.

Suppose that $\mathbf{n}$ and $\mathbf{d}$ are bounded. Since x and e_{p^*} are bounded, x_C and x_E must be bounded because of (14) and (15). So must be v and $\mathbf{e_T}$ because of (24). In view of (5), y must be bounded. Thus W must be bounded because of (9) and y_c must be bounded because $y_c = y - \mathbf{n}$. Finally note that u must be bounded because of the boundedness of y_c and v and because of the observability of the cascade interconnection of (4) with any minimal state space representation of Σ_P [2]. This proves claim 1 of Theorem 1.

Now suppose that $\mathbf{n}$ and $\mathbf{d}$ are both constant. Since $x \to 0$ and $e_{p^*} \to 0$ as fast as $e^{-\lambda t}$, so must the pairs $\{x_C, \bar{x}_E\}$ and $\{v, \mathbf{e_T}\}$ because of (14) and (24) respectively. In view of (5) and (15), y must tend to r and x_E must tend to a finite limit, each as fast at $e^{-\lambda t}$. Thus W must tend to a finite limit because of (9) and y_c must tend to $r - \mathbf{n}$ because $y_c = y - \mathbf{n}$. Finally note that u must tend to a finite limit because y_c and v do and because of the observability of the cascade interconnection of (4) with any minimal state space representation of Σ_P. This proves claim 2 of Theorem 1. ■

VIII. Dwell-Time Switching

The ultimate goal of this section is to prove Lemma 5. This will be done in three steps. First in §VIII-A we will derive several key inequalities which are consequences of dwell-time switching. Subsection VIII-A is almost completely self-contained and can be read without reference to anything else in the paper other that the definition of the performance signals π_p in (8) and (10), key equation (18) which relates these performance signals to the exponentially weighted 2-norm $||\cdot||_{\{t_1,t_2\}}$, and of course the description given in §II of the dwell-time switching logic itself. Second in §VIII-B we will briefly outline several elementary algebraic ideas under the heading of "strong bases". §VIII-B is completely self-contained and can be read without reference to anything else in the paper. Finally we will carry out the proof of Lemma 5 in §VIII-C.

A. Consequences of Dwell-time Switching

The purpose of this section is to derive certain key properties characteristic of dwell-time switching. In the sequel, $t_0 \triangleq 0$, t_i denotes the ith time at which σ switches and p_i is the value of σ on $[t_{i-1}, t_i)$; if σ switches at most $k < \infty$ times then $t_{k+1} \triangleq \infty$ and p_{k+1} denotes σ's value on $[t_k, \infty)$. We write φ for that piecewise-constant signal on $[0, \infty)$ whose value on each switching interval $[t_{i-1}, t_i)$ is i. Thus $\sigma = p_\varphi$. Any time X takes on the current value of W is called a *sample time*. We shall also call $T = 0$ a sample time, even though W is not actually sampled at time zero. For $i > 1$, we write $\mathcal{S}_i$ for the set consisting of sample time $t_{i-1} - \tau_C$ and all sample times in the open half interval $[t_{i-1} + \tau_D - \tau_C, t_i - \tau_C)$ - and $\mathcal{S}_1$ is the set consisting of the time $T = 0$ and all sample times in the open half interval $[\tau_D - \tau_C, t_1 - \tau_C)$.

A typical time segment, partitioned in this manner, is shown in Figure 6. Here dots $\{\cdot\}$, circles $\{\circ\}$, stars $\{\star\}$, and squares $\{\blacksquare\}$ represent sample times in $\mathcal{S}_{i-1}$, $\mathcal{S}_i$, $\mathcal{S}_{i+1}$, and $\mathcal{S}_{i+2}$ respectively.

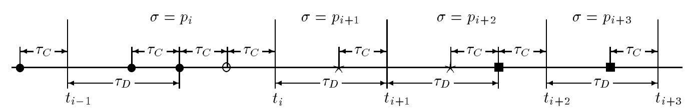

Fig. 6. Segment of Partitioned Time Axis

The utility of this partition stems from the fact that each $\mathcal{S}_i$ is the set of times at which it is known for sure that

$$\pi_{p_i}(T) \le \pi_p(T), \quad p \in \mathcal{P}, \quad T \in \mathcal{S}_i, \qquad (56)$$

This is true for all T in all $\mathcal{S}_i$ with the exception of $T = 0$. The validity of (56) is a direct consequence of the definition of Σ_D.

The following inequality captures the essence of dwell-time switching.

Lemma 6: For each $i \ge 1$

$$||e_{p_i}||_{\{0,T\}} \le ||e_p||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|}, \quad p \in \mathcal{P},\ T \in \mathcal{S}_i, \qquad (57)$$

where $\mathfrak{w} = \sqrt{2}\sup_{p\in\mathcal{P}} |[\, c_p \quad -1 \,]|$

Proof: Fix $i > 0$ and $T \in \mathcal{S}_i$. If $T = 0$, (57) holds because $||\cdot||_{\{0,T\}} = 0$. Suppose $T \ne 0$ in which case (56) is valid. By multiplying both sides of this inequality by $e^{2\lambda T}$ and then using key equation (18) there results

$$||e_{p_i}||^2_{\{0,T\}} + \pi_{p_i}(0) \le ||e_p||^2_{\{0,T\}} + \pi_p(0), \quad p \in \mathcal{P}$$

But

$$|\pi_p(0)| \le \frac{\mathfrak{w}^2}{2}|W(0)|, \ \ p \in \mathcal{P}$$

because of (8), (10) and the definition of $\mathfrak{w}$. It thus follows that

$$\begin{aligned} ||e_{p_i}||^2_{\{0,T\}} &\le ||e_p||^2_{\{0,T\}} + (|\pi_{p_i}(0)| + |\pi_p(0)|) \\ &\le ||e_p||^2_{\{0,T\}} + \mathfrak{w}^2|W(0)| \end{aligned}$$

This implies (57) which completes the proof. ■

The problem with the inequality in Lemma 6 is that it does not typically provide a norm bound for e_{p_i} during the time period when $\sigma = p_i$, which is just when controller p_i is in the feedback-loop. The reason for this stems from causality and cannot be avoid unless computation time is ignored and $\tau_D = 0$. The consequences of this are deep and would be encountered with *any* certainty equivalence based adaptive control algorithm, were one to take into account computation time. For the problem at hand, it is possible to circumvent this obstacle by using suitably defined projections with finite supports. This will be demonstrated in a moment. But first we need the following.

For $t \ge 0$, let $t^\diamond$ denote the largest sample time less than or equal to t. For $i \ge 1$ and each finite time $t \in [t_i - \tau_C, t_i]$, let us write $t^\diamond$ for the largest sample time less than $t_i - \tau_C$. Note that {for finite t}

$$\left.\begin{array}{lll} t \in [t_{i-1}, t_i - \tau_C) & \Longrightarrow & t^\diamond \in \mathcal{S}_i \\ t \in [t_i - \tau_C, t_i] & \Longrightarrow & t^\diamond \in \mathcal{S}_i \end{array}\right\} \quad i \ge 1 \qquad (58)$$

[2] Observability of this interconnection is a consequence of the standing assumption that the numerator of the transfer function of Σ_P from u to y is nonzero at $s = 0$.

and that

$$\left.\begin{array}{lll} t\in[t_{i-1},t_i-\tau_C) & \Longrightarrow & t-t^{\diamond}\leq\tau_D+\tau_C \\ t\in[t_i-\tau_C,t_i] & \Longrightarrow & t-t^{\circ}\leq\tau_D+\tau_C \end{array}\right\}\quad i\geq 1 \tag{59}$$

We can now state and prove

Lemma 7: For each switching interval $[t_{i-1},t_i)$ and each finite time $T\in[t_{i-1},t_i]$, there is a piecewise-constant signal $\Psi_{iT}:[0,\infty)\rightarrow\{0,1\}$ such that

$$||(1-\Psi_{iT})e_{p_i}||_{\{0,T\}} \leq ||e_p||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|},\quad p\in\mathcal{P} \tag{60}$$

$$\int_0^{\infty}|\Psi_{iT}|ds \leq \tau_D+\tau_C \tag{61}$$

Proof: If $T\in[t_{i-1},t_i-\tau_C)$, set $\bar{T}=T^{\diamond}$ and define $\Psi_{iT}(t)=1,\ t\in[T^{\diamond},T)$ and $\Psi_{iT}(t)=0$ otherwise. If $T\in[t_i-\tau_C,t_i]$, set $\bar{T}=T^{\circ}$ and define define $\Psi_{iT}(t)=1,\ t\in[T^{\circ},T)$ and $\Psi_{iT}(t)=0$ otherwise. In either case, (61) must hold because of (59), $\bar{T}$ must be in $\mathcal{S}_i$ because of (58), and $||(1-\Psi_{iT})e_{p_i}||_{\{0,T\}}=||e_{p_i}||_{\{0,\bar{T}\}}$ because of the definitions of Ψ_{iT} and $\bar{T}$. Since $\bar{T}\in\mathcal{S}_i$, we can use (57), with $\bar{T}$ playing the role of T, to obtain $||e_{p_i}||_{\{0,\bar{T}\}}\leq||e_p||_{\{0,\bar{T}\}}+\mathfrak{w}\sqrt{|W(0)|},\ p\in\mathcal{P}$. Therefore $||(1-\Psi_{iT})e_{p_i}||_{\{0,T\}}\leq||e_p||_{\{0,\bar{T}\}}+\mathfrak{w}\sqrt{|W(0)|},\ p\in\mathcal{P}$. But $||e_p||_{\{0,\bar{T}\}}\leq||e_p||_{\{0,T\}}$ because $\bar{T}\leq T$ so (60) must be true. ■

Let us note that for each T and i, Ψ_{iT} is idempotent and thus a projection with finite support. In the following we will make use of a more complicated projection, constructed from such Ψ_{iT}, which also has finite support. The significance of this technical result will become apparent in §VIII-C.

Lemma 8: Let $[t_{m-1},t_m)$ be a switching interval and T a time in $[t_{m-1},t_m)$. Suppose there are $\bar{m}\leq m$ positive integers $i_1,\ i_2,\ \ldots,\ i_{\bar{m}}$, such that

$$i_1<i_2<\ldots<i_{\bar{m}}=m, \tag{62}$$

There exists a piecewise constant function $\bar{\Psi}:[0,\ \infty)\rightarrow\{0,1\}$, namely

$$\bar{\Psi}\triangleq 1-\left\{(1-\Psi_{i_{\bar{m}}T})\prod_{j=1}^{\bar{m}-1}(1-\Psi_{i_jt_{i_j}})\right\}, \tag{63}$$

which satisfies

$$\int_0^{\infty}|(\bar{\Psi})(s)|ds\leq\bar{m}(\tau_D+\tau_C), \tag{64}$$

and has the following property. For any set of real numbers

$$\mathcal{G}\triangleq\{g_{ij}:\ i\geq 1,\ j\in\{1,2,\ldots,\bar{m}\}\}$$

satisfying

$$g_{ij}=0,\quad i>i_j,\ j\in\{1,2,\ldots\bar{m}-1\}, \tag{65}$$

the inequality

$$||(1-\bar{\Psi})\sum_{j=1}^{\bar{m}}g_{\varphi j}e_{p_{i_j}}||_{\{0,T\}} \leq \bar{m}\{\sup\mathcal{G}\}\Big\{||e_p||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|}\Big\} \tag{66}$$

is valid for all $p\in\mathcal{P}$.

Proof: The definition of $\bar{\Psi}$ ensures that its codomain is $\{0,1\}$ and that its support is contained in the union of the supports of the $\bar{m}$ functions $\Psi_{i_1t_{i_1}},\ \Psi_{i_2t_{i_2}},\ldots,\Psi_{i_{\bar{m}-1}t_{i_{\bar{m}-1}}}$, $\Psi_{i_{\bar{m}}T}$. But by Lemma 7, each of these functions has support of length no greater $\tau_D+\tau_C$. It therefore follows that (64) is true.

To prove (66), it will first be shown that

$$||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,T\}} \leq \{\sup\mathcal{G}\}\Big\{||e_p||_{\{0,t_{i_j}\}} + \mathfrak{w}\sqrt{|W(0)|}\Big\},\quad p\in\mathcal{P},\quad j\in\{1,2,\ldots\bar{m}-1\} \tag{67}$$

For this, fix $j\in\{1,2,\ldots\bar{m}-1\}$. Then $t_{i_j}\leq t_{m-1}$ because of (62). Since $T\in[t_{m-1},t_m)$, it follows that $t_{i_j}\leq T$. Next observe that $g_{\varphi j}=0,\ t\in[t_{i_j},T)$ because of (65) and the definition of φ. Therefore

$$||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,T\}}=||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,t_{i_j}\}} \tag{68}$$

From (63)

$$1-\bar{\Psi}=(1-\Psi_{i_{\bar{m}}T})\prod_{j=1}^{\bar{m}-1}(1-\Psi_{i_jt_{i_j}}) \tag{69}$$

But each factor in this product is norm bounded by 1 on $[0,\infty)$. Therefore

$$||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,t_{i_j}\}}\leq||(1-\Psi_{i_jt_{i_j}})g_{\varphi j}e_{p_{i_j}}||_{\{0,t_{i_j}\}}$$

so

$$||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,t_{i_j}\}}\leq\{\sup\mathcal{G}\}||(1-\Psi_{i_jt_{i_j}})e_{p_{i_j}}||_{\{0,t_{i_j}\}} \tag{70}$$

Moreover by Lemma 7

$$||(1-\Psi_{i_jt_{i_j}})e_{p_{i_j}}||_{\{0,t_{i_j}\}}\leq||e_p||_{\{0,t_{i_j}\}}+\mathfrak{w}\sqrt{|W(0)|},\ p\in\mathcal{P}$$

From this, (70) and (68) it now follows that (67) is true.

Similar reasoning applies to the case when $j=\bar{m}$. In particular

$$\begin{aligned}||(1-\bar{\Psi})g_{\varphi\bar{m}}e_{p_{i_{\bar{m}}}}||_{\{0,T\}} &\leq ||(1-\Psi_{i_{\bar{m}}T})g_{\varphi\bar{m}}e_{p_{i_{\bar{m}}}}||_{\{0,T\}}\\ &\leq \{\sup\mathcal{G}\}||(1-\Psi_{i_{\bar{m}}T})e_{p_{i_{\bar{m}}}}||_{\{0,T\}}\\ &\leq \{\sup\mathcal{G}\}\Big\{||e_p||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|}\Big\},\ p\in\mathcal{P}\end{aligned} \tag{71}$$

Here we've used (69) and Lemma 7, just as before.

Using (67) and (71) we can now write

$$\begin{aligned}||(1-\bar{\Psi})\sum_{j=1}^{\bar{m}}g_{\varphi j}e_{p_{i_j}}||_{\{0,T\}} &\leq \sum_{j=1}^{\bar{m}}||(1-\bar{\Psi})g_{\varphi j}e_{p_{i_j}}||_{\{0,T\}}\\ &\leq \sum_{j=1}^{\bar{m}}\{\sup\mathcal{G}\}\Big\{||e_p||_{\{0,t_{i_j}\}} + \mathfrak{w}\sqrt{|W(0)|}\Big\}\\ &= \bar{m}\{\sup\mathcal{G}\}\Big\{||e_p||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|}\Big\},\quad p\in\mathcal{P}\end{aligned}$$

Thus (66) is true. ■

B. Strong Bases

The definitions of the the functions $\bar{c},\tilde{c}$ and $\bar{g}$ which appear in the statement of Lemma 5 depend on the projections defined in the preceding lemmas as well as upon several elementary algebraic constructions. Here we digress briefly to discuss what these constructions are.

Let $\mathcal{X}$ be a subset of a real, finite dimensional linear space with norm $|\cdot|$ and let ϵ be a positive number. A nonempty list of vectors $\{x_1,\ x_2,\ \ldots,\ x_{\bar{n}}\}$ in $\mathcal{X}$ is *ϵ-independent* if

$$|x_{\bar{n}}|\geq\epsilon, \tag{72}$$

and, for $k \in \{1, 2, \ldots, \bar{n}-1\}$,

$$\left| x_k + \sum_{j=k+1}^{\bar{n}} \mu_j x_j \right| \geq \epsilon, \quad \forall \mu_j \in \mathbb{R} \tag{73}$$

$\{x_1,\ x_2,\ \ldots,\ x_{\bar{n}}\}$ *ϵ-spans* $\mathcal{X}$ if for each $x \in \mathcal{X}$ there is a set of real numbers $\{a_1, a_2, \ldots, a_{\bar{n}}\}$, called *$\epsilon$-coordinates*, such that

$$\left| x - \sum_{i=1}^{\bar{n}} a_i x_i \right| \leq \epsilon \tag{74}$$

The following lemma gives an estimate on how large these ϵ-coordinates can be assuming $\mathcal{X}$ is a bounded subset.

Lemma 9: Let $\mathcal{X}$ be a bounded subset which is ϵ-spanned by an ϵ-independent list $\{x_1, x_2, \ldots, x_{\bar{n}}\}$. Suppose that x is a vector in $\mathcal{X}$ and that $a_1, a_2, \ldots a_{\bar{n}}$ is a set of ϵ-coordinates of x with respect to $\{x_1,\ x_2,\ \ldots,\ x_{\bar{n}}\}$. Then

$$|a_i| \leq \left(1 + \frac{\sup \mathcal{X}}{\epsilon}\right)^{\bar{n}}, \quad i \in \{1, 2, \ldots, \bar{n}\} \tag{75}$$

This lemma is proved in the Appendix.

Now suppose that $\mathcal{X}$ is an finite list of vectors $x_1,\ x_2,\ \ldots,\ x_m$ in a real n-dimensional vector space. Suppose, in addition, that $|x_m| \geq \epsilon$. It is possible to extract from $\mathcal{X}$ an ordered subset $\{x_{i_1}, x_{i_2}, \cdots, x_{i_{\bar{n}}}\}$, with $\bar{n} \leq n$, which is ϵ-independent and which ϵ-spans $\mathcal{X}$. Moreover the i_j can always be chosen so that

$$i_1 < i_2 < i_3 < \cdots < i_{\bar{n}} = m \tag{76}$$

and also so that for suitably defined $a_{ij} \in \mathbb{R}$

$$\left| x_i - \sum_{j=k+1}^{\bar{n}} a_{ij} x_{i_j} \right| \leq \epsilon,\ i \in \{i_k+1,\ i_k+2,\ \ldots,\ i_{k+1}\}, \quad k \in \{1, 2, \ldots, \bar{n}-1\} \tag{77}$$

$$\left| x_i - \sum_{j=1}^{\bar{n}} a_{ij} x_{i_j} \right| \leq \epsilon, \quad i \in \{1, 2, \ldots, k_1\} \tag{78}$$

In fact, the procedure for doing this is almost identical to the familiar procedure for extracting from $\{x_1,\ x_2,\ \ldots,\ x_m\}$, an ordered subset which is linearly independent {in the usual sense} and which spans the span of $\{x_1,\ x_2,\ \ldots,\ x_m\}$. The construction of interest here begins by defining an integer $j_1 \triangleq m$. j_2 is then defined to be the greatest integer $j < j_1$ such that

$$|x_j - \mu x_{j_1}| \geq \epsilon \quad \forall \mu \in \mathbb{R},$$

if such an integer exists. If not, one defines $\bar{n} \triangleq 1$ and $i_1 \triangleq j_1$ and the construction is complete. If j_2 exists, j_3 is then defined to be the greatest integer $j < j_2$ such that

$$|x_j - \mu_1 x_{j_1} - \mu_2 x_{j_2}| \geq \epsilon \quad \forall \mu_i \in \mathbb{R},$$

if such an integer exists. If not, one defines $\bar{n} \triangleq 2$ and $i_k \triangleq j_{\bar{m}+1-k},\ k \in \{1, 2\}$ .... and so on. By this process one thus obtains an ϵ-independent, ϵ-spanning subset of $\mathcal{X}$ for which there exist numbers a_{ij} such that (76)-(78) hold. Since such $a_{ij},\ j \in \{1, 2, \ldots, \bar{n}\}$, are ϵ-coordinates of $x_i,\ i \in \{1, 2, \ldots, \bar{n}\}$, each coordinate must satisfy the same bound inequality as the a_i in (75). Moreover, because $\bar{n}$ cannot be larger than the dimension of the smallest linear space containing $\mathcal{X}$, $\bar{n} \leq n$.

With these preliminaries complete, we now turn to the proof of Lemma 5

C. Proof of Lemma 5

Set $\psi \triangleq \Psi_{qT}$ where Ψ_{qT} is the projection defined in the proof of Lemma 7. Then ψ satisfies (36) and (43) is true. To complete the proof it is therefore enough to construct functions $\bar{c}$, $\tilde{c}$, and $\bar{g}$ such that

$$\sup_{t\geq 0} |\bar{c}(t)| \leq \mathfrak{d}_{p^*} \tag{79}$$

$$\int_0^\infty |\tilde{c}(s)| ds \leq n_E(\tau_D + \tau_C)\mathfrak{v}_{p^*}(\sup_{p\in\mathcal{P}} |c_{pp^*}|) \tag{80}$$

$$\sup_{t\geq 0} |\bar{g}(t)| \leq \mathfrak{v}_{p^*} \tag{81}$$

and

$$||\hat{e}||_{\{0,T\}} \leq ||e_{p^*}||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|} \tag{82}$$

where

$$\hat{e} \triangleq \frac{1}{\mathfrak{d}_{p^*}} \{e_\sigma - (1-\bar{g})e_{p^*} - (\bar{c}+\tilde{c})x\}, \quad t \geq 0 \tag{83}$$

For clarity, we write c_i for $c_{p_i p^*}$ throughout the remainder of this proof. In view of the key equation for e_p in (24),

$$e_\sigma - e_{p^*} = c_\varphi x,\ t \geq 0 \ \text{ and } \ e_{p_i} - e_{p^*} = c_i x,\ i \geq 1 \tag{84}$$

where, as before, φ is that piecewise-continuous signal whose value is i on switching interval $[t_{i-1}, t_i)$.

Suppose $T \in [t_{m-1}, t_m)$. Then $p_m = q$. Thus $|c_m| \geq \mathfrak{d}_{p^*}$ because of the definition of $\mathfrak{d}_{p^*}$ in (31) and (32). It is therefore possible to extract from the list $\{c_1, c_2 \ldots, c_m\}$, a $\mathfrak{d}_{p^*}$-independent sublist $\{c_{i_1}, c_{i_2} \ldots, c_{i_{\bar{m}}}\}$ which $\mathfrak{d}_{p^*}$-spans $\{c_1, c_2 \ldots, c_m\}$ in such a way that $\bar{m} \leq n_E$ and

$$i_1 < i_2 < i_3 < \cdots < i_{\bar{m}} = m \tag{85}$$

Thus there must be numbers g_{ij} such that

$$\left| c_i - \sum_{j=k+1}^{\bar{m}} g_{ij} c_{i_j} \right| \leq \mathfrak{d}_{p^*},\ i \in \{i_k+1,\ i_k+2,\ \ldots,\ i_{k+1}\}, \quad k \in \{1, 2, \ldots, \bar{m}-1\}$$

$$\left| c_i = \sum_{j=1}^{\bar{m}} g_{ij} c_{i_j} \right| \leq \mathfrak{d}_{p^*}, \quad i \in \{1, 2, \ldots, k_1\}$$

Moreover because of the bound given by (75) in Lemma 9 and the fact that $\bar{m} \leq n_E$,

$$|g_{ij}| \leq \left(1 + \frac{\sup_{p\in\mathcal{P}} |c_{pp^*}|}{\mathfrak{d}_{p^*}}\right)^{n_E}$$

This can be written as

$$|g_{ij}| \leq \frac{\mathfrak{v}_{p^*}}{n_E} \tag{86}$$

because of the definition of $\mathfrak{v}_{p^*}$ in (33). It follows that if we extend the domain of definition of g_{ij} to $\{1, 2, 3 \ldots \infty\} \times \{1, 2, \ldots, \bar{m}\}$ by setting

$$g_{ij} = 0,\ \ i > i_j,\ j \in \{1, 2, \ldots \bar{m}-1\} \tag{87}$$

then (86) must hold for all such i and j and

$$\left| c_i - \sum_{j=1}^{\bar{m}} g_{ij} c_{i_j} \right| \leq \mathfrak{d}_{p^*}, \quad i \in \{1, 2, \ldots, m\} \tag{88}$$

Since (85) and (87) are the same as conditions (62) and (65) of Lemma 8 respectively, it is possible to define via (63), a projection $\bar{\Psi}$ in such a way that

$$\int_0^\infty |(\bar{\Psi})(s)| ds \leq n_E(\tau_D + \tau_C), \tag{89}$$

and

$$||(1-\bar{\Psi}) \sum_{j=1}^{\bar{m}} g_{\varphi j} e_{p_{i_j}} ||_{\{0,T\}} \leq \mathfrak{v}_{p^*} \left\{ ||e_{p^*}||_{\{0,T\}} + \mathfrak{w}\sqrt{|W(0)|} \right\} \tag{90}$$

In writing these inequalities, we've made use of (86) and the fact that $\bar{m} \leq n_E$.

Define

$$\bar{g} \stackrel{\Delta}{=} (1-\bar{\Psi})\sum_{j=1}^{\bar{m}} g_{\varphi j}, \quad \bar{c} \stackrel{\Delta}{=} c_\varphi - \sum_{j=1}^{\bar{m}} g_{\varphi j} c_{i_j}, \quad \tilde{c} \stackrel{\Delta}{=} \bar{\Psi}\sum_{j=1}^{\bar{m}} g_{\varphi j} c_{i_j} \tag{91}$$

Then $\bar{c}$ satisfies (79) and $\bar{g}$ satisfies (81) because of (88) and (86) respectively - and $\tilde{c}$ satisfies (80) because of (86) and (89).

The definitions of $\bar{c}$ and $\tilde{c}$ in (91) imply that

$$c_\varphi = (1-\bar{\Psi})\sum_{j=1}^{\bar{m}} g_{\varphi j} c_{i_j} + \bar{c} + \tilde{c}$$

Multiplying both sides by x and then using (84) we get

$$e_\sigma - e_{p^*} = (1-\bar{\Psi})\sum_{j=1}^{\bar{m}} g_{\varphi j}(e_{p_{i_j}} - e_{p^*}) + \bar{c}x + \tilde{c}x$$

Therefore, in view of the definitions of $\hat{e}$ in (83) and $\bar{g}$ in (91) we get

$$\hat{e} = \frac{1}{\mathfrak{v}_{p^*}}(1-\bar{\Psi})\sum_{j=1}^{\bar{m}} g_{\varphi j} e_{p_{i_j}}$$

From this and (90) it follows that (82) is true. ■

IX. Concluding Remarks

Problems similar to the one considered here have been under study within the adaptive control field for at least fifteen years. During this period a number of clever ideas have been devised to deal with non-parametric modeling errors, disturbances and measurement noise. Typical approaches include the use of various types of tuner modifications, deadzones, and parameter projection techniques. Many of these ideas are discussed in [3]. The approach taken in [4] is perhaps the closest to the one followed here.

Despite the similarities, the ideas developed in this paper and in [1] differ in many ways from those found in mainstream adaptive control. At the formulational stage, conventional adaptive control emphasizes identification and invariably focuses on parameter update algorithms aimed at reducing parameter errors or some combination of parameter errors and augmented errors. Parameter space is typically a compact, convex continuum and the class of admissible plant models is usually linearly parameterized on this space. Parameter tuning is generally done recursively or dynamically and is based on some type of pseudo-gradient algorithm - and results for the constant disturbance noise-free case, do not typically establish *exponential* convergence of the process output to its set-point. In contrast, the approach taken is this paper and in [1] focuses primarily on controller update algorithms aimed at minimizing every so often, a continuously evolving, normed output estimation error. Parameter space may be either a finite set or a compact continuum but convexity is not crucial, and the class of admissible nominal plant models need not be linearly parameterized. In sharp contrast with conventional adaptive control, loop-controller updating is done via switching rather than by recursive or dynamic adjustment - and for the constant disturbance noise-free case, the process output converges to its set-point exponentially fast.

From the results obtained in this paper, it is possible {but tedious} to derive positive gains $\mathfrak{G}_1$, $\mathfrak{G}_2$, and $\mathfrak{G}_3$, such that

$$||\mathbf{e_T}||_{\{0,t\}} \leq \mathfrak{G}_1||\mathbf{n}||_{\{0,t\}} + \mathfrak{G}_2||\mathbf{d}||_{\{0,t\}} + \mathfrak{G}_3|x_\Sigma(0)|, \ t \geq 0$$

for every for every initialization $x_\Sigma(0)$ of the overall system Σ shown in Figure 5, and for every pair of piecewise-continuous noise and disturbance inputs $\mathbf{n}$ and $\mathbf{d}$ respectively. These gains could then be used as measures of performance, and one might try to develop a synthesis theory for choosing system components to reduce or minimize their values. This approach would thus be following in the footsteps of robust, non-adaptive control.

While such an approach would have merit, it would not be taking into account the fact that an adaptive system such as the one we've been discussing, can have two kinds of behavior depending on whether or not the system is being perfectly supervised. It might, for example, be more useful to have two measures of performance, one for when a system is being perfectly supervised and the other for when it is not. The development of a performance theory for perfectly supervised systems should not be too difficult. On the other hand, for imperfectly supervised systems, it is not clear at this point how to formalize what good performance ought to mean, let alone how one might seek to achieve it. If one adopts the point of view that imperfect supervision will occur intermittently, in response to either a process change or large intermittent disturbance inputs, then one might aim to develop a performance theory which seeks to limit excursions from desired loop-controllers while at the same time trying to induce a quick recovery from the imperfectly supervised mode. The concept of bursting [5], may prove relevant in this regard.

X. Appendix

Proof of Proposition 1:

Define $\bar{\sigma} : [0,\infty) \to \mathcal{Q}_{p^*}$ and $\bar{e} : [0,\infty) \to \mathbb{R}$ so that

$$\bar{\sigma} = \begin{cases} \sigma & \text{for } t \in [t_a, t_b) \\ p^* & \text{otherwise} \end{cases}$$

$$\bar{e} = \begin{cases} e_{p^*} & \text{for } t \in [t_a, t_b) \\ 0 & \text{otherwise} \end{cases}$$

Then $\bar{\sigma}$ is a piecewise constant switching signal taking values only in $\mathcal{Q}_{p^*}$. In addition

$$\dot{x} = (A_{p^*} + bf_{\bar{\sigma}p^*})x + (bg_{\bar{\sigma}} + d)\bar{e}, \quad t \in [t_a, t_b) \tag{92}$$

and

$$v = f_{\bar{\sigma}p^*}x + g_{\bar{\sigma}}\bar{e}, \quad t \in [t_a, t_b)$$

because of the equations for x and v in (24). Since $\bar{e} = 0$ for $t < t_a$,

$$v = \Gamma \circ \bar{e} + \Phi x(t_a), \quad t \in [t_a, t_b) \tag{93}$$

where

$$\Gamma \stackrel{\Delta}{=} \left\{\!\!\left\{ \begin{array}{cc} A_{p^*} + bf_{\bar{\sigma}p^*} & bg_{\bar{\sigma}} + d \\ f_{\bar{\sigma}p^*} & g_{\bar{\sigma}} \end{array} \right\}\!\!\right\}$$

and

$$\Phi \stackrel{\Delta}{=} f_{\bar{\sigma}p^*} [\![A_{p^*} + bf_{\bar{\sigma}p^*}]\!]_{t_a}$$

Therefore for $t \in [t_a, t_b)$,

$$\begin{aligned} ||v||_{\{t_a,t\}} &\leq ||\Gamma \circ \bar{e}||_{\{t_a,t\}} + ||\Phi x(t_a)||_{\{t_a,t\}} \\ &\leq ||\Gamma \circ \bar{e}||_{\{0,t\}} + ||\Phi||_{\{t_a,\infty\}}|x(t_a)| \\ &\leq ||\Gamma||\,||\bar{e}||_{\{0,t\}} + ||\Phi||_{\{t_a,\infty\}}|x(t_a)| \\ &\leq \mathfrak{g}_{p^*}||\bar{e}||_{\{0,t\}} + \mathfrak{b}_{p^*}|x(t_a)| \end{aligned}$$

Since $||\bar{e}||_{\{0,t\}} = ||e_{p^*}||_{\{t_a,t\}}$ for $t \in [t_a, t_b)$, it follows that

$$||v||_{\{t_a,t\}} \leq \mathfrak{g}_{p^*}||e_{p^*}||_{\{t_a,t\}} + \mathfrak{b}_{p^*}|x(t_a)| \qquad (94)$$

and thus that

$$||v||_{\{0,t\}} \leq \mathfrak{g}_{p^*}||e_{p^*}||_{\{t_a,t\}} + ||v||_{\{0,t_a\}} + \mathfrak{b}_{p^*}|x(t_a)|$$

Inequality (29) follows immediately from this and (23) while the inequality

$$\begin{aligned} ||e_{p^*}||_{\{t_a,t\}} \quad &\leq \quad \frac{1}{(1-\epsilon_{p^*}\mathfrak{g}_{p^*})}\Big\{\epsilon_{p^*}||v||_{\{0,t_a\}} + \epsilon_{p^*}\mathfrak{b}_{p^*}|x(t_a)| \\ &\quad +||\mathbf{b}||_{\{t_a,t\}}\Big\}, \qquad t \in [t_a, t_b) \end{aligned} \qquad (95)$$

is a consequence of (94) and (23).

Using the differential equation for x in (24) and the preceding definitions of $\bar{\sigma}$ and $\bar{e}$ we can write, for $t \in [t_a, t_b)$,

$$\frac{d}{dt}\{e^{\lambda t}x\} = (\lambda I + A_{p^*} + bf_{\bar{\sigma}p^*})\{e^{\lambda t}x\} + (bg_{\bar{\sigma}} + d)\{e^{\lambda t}\bar{e}\}$$

Thus for such values of t

$$\{e^{\lambda t}x\} = \Upsilon \circ \{e^{\lambda t}\bar{e}\} + \Theta\{e^{\lambda t_a}x(t_a)\}$$

where

$$\Upsilon \triangleq \left\{\!\!\left\{ \begin{array}{cc} \lambda I + A_{p^*} + bf_{\bar{\sigma}p^*} & bg_{\bar{\sigma}} + d \\ I & 0 \end{array} \right\}\!\!\right\}$$

and

$$\Theta \triangleq [\![\, \lambda I + A_{p^*} + bf_{\bar{\sigma}p^*} \,]\!]_{t_a}$$

Therefore

$$\begin{aligned} e^{\lambda t}|x(t)| \quad &\leq \quad \left|\Upsilon \circ \{e^{\lambda t}\bar{e}\}\right| + |\Theta\{e^{\lambda t_a}x(t_a)\}| \\ &\leq \quad |\Upsilon|\,||\bar{e}||_{\{0,t\}} + |\Theta|e^{\lambda t_a}|x(t_a)| \\ &\leq \quad \mathfrak{a}_{p^*}||\bar{e}||_{\{0,t\}} + \mathfrak{c}_{p^*}e^{\lambda t_a}|x(t_a)| \\ &= \quad \mathfrak{a}_{p^*}||e_{p^*}||_{\{t_a,t\}} + \mathfrak{c}_{p^*}e^{\lambda t_a}|x(t_a)|, \quad t \in [t_a, t_b) \end{aligned}$$

From this and (95) it follows that (28) is true. ■

Proof of Lemma 9: For $k \in \{1, 2, \ldots, \bar{n}\}$ let

$$y_k \triangleq \sum_{i=k}^{\bar{n}} a_i x_i \qquad (96)$$

We claim that

$$|a_k| \leq \frac{|y_k|}{\epsilon}, \quad k \in \{1, 2, \ldots, \bar{n}\} \qquad (97)$$

Now (97) surely holds for $k = \bar{n}$, because of (72) and the formula $|y_{\bar{n}}| = |a_{\bar{n}}||x_{\bar{n}}|$ which, in turn, is a consequence of (96). Next fix $k \in \{1, 2, , \ldots, \bar{n} - 1\}$. Now (97) is clearly true if $a_k = 0$. Suppose $a_k \neq 0$ in which case

$$y_k = a_k \left(x_k + \sum_{j=k+1}^{\bar{n}} \mu_j x_j \right)$$

where $\mu_j \triangleq \frac{a_j}{a_k}$. From this and (73) it follows that $|y_k| \geq |a_k|\epsilon$, $k \in \{1, 2, \ldots, \bar{n}\}$, so (97) is true.

Next write $y_1 = (y_1 - x) + x$. Then $|y_1| \leq |y_1 - x| + |x|$. But $|x| \leq \sup \mathcal{X}$ because $x \in \mathcal{X}$ and $|y_1 - x| \leq \epsilon$ because of (74) and the definition of y_1 in (96). Therefore

$$\frac{|y_1|}{\epsilon} \leq \left(1 + \frac{\sup \mathcal{X}}{\epsilon}\right) \qquad (98)$$

From (96) we have that $y_{k+1} = y_k - a_k x_k$, $k \in \{1, 2, \ldots, \bar{n} - 1\}$. Thus $|y_{k+1}| \leq |y_k| + |a_k||x_k|$, $k \in \{1, 2, \ldots, \bar{n} - 1\}$. Dividing both sides of this inequality by ϵ and then using (97) and $|x_k| \leq \sup \mathcal{X}$, we obtain the inequality

$$\frac{|y_{k+1}|}{\epsilon} \leq \left(1 + \frac{\sup \mathcal{X}}{\epsilon}\right)\frac{|y_k|}{\epsilon}, \quad k \in \{1, 2, \ldots, \bar{n} - 1\}$$

This and (98) imply that

$$\frac{|y_k|}{\epsilon} \leq \left(1 + \frac{\sup \mathcal{X}}{\epsilon}\right)^k, \quad k \in \{1, 2, \ldots, \bar{n}\}$$

In view of (97), it follows that (75) is true. ■

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