Forming Stable Coalitions: The Process Matters

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Abstract

Players are assumed to rank each other as coalition partners. Two processes of coalition formation are defined and illustrated:

• *Fallback (FB)*: Players seek coalition partners by descending lower and lower in their preference rankings until some majority coalition, all of whose members consider each other mutually acceptable, forms.

• *Build-up (BU)*: Same descent as FB, except only majorities whose members rank each other highest form coalitions.

BU coalitions are *stable* in the sense that no member would prefer to be in another coalition, whereas FB coalitions, whose members need not rank each other highest, may not be stable. BU coalitions are bimodally distributed in a random society, with peaks around simple majority and unanimity; the distributions of majorities in the US Supreme Count and in the US House of Representatives follow this pattern. The dynamics of real-life coalition-formation processes are illustrated by two Supreme Court cases.

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1. Introduction

Coalitions are collections of players. Their stability is usually defined in terms of outcomes and the incentives coalition members have to sustain them. In this paper, we show that the process by which the players come together and form coalitions also may critically affect how enduring coalitions will be.

To determine which coalitions are likely to form and be stable, we assume that each player ranks all other players as coalition partners. At the outset, we assume that players report their rankings truthfully, but we reconsider this assumption later. A coalition of k players, or k-coalition, is *stable* if no member would prefer to be in another k-coalition.

It is apparent that there is always at least one stable coalition—the grand coalition, or n-coalition, that comprises all n players—because there is no other n-coalition. But below the grand coalition, what coalitions will form, and how stable they will be, is unclear. The coalition-formation processes we postulate clarify this question and also enable us to distinguish two levels of stability.

To rule out strategic issues that arise because of differences in player size or capability,[1] we assume that (i) all players are of equal weight (as in a legislature in which each member has one vote) and (ii) winning coalitions are those with at least a simple majority, m, of members. While we focus on nonstrategic processes of coalition formation, we later consider the manipulability of these processes.

Coalitions form according to two processes:

• *Fallback (FB)*: Players seek coalition partners by descending lower and lower in their preference rankings until some majority coalition, all of whose members consider each other mutually acceptable, forms.

• *Build-up (BU)*: Same descent as FB, except only majorities whose members rank each other highest form coalitions.

Both these processes are driven by players' mutual preferences, not their evaluations of coalitions.[2]

We begin with notation and definitions in section 2. In section 3 we show that several FB majority coalitions with $k \geq m$ members may form simultaneously. Call the set of FB coalitions that form first (i.e., at the lowest value of k) FB_1. The analogous set, BU_1, comprises a unique coalition, which is stable. If the preferences of the players are single-peaked, FB_1 coalitions may be disconnected, but the BU_1 coalition is always connected (in a sense to be made precise later).

BU_1 contains all coalitions in FB_1, which may have fewer members than BU_1. This raises the question of which majority coalition is most likely to form—smaller FB_1 majority coalitions, in which some members may prefer players outside the coalition to players inside, or the BU_1 coalition, in which this cannot happen? We call smaller FB_1 majority coalitions *semi-stable* because at least some of their members are attracted to outside players, whereas BU_1 coalitions are stable.

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[1] For example, two ideologically distant players might join together if it would enable them to win, but neither would join a smaller more centrally located player if the resulting coalition were not winning.

[2] In Cechlárová and Romero-Medina (2000), each player uses its preference rankings of all other players to evaluate coalitions according to two criteria—the most-preferred, and the least-preferred, members that they contain. Other criteria are postulated in an agent-based simulation model in a neural-network framework, wherein political parties seek to attract a majority of players in a spatial voting game (Iizuka, Yamamoto, Suzuki, and Ohuchi, 2002). Related work on coalition-formation models is discussed in section 3.

In section 4 we show that semi-stable FB_1 coalitions are *manipulable* in that a player, by announcing a false preference ranking, can induce a majority coalition that it prefers. By contrast, BU_1 semi-stable coalitions are not manipulable: A manipulator may be able to induce a smaller majority coalition with a false announcement, but this coalition will not necessarily be preferred. The reason is that the larger BU_1 coalition, which forms when the manipulator is truthful, must contain at least one member the manipulator prefers to some player in the smaller majority coalition—so the manipulator will not assuredly prefer the smaller coalition.

In section 5 we investigate the properties of stable coalitions. BU_1 may be a simple-majority coalition, the grand coalition, or any size in between. More generally, stable coalitions of any size between m and $n - 1$ may or may not exist. Two stable coalitions (of any size) are either disjoint, or one contains the other. A *bandwagon strategy* may enable a player to be a member of a winning coalition sooner than it would be otherwise, but it will not necessarily be a winning coalition that it prefers.

In section 6 we show that if all player preferences are equally likely, the probability that a randomly chosen majority coalition is stable first decreases to some minimum between m and $n - 1$, then increases to 1 when the grand coalition forms, yielding a bimodal distribution, with peaks at minimal majority and unanimity. This finding also holds for the distribution of first-forming majority coalitions when preferences, whether single-peaked or not, are randomly chosen.

Empirical data on the size of US Supreme Court majorities that we present in section 7 show the distribution to be bimodal, with most being either minimal (5-person) or maximal (9-person) majorities. We illustrate the formation of majorities on the Court with an 8-0 decision (one justice recused himself) and a 5-4 decision. Data we present on the size of majorities in the US House of Representatives also show the distribution to be bimodal.

We conclude that FB and BU mirror different real-life coalition-formation processes. BU yields larger and less manipulable majority coalitions, compared to the more wieldy but vulnerable coalitions of FB. Together these models show how the stability of outcomes is inextricably linked to the processes that generate them.

2. Notation and Definitions

We assume that all players, named 1, 2, …, *n*, strictly rank each other as coalition partners, as illustrated in Example A.

Example A. **1**: 2 3 4 5 **2**: 1 3 4 5 **3**: 4 5 2 1 **4**: 3 2 1 5 **5**: 4 3 2 1.

We further assume that each player ranks itself first—that is, it most desires to be included in any majority coalition that forms. In Example A, player 1, after itself, most prefers player 2 as a coalition partner, followed by players 3, 4, and 5 in that order. A complete listing of all players' preferences, as illustrated in Example A, is called a *preference profile*.

It is clear that if there are *n* players, there are $[(n-1)!]^n$ preference profiles. In our model of a random society used later, all preference profiles are assumed to be equiprobable.

Sometimes we will assume that the players can be placed along a line—in order 1, 2, 3, …, *n*, from left to right—so that the preference profile is single-peaked. That is, each player's preference for coalition partners declines monotonically to the left and right of its position in this ordering. A preference profile that satisfies this condition is called *ordinally single-peaked* (Brams, Jones, and Kilgour, 2002). Such profiles are commonly assumed in spatial models of candidate and party ompetition.

To express single-peakedness in another way, consider the set of players in a coalition; call the left-most player l and the right-most player r. The set is *connected* if it is of the form $\{l, l + 1, \ldots, r\}$: It contains exactly the players from l to r, inclusive. Then a preference profile is single-peaked if and only if, for each $k = 1, 2, \ldots, n$, every player's k most-preferred coalition partners, including itself, form a connected set. Thus in Example A, when $k = 3$, the most-preferred 3-coalitions of players 1 (123), 2 (213), 3(345), 4 (432), and 5 (543) are all connected sets. For all other k between 1 and 5, it is easy to see that all most-preferred k-coaltions are connected, so the preference profile of Example A is ordinally single-peaked.

In fact, such a preference profile may or may not be geometrically realizable in the following sense: If n points can be positioned along a line such that a player's preference decreases as distance from its position increases, then the preference profile is called *cardinally single-peaked*. To see that this condition is not satisfied in Example A, assume that player i is located at position p_i on the line. Define the distance between two positions, p_i and p_j, to be $d_{ij} = |p_i - p_j|$. From player 3's preference ordering, $d_{54} < d_{53} < d_{32}$, whereas from player 4's ordering, $d_{32} < d_{42} < d_{54}$. This contradiction shows that the preference profile of Example A is ordinally but not cardinally single-peaked.[3]

3. The Fallback and Build-Up Processes

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[3] In other words, the players' ordinal rankings are inconsistent with every possible cardinal representation of their positions.

The *fallback (FB)* process of coalition formation unfolds as follows (Brams, Jones, and Kilgour, 2002; Brams and Kilgour, 2001):

1. The most preferred coalition partner of each player is considered. If two players mutually prefer each other, and this is a majority of players, then this is the majority coalition that forms. The process stops, and we call this a level 1 majority coalition because only first-choice partners are considered.

2. If there is no level 1 majority coalition, then the next-most preferred coalition partners of all players are also considered. If there is a majority of players that mutually prefer each other at this level, then this is the majority coalition (or coalitions) that forms. The process stops, and we call this a level 2 majority coalition.

3. The players successively descend to lower and lower levels in their reported rankings until a majority coalition (or coalitions), all of whose members mutually prefer each other, forms *for the first time*. The process stops, with the set of *largest* majority coalition(s)—not contained in any others at this level—designated FB_1.

What does FB yield in Example A? At level 1, observe that player 1 prefers player 2, and player 2 prefers 1, so we designate 12 as a level 1 coalition, as is coalition 34 also.[4] Descending one level, player 3 likes player 5 and player 5 likes player 3, yielding 35 as a coalition at level 2. Descending one more level, majority coalitions 124 and 234 form for the first time: Each player in these coalitions finds the other two players acceptable at level 3. In summary, we have the following coalitions at each level:

Level 1: 12, 34 *Level 2*: 35 *Level 3*: 124, 234.

Notice that coalitions are listed at the level at which they form, except that subcoalitions are never listed. Thus at level 3, pairs 14, 23, and 24 form but do not appear in our listing, because they are proper subsets of coalitions 124 or 234.

Since coalitions 124 and 234 are the first majority coalitions to form, the process stops, rendering FB_1 = {124, 234}. Observe that players 2 and 4 are common to both coalitions; player 2 prefers coalition 124, and player 4 prefers coalition 234. Obviously, players 1 and 3 prefer the coalition of which each is a member.

Despite the players' preferences being single-peaked, one of the two FB_1 coalitions (124) is *disconnected*: There is a "hole" due to the absence of player 3. The reason that player 3 is excluded from coalition 124 is that whereas players 1 and 2 necessarily rank player 3 higher than player 4 (because of single-peakedness), player 3 ranks players 2 and 1 at the bottom of its preference order. In particular, player 3 does not consider player 1 acceptable at level 3, which excludes player 3 from coalition 124.

While FB is grounded in preferences of players for each other, it could as well be based on their preferences for different features that a policy might include. Thus in Example A, assume players rank five features, {*a*, *b*, *c*, *d*, *e*}, in the same way that they do each other. Then at level l player 1 would find feature *a* acceptable, and at level 2 feature *b*; likewise, player 2 would find both *a* and *b* acceptable at level 2. Consequently, at level 2 (rather than level 1) the coalition 12 would form because of the two players' concurrence on both *a* and *b*. In this example, the level at which coalitions form changes,

[4] These preferences are truthful; we will consider later the possibility that the players strategically misreport their preferences.

but not their membership, as players switch from ranking each other to ranking policy features.[5]

The *build-up* (BU) process of coalition formation is the same as FB, with one major difference. As players descend to lower and lower levels, coalitions form if and only if two or more players consider each other mutually desirable *and consider players not in the coalition less desirable*. In other words, all players in a BU coalition rank each other—and no players outside the coalition—highest. In Example A, this yields the following coalitions at each level:[6]

Level 1: 12, 34 *Level 4*: 12345.

At levels 2 and 3, no new BU coalitions form after coalitions 12 and 34 form at level 1. Only at level 4 does the first majority coalition appear; it is the grand coalition, so BU_1 = {12345}, or just 12345. Note that no member would prefer to be in another 5-coalition—there is none!—proving that this majority coalition is not only *stable* but uniquely so.

Compare this outcome with that produced by FB, which gave FB_1 coalitions 124 and 234 at level 3. These coalitions are *semi-stable*: Even though all their members consider each other acceptable at level 3, some members of each coalition consider some excluded players more desirable as coalition partners. For coalition 124, players 1 and 2 prefer excluded player 3 to included player 4; for coalition 234, player 2 prefers excluded

[5] The number of policy features need not match the number of players. If there are more features than players, coalitions will form later than if there are fewer features than players. For examples, see Brams and Kilgour (2001).

[6] In Brams, Jones, and Kilgour (2002), a different BU model is proposed in a cardinal-utility context. Coalition members fuse into a single player whose position is the average of its members when preferences are defined by points on the real line.

player 1 to included players 3 and 4, and player 3 prefers excluded player 5 to included player 2.[7]

Proposition 1. *BU_1 contains a unique stable coalition. If FB_1 forms at the same level as BU_1, $FB_1 = BU_1$. Otherwise, FB_1 forms at a lower level, in which case all FB_1 coalitions are semi-stable and proper subsets of the BU_1 coalition.*

Proof. Because the grand coalition is a BU coalition, BU_1 is well-defined and never empty. Suppose it contains two majority coalitions. Because both are of the same size, say *k*, they must contain at least one common member *i*.[8] Because the other members of both coalitions must be exactly *i*'s *k* most-preferred coalition partners, the two coalitions in BU_1 must be identical. Hence, BU_1 contains a unique coalition, which we henceforth call BU_1. Moreover, because all members of BU_1 rank each other highest, BU_1 is stable.

Every BU coalition is an FB coalition since the process of descent is the same. If the level of FB_1 is the same as the level of BU_1, then BU_1 belongs to FB_1. Because there cannot be any other coalition in FB_1, then $FB_1 = BU_1$.

Now suppose that the level at which BU_1 forms is *k*, and the level at which FB_1 forms is $j < k$. Consider any coalition *C* in FB_1. While the members of *C* consider each other acceptable at some level, there is at least one player in *C* that prefers some player not in *C*. (If this were not the case, then *C* would be BU_1, and *j* would equal *k*.) This makes *C* semi-stable. Moreover, because both *C* and BU_1 are majority coalitions, they

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[7] The exclusion of preferred players from a coalition, and its manipulability (section 4), are two indicators of its instability. While "there is only a relatively small number of results that guarantee the existence of a 'stable' coalition structure" (Greenberg and Weber, 1993, p. 60), even fewer models offer insight into the step-by-step processes of coalition formation that may (or may not) contribute to stability (Brams, Jones, and Kilgour, 2002).

must have a member in common, say *i*. But BU_1 contains *i* and *i*'s *k* most-preferred coalition partners, whereas *C* contains *i* and a proper subset, with *j* members, of *i*'s most-preferred coalition partners. Therefore, *C* is properly contained in BU_1. Q.E.D.

Example A illustrates Proposition 1. Semi-stable FB_1 coalitions 124 and 234 are contained in stable BU_1 coalition 12345. There are no stable majority coalitions smaller than this grand coalition. Our next example illustrates that BU_1 need not be the grand coalition.

Example B. **1**: 2 3 4 5 **2**: 3 4 1 5 **3**: 4 2 1 5 **4**: 1 2 3 5 **5**: 4 3 2 1.

The FB coalitions at each level are:

Level 2: 13, 23, 24 *Level 3*: 1234.

Whereas no two players consider each other mutually acceptable at level 1, at level 2 two pairs do. At level 3, the first majority coalition forms, so FB_1 = {1234}. But this 4-player coalition is also BU_1, because all its members consider each other, and no others, acceptable. Thus in Example B, the FB and BU processes produce exactly the same majority coalition, which is neither minimal nor grand. To be sure, the grand coalition is also stable, but it seems unlikely to form since players 1 - 4 are united in their opposition to player 5, which they all rank last.

If $FB_1 \neq BU_1$, smaller FB_1 coalitions, which are semi-stable, form earlier in the descent, only later to be subsumed by a larger BU_1 coalition that is stable. Thus in

[8] If preferences are single-peaked and this common member is unique, it must be the median player.

Example A, semi-stable FB_1 coalitions 124 and 234 are proper subsets of stable BU_1 coalition 12345.

Proposition 2. *If preferences are single-peaked, at least one FB coalition of two players must form at level 1.*

Proof. Single-peakedness requires that every player rank an adjacent player highest. Let *C* be the subset of players whose top-ranked coalition partners are players to their right—that is, all players *i* for which $i + 1$ is first choice. Note that $1 \in C$ (because there is no player to the left of 1) and that $n \notin C$ (because there is no player on *n*'s right). Let *r* be the right-most (highest-numbered) player in *C* and note that $r < n$. Then $r + 1$ must be *r*'s top choice, and *r* must be $(r + 1)$'s top choice, so the coalition $\{r, r + 1\}$ must form at level 1. Q.E.D.

In Example A, two coalitions, 12 and 34, form at level 1, whereas in Example B no coalitions form at level 1 because its preference profile is not single-peaked.

Proposition 3. *If preferences are single-peaked, then (i)* FB_1 *coalitions may be disconnected, but (ii)* BU_1 *is connected.*

Proof. Example A, with disconnected FB_1 coalition124, proves (i). To prove (ii), assume that the left-most (lowest-numbered) member of BU_1 is player *l*, and the right-most (highest-numbered) player is *r*, where $l < r$. We next show that BU_1 must also contain any *i* satisfying $l < i < r$. If the level of BU_1 is *k*, then BU_1 comprises *l* and *l*'s *k* most-preferred coalition partners. By single-peakedness, these must be players $l + 1$, $l + 2$, …, $l + k$. It follows that $l + k = r$, and $i \in BU_1$, rendering BU_1 connected. Q.E.D.

It is worth mentioning linkages to other work on coalition-formation processes. Grofman (1982) and Straffin and Grofman (1984) show, in a dynamic model of coalition formation that somewhat resembles our BU model, that coalitions will always be connected in one dimension but not necessarily in two or more dimensions. But under FB, as we illustrated in Example A, coalitions need not be connected, even in one dimension, if preferences are ordinally single-peaked.[9]

We now turn to the question of whether players can manipulate either the FB or the BU processes to their advantage. FB, as we will see, is vulnerable to manipulation, but BU is quite robust.

4. The Manipulability of FB and BU

Call a process *manipulable* if one player, by reporting a preference ranking different from its true preference ranking, can induce a majority coalition that it prefers.

Proposition 4. *FB is manipulable.*

Proof. Consider the following example:

Example C. **1**: 2 3 4 5 **2**: 3 4 1 5 **3**: 2 4 1 5 **4**: 3 5 2 1 **5**: 4 3 2 1.

The FB coalitions at each level are:

Level 1: 23 *Level 2*: 34, 45 *Level 3*: 123, 234 *Level 4*: 12345.

Now assume player 4 misrepresents its preferences as follows:

4: 3 2 5 1.

[9] For references to more recent models in this vein, and tests of these models in party-coalition formation in parliamentary systems, see Brams, Jones, and Kilgour (2002).

Then FB gives the following:

Level 1: 23 *Level 2*: 234 *Level 3*: 123 *Level 4*: 12345.

When player 4 is truthful, FB_1 = {123, 234}, whereas when player 4 misrepresents its preferences, FB_1 = {234}. Because player 4 prefers coalition 234 to coalition 123, misrepresentation, which precludes the possibility of coalition 123, is rational, rendering FB manipulable.[10] Q.E.D.

By comparison, misrepresentation will not be rational for player 4 if the comparison is between the (apparent) BU_1 coalition that forms with misrepresentation and one that forms without misrepresentation. With misrepresentation, BU_1 = 234; without misrepresentation, BU_1 = 12345. Because the larger coalition, 12345, includes both a preferred player (5) and a non-preferred player (1) compared to player 2 in the smaller coalition, 234, one cannot say that player 4 prefers 234 to 12345 or vice versa. Thus, by reporting a preference ranking different from its true preference ranking, player 4 cannot induce a majority coalition that it *assuredly* prefers, illustrating the nonmanipulability of BU.[11]

Proposition 5. *BU is not manipulable.*

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[10] Thus, truthful reporting is not a Nash equilibrium under FB, given the strategies of players are to be truthful or not in a noncooperative game (player 4 would have an incentive not to be truthful in Example C). As we will show next, however, a player cannot assuredly do better by misrepresenting its preferences under BU. Thereby *when* the process of coalition formation terminates affects the stability of outcomes generated under it, underscoring our contention that "the process matters."

[11] To be sure, if there were more information about preferences—in particular, cardinal valuations of different coalitions by each player—it would be possible to say whether player 4 prefers 234 to 12345 or vice versa. In the absence of such information, however, we assume that player 4 does *not* have an incentive to depart from reporting its true preference that yields 12345.

Proof. Assume BU_1 has *k* members. Then a majority coalition that any member *i* of BU_1 prefers cannot have more than *k* members, because it would contain members that *i* ranks lower than those in BU_1 and no members that *i* ranks higher.

Suppose that *i* prefers a majority coalition with fewer members—specifically with *j* members such that $m \leq j < k$, where *m* is a simple majority. To induce this smaller majority coalition through misrepresentation, *i* would have to reduce its ranking of some player *P* not in the *j*-coalition, and raise some player *Q* into the *j*-coalition.

The resulting *j*-coalition, though an apparent BU coalition, does not contain *i*'s $j - 1$ most-preferred coalition partners since it contains player *Q*. Moreover, *i* will not necessarily prefer the *j*-coalition to BU_1, because although it is smaller, it does not contain player *P*, which *i* prefers to *Q*. Thus, *i* is not able to induce through misrepresentation a smaller coalition that it definitely prefers. Q.E.D.

In Example C, as we saw earlier, player 4 can induce through misrepresentation coalition 234—by raising player 2 (*Q*) and lowering player 5 (*P*) in its reported ordering—making it an (apparent) BU_1 coalition at level 2. But player 4 will not necessarily prefer coalition 234, which is an FB_1 coalition, to the grand coalition, 12345.

5. Properties of Stable Coalitions

After the appearance of BU_1, larger and larger BU majority coalitions may—or may not—appear at subsequent levels of descent. Each larger BU majority coalition contains all smaller BU majority coalitions, as illustrated next with a cardinally single-peaked example.

Example D. **1**: 2 3 4 5 6 7 **2**: 1 3 4 5 6 7 **3**: 2 1 4 5 6 7 **4**: 3 2 1 5 6 7

5: 6 4 3 2 1 7 **6**: 5 4 3 2 1 7 **7**: 6 5 4 3 2 1.

Geometrically, we can represent the preferences of these players by placing them at points along the real line:

1 2 3 4 5 6 7.

Thus, for example, the members of pairs 12 and 56 are each other's most-preferred coalition partners, for they are closer to each other in distance to each other than to any other players. Because player 3 prefers players 2 and 1 to player 4, player 3 is farther from player 4 than from player 1. Likewise, players 5 and 6 are farther from player 7 than from player 1, because they rank player 7 last.

We list below all the FB coalitions, not contained in any others at each level, distinguishing those that are also BU coalitions:

Level 1: 12 (BU), 56 (BU) *Level 2*: 13, 23 *Level 3*: 1234 (BU)

Level 4: 2345 *Level 5*: 123456 (BU) *Level 6*: 1234567 (BU).

Observe that the first FB majority coalition to appear, 1234 at level 3, is also a BU majority coalition, so $FB_1 = BU_1 = 1234$. As the descent continues, there is no BU coalition at level 4, but at level 5 a 6-member BU coalition forms. Finally, the grand coalition, which is always a BU coalition, appears at level 6.

Given a cardinally single-peaked preference profile, define the *spread* of a coalition to be the distance between its extreme players. Thus, the spread of coalition 1234 is the distance between player 1 on the left and player 4 on the right, or d_{14}. That this distance is less than d_{45} ensures that coalition 1234 forms before player 5 is brought into the fold. But because player 5 ranks player 6 above all other players, player 5 does not find player 1 acceptable at level 4—only players 2, 3, 4, and 6 are acceptable at this level.

Hence, coalition 12345 is not a BU coalition. On the other hand, because the spread of coalition 123456 is less than the distance between player 6 and player 7, coalition 123456 is a BU coalition at level 5, as is the grand coalition, 1234567, at level 6.

If players' preferences are cardinally single-peaked, it is easy to discern the stable coalitions that form from the players' positions and distances between them.

Proposition 6. *If preferences are cardinally single-peaked, then a subset of players is a BU coalition if and only if it is connected and its spread is less than the distances from each extreme member (other than 1 and n) to the nearest player not in the subset.*

Proof. Suppose that the connected subset $C = \{l, l + 1, \ldots, r\}$ has the properties that either $l = 1$ or $d_{l(l\text{-}1)} > d_{rl}$ and either $r = n$ or $d_{(r+1)r} > d_{rl}$. Clearly, the remaining members of C are l's top choices as coalition partners, and similarly for r. Also, if $i \in C$, $j \in C$, and $k \notin C$, then $d_{ij} < d_{rl} < \min\{d_{ik}, d_{rk}\}$, which shows that the remaining members of C are also i's top choices as coalition partners, making C a BU coalition. The converse is obvious. Q.E.D.

Put more informally, a coalition that is disconnected cannot be a BU coalition, because members would rank the left-out member higher than some members of the coalition. Now assume a coalition is connected but that the distance of an extreme member to an adjacent non-member—either on the left or on the right—is less than the spread. Then the adjacent non-member will be ranked higher by the extreme member than some player in the coalition, so the coalition cannot be a BU coalition.

Proposition 6 provides a characterization of BU coalitions if the players have cardinally single-peaked preferences, thereby enabling one to "read" the BU coalitions from the geometric representation. In general, members of a BU coalition must be sufficiently isolated from players outside it to rank only each other tops.

Whether players' preferences are cardinally single-peaked or not, it is always possible to ensure the existence—or nonexistence—of BU majority coalitions at any level from $m - 1$ (simple majority coalition) to $n - 1$ (grand coalition).

Proposition 7. *BU majority coalitions may appear—or not appear—at any level, up to the appearance of the grand coalition.*

Proof. See Appendix.

In the proof of Proposition 7, we show that, with one exception, it is possible to construct a cardinally single-peaked preference profile whose majority BU coalitions are of any size, or combination of sizes. The exception occurs when $n = 3$; in this case, ordinally and cardinally single-peaked preferences are identical and produce a BU_1 coalition of size 2. When preferences are not single-peaked, BU_1 is of size 3. We will describe this case in detail in section 6.

The algorithm used to prove Proposition 7 yields, in the case of Example D, the following positions p_i of players i:

i:	1	2	3	4	5	6	7

p_i:	0	1	3	7	16	16.5	34

These positions are approximated by the representation given on the real line earlier.

Our construction in the proof of Proposition 7 is a quantitative one that yields the stable majority coalitions (1234, 123456, and 1234567) in this example. But we emphasize that it is the ordinal rankings that determine the stability of coalitions. Thus, BU_1 = 1234 in Example D, because players 1 - 4 all rank each other highest. There is no 5-member BU coalition, because player 5 does not rank players 1 - 4 in its top four places (it ranks player 1 lower, and player 6 higher, than 4th place). There is next a 6-member BU coalition, 123456, because players 1 - 6 rank each other highest. The grand coalition, 1234567, as always, is a BU coalition.

Notice that less-than-majority FB coalitions, but not BU coalitions, form at levels 2 and 4 in Example D. The two 2-member coalitions at level 2 become part of BU coalition 1234 at level 3, and the 4-member FB majority coalition, 2345, at level 4 becomes part of BU majority coalition 123456 at level 5.

The level 1 BU coalition 56 remains apart until level 5. Although player 5 is acceptable to players 1 - 4 at level 4, player 5 does not find player 1 acceptable at this level. Consequently, player 5 does not get absorbed into a majority coalition until the descent reaches level 5, when player 6—player 5's most-preferred coalition partner—also gets absorbed.

Example D illustrates that a less-than-majority BU coalition (56) and a majority BU coalition (1234) can co-exist. However, two different BU majority coalitions, which of necessity overlap, cannot co-exist, as is possible under FB (see Example A for an illustration).

Proposition 8. *If two BU coalitions intersect, then one contains the other.*

Proof. Suppose that two BU coalitions—one with *j* members and the other with $k \geq j$ members—have a member *i* in common. The members of the *j*-coalition must be player *i* and the first $j - 1$ players in *i*'s preference ranking. The members of the *k*-coalition must be player *i* and the first $k - 1$ players in *i*'s preference ranking. Clearly, the *k*-coalition contains the *j*-coalition. Q.E.D.

A consequence of Proposition 8 is that any majority BU coalition of a specific size is unique. In particular, BU_1 contains only a single coalition, as already noted in Proposition 1. In Example D, the BU coalitions that form at levels 1 and 3 are contained in the level 5 BU coalition, which in turn is contained in the level 6 BU coalition. But BU coalition 56, which forms at level 1, is disjoint from BU majority coalition 1234 that forms at level 3. In general, if BU coalitions co-exist, then at most one is of majority size.

In section 4, we showed that members of BU_1 cannot, in general, induce a preferred majority coalition, although they might be able to speed up the formation of an apparent (smaller) BU_1 coalition. But what if a non-member of BU_1 desires to be part of a BU coalition? We next show that such a player can conceivably benefit from a *bandwagon strategy*, which enables it, by misrepresenting its preferences, to be part of a larger BU majority coalition sooner than it would be if it were truthful.

To illustrate, suppose that player 5 in Example D reports its preference ranking to be

5: 4 3 2 1 6 7.

At level 5, BU majority coalition 12345 will form, which includes player 5. By comparison, if player 5 were truthful, the next BU majority coalition to form—after BU_1 = 1234—would be 123456. Because player 6 is player 5's most-preferred coalition partner, player 5 does not necessarily benefit from a bandwagon strategy, even though this strategy puts it into a smaller BU majority coalition at level 4 rather than level 5.

If there is a benefit, it would come by misrepresenting one's preferences in order to join the winning coalition early (i.e., "jumping on the bandwagon"). Indeed, there is evidence from US national party conventions of delegates' shifting to the expected winner—allegedly to demonstrate party unity—as soon as the handwriting of victory is on the wall. Such proclamations of support may well be motivated by cold-blooded calculations of the direct benefit (e.g., a government appointment) that sometimes accrues to former opponents (Brams, 1978).

6. The Probability of Stable Coalitions

Because BU coalitions may or may not exist at every level from simple majority to grand, it is useful to ask when they are most likely to form. To illustrate in a simple case, assume there are three players. Then each player can rank the two others in two ways. For example, player 1 can rank players 2 and 3 as follows:

(i) **1**: 2 3 (ii) **1**: 3 2.

Suppose (i) obtains. If player 2 has the following ranking,

2: 1 3,

BU_1 = 12, whatever the ranking of player 3 (2 cases). Suppose (ii) obtains. If player 3 has the following ranking,

3: 1 2,

BU_1 = 13, whatever the ranking of player 2 (2 cases). Whether the preferences of player 1 are (i) or (ii) (2 cases), BU_1 = 23 if

2: 3 1

3: 2 1.

Altogether, there are 6 cases in which a 2-member BU coalition forms.

By comparison, there are 2 cases in which BU_1 = 123:[12]

1: 2 3	**1**: 3 2
2: 3 1	**2**: 1 3
3: 1 2	**3**: 2 1.

If the $2^3 = 8$ cases are equally likely, the probability that there is a 2-member stable coalition is 6/8 = 3/4. On the other hand, because the grand coalition is always stable, the probability that there is a 3-member stable coalition is 1.

We next generalize this result by finding a formula, $P(n, k)$, for the probability that a k-coalition ($k \geq m$) is stable if all strict preference rankings of n players are equally likely, which we call a *random society*. The following proposition describes the behavior of this probability as the size of a majority coalition increases from m to n.

[12] The preferences of players in these two cases lead to a Condorcet voting paradox, or cyclical majorities.

Proposition 9. *The probability of a BU coalition, starting at $k = m$, decreases to a minimum at some intermediate value of k before increasing to 1 at $k = n$. More precisely, for each $n \geq 3$, there exists an integer $k_0(n) = k_0$, satisfying $m \leq k_0 < n$, such that $P(n, k + 1) < P(n, k)$ if $m \leq k < k_0$, $P(n, k_0 + 1) \geq P(n, k_0)$, and $P(n, k + 1) > P(n, k)$ if $k > k_0$. Moreover, $k_0(n) > m$ whenever $n \geq 5$.*

Proof. See Appendix.

For small values of n and k, we have calculated not only $P(n, k)$ but also $Q(n, k)$, the probability that a k-coalition ($k \geq m$) is stable when all preference rankings of the n players are ordinally single-peaked and equally likely to occur. In addition, using the method of inclusion-exclusion (Brualdi, 1999, pp. 159-168), we have made analogous calculations of the probabilities, $P_1(n, k)$ and $Q_1(n, k)$, that stable majority coalitions form *for the first time*—that is, form at size k but not earlier. All these probabilities are given in Table 1 for values of n between 3 and 9, and all values of k between m and n.

Table 1 about here

In the 3-player case we just described, $P(3, 2) = 0.75$ and $P(3, 3) = 1$, as we showed. When preferences are restricted to those that are ordinally single-peaked, $Q(3, 2) = Q(3, 3) = 1$, because there are no instances in which BU_1 coalitions do not form at level 1.

The probabilities of BU_1 coalitions appearing for the first time are $P_1(3, 2) = 0.75$ and $P_1(3, 3) = 0.25$, because 2 of the 8 preference profiles yield BU_1 coalitions when $k =$

3. But when preferences are ordinally single-peaked, $Q_1(3, 2) = 1$ and $Q_1(3, 3) = 0$, because all 6 preference profiles yield BU_1 coalitions when $k = 2$.

Now consider the $\underline{P}$ values in the Table 1A. For fixed *n*, these probabilities are virtually identical when $n = 7$ and $n = 9$. They first decrease going from $k = m$ to some intermediate value of *k*, and then increase to almost 1 in the case of $P_1(n, k)$, and to 1 in the case of $P(n, k)$.

What Proposition 9 does not indicate, though the numerical values of both $P(n, k)$ and $P_1(n, k)$ do, is that even when $k = m$, these probabilities are very small compared with their values when $k = n$. In other words, almost all BU_1 coalitions in a random society form—in fact, form for the first time—only when the grand coalition appears.

It is evident that the probability that *any* BU majority coalition (except the grand coalition) forms in a random society becomes vanishingly small as *n* increases. This reflects the fact that there is at most one BU coalition at each majority size, and that stability is a certainty only for the grand coalition.

While the probability values in Table 1 may not be empirically accurate, the *distributions* may be qualitatively correct in many situations. As we will see later, majority coalitions in real-life voting bodies often do cluster around simple majority and grand—that is, their distribution is V-shaped between $k = m$ and $k = n$, as the BU model predicts.

To be sure, the bimodal distribution of the probability values for general preferences concentrates almost all the support on the grand coalition. This support is dampened somewhat if preferences are restricted to profiles that are ordinally single-peaked (see the *Q* values in Table 1B). When $n = 5$, for example, $Q_1(5, 3) = 0.333$ and

$Q_1(5, 4) = 0.104$, compared with $P_1(5, 3) = 0.046$ and $P_1(5, 4) = 0.016$. Thus, in the former case there is a 44% chance that BU_1 will not be the grand coalition, whereas in the latter case there is only a 6% chance.

Of course, coalition formation does not generally occur in a random society. Subsets of players, such as political parties in a national legislature, will have members with similar preferences. In such situations, we would expect less-than-grand coalitions to form more frequently and be stable.

We conjecture that the distribution of FB_1 semi-stable majority coalitions in a random society, for which we have not yet made detailed calculations, is also *V*-shaped, whether preferences are general or ordinally single-peaked. But instead of the *V*'s being so heavily weighted on the side of the grand coalition—that the *V* looks more like a *J*—our preliminary calculations indicate that the FB_1 distribution will be considerably flattened, so there will be more weight in the middle as well as around a simple majority.

In the next section, we present empirical data on the distribution of majorities in the Supreme Court and illustrate coalition formation on the Court with two cases. In addition, we present data on the distribution of majorities in the House of Representatives, showing that, like the Court, the distribution is bimodal.

7. The Formation of Majorities on the Supreme Court

In the 9-person US Supreme Court, majority coalitions fit the bimodal probability distribution we found under BU, with majorities tending to be either minimal winning or unanimous. Between 1962 and 1997, we have the following distribution, with the minimum occurring at majority size 7:[13]

Majority size:	5	6	7	8	9
Percent of cases:	24	21	13	14	27

These statistics, however, do not elucidate the process by which justices actually coalesce, either in divided 5-4 majorities and in consensual 9-0 decisions. For this purpose we consider two examples, one in which the Court was unanimous and the other in which it was sharply divided.

We start with the unanimous decision, *United States v. Nixon*, 418 U.S. 683 (1974), the infamous White House tapes case, which was actually decided by an 8-0 vote.[14] Before this case reached the Court, it looked like it would be contentious, based on an unofficial poll by Justice William Brennan. The four Nixon appointees favored the president's claim of executive privilege on withholding the White House tapes, and four took the other side, with Justice Byron White, who usually kept his counsel, inscrutable (Woodward and Armstrong, 1979, p. 289). Because the decision in this case triggered an unprecedented event—the resignation of a president—Schwartz (1996, p. 145) views it as "the most spectacular case decided by the Burger Court."

In the Court's deliberations, Chief Justice Warren Burger, a Nixon appointee, initially sided with the president on executive privilege but was opposed by the rest of the Court. Acting as a kind of rump committee, the other justices redrafted Burger's original

[13] These data are drawn from Edelman and Sherry (2000), who also note the bimodal character of the Supreme Court majority decisions. They explain it in terms of a Markov process of coalition formation, using the Supreme Court voting data to calculate the probability of different absorbing states. By contrast, our work suggests a V-shaped distribution on theoretical grounds, independent of any data.

opinion, with different twosomes and threesomes rewriting its seven parts (Schwartz, 1996, p. 147). While *Nixon* was not the only such case of "decision by committee" during the Burger reign—another exception was *Buckley v. Valeo*, 424 U.S. 1 (1976), an important campaign-finance case (Schwartz, 1996, pp. 143-144)—*Nixon* is particularly insightful on how the build-up toward a final consensus was achieved.

The analysis of *Nixon* that follows does not do justice to Woodward and Armstrong's (1979) 63-page blow-by-blow account, or even Schwartz's (1996) 4-page insider account that includes a page from the personal files of one justice. Although the justices agreed that executive privilege was neither absolute nor unreviewable (especially *in camera*), they differed on how much confidentiality should be accorded presidential conversations and papers.

Because of the paramount importance and extreme public interest in the case, most justices believed that the decision should be the strongest possible—in particular, one delivered as a joint opinion, not written by a particular justice.[15] But Chief Justice Burger refused to go along, saying, "The responsibility is on my shoulders." Schwartz (1996, pp. 145-146) summarizes the situation that then developed:

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[14] Justice William Rehnquist recused himself because of his earlier service in the Nixon administration.

[15] It was not just a matter of writing a strong opinion; the justices were also extremely concerned that President Nixon would not abide by their decision unless it was "definitive," a term used but never defined by Nixon that was widely interpreted to mean unanimous. Brams (1978, ch. 5) argues that Nixon's implicit threat set up a game between Nixon and his two appointees, Burger and Harry Blackmun, who had to decide whether to make the decision unanimous by siding with the 6-person majority; Nixon, in turn, had to decide whether to comply with the decision or not. Rational strategies in the game are for Burger and Blackmun to side with the majority, and for Nixon to comply, which is, of course, what happened and forced Nixon's resignation 17 days later. But treating Burger and Blackmun as a single player belies newer evidence, cited here, indicating that Burger alone was the only significant holdout. For a more informal treatment of strategizing on the Supreme Court that includes a statistical analysis of cases, see Maltzman, Spriggs, and Wahlbeck (2000).

> He [Burger] would prepare the opinion and would circulate its different parts as he finished them. But the drafts he sent around took a more expansive view of presidential power than the others were willing to accept. The Justices refused to go along and virtually wrested the opinion-drafting process from the Chief Justice in order to secure an opinion that they could join.

In granting *certiorari* under a provision that allowed for expedited review of cases "of imperative public importance," five—rather than the usual four—justices needed to agree to review the case. Justice William Brennan took the lead in putting together the votes, making the following calculation (Woodward and Armstrong, p. 290):

> He could count on [William] Douglas and [Thurgood] Marshall. Douglas was eager to come to grips with his long-time antagonist…. They might well be joined by [Potter] Stewart…. [Byron] White could be within reach. Burger was beyond hope…. Blackmun was a possible cert vote…. It was difficult to tell where [Lewis] Powell stood.

In the end, the expedited *certiorari* decision received six votes.

Brennan and Douglas, the core of the coalition against Nixon, worried that the other justices might resent their doing most of the writing if Burger did not go along. Marshall signed on next, and Stewart seemed receptive. But even at the start of deliberations, Brennan found Douglas's draft opinion "rais[ed] issues that were likely to derail consensus" that "did not need to be addressed" (Woodward and Armstrong, 1979, p. 297).

Subsequently, these drafts were modified, but Powell and Stewart still had misgivings, not wanting to put the functionings of government "into a goldfish bowl," exposed to all and undermining the principle of confidentiality. Blackmun had another misgiving, fretting that if the Court reached a consensus against the president, Burger might assign the case to himself. This would raise serious questions about his impartiality if the president were impeached and, subsequently, there was trial in the Senate, over which he as chief justice would preside (Woodward and Armstrong, 1979, pp. 298, 301).

In the end, a liberal coalition comprising Brennan, Douglas, and Marshall, and a moderate coalition comprising Blackmun, Powell, and White, formed, with Stewart the linchpin that brought the two sides together.[16] He was assisted by Brennan, the greatest consensus builder then sitting on the Court.

It is reasonable to suppose that each of the 3-member coalition members ranked Stewart fourth, followed by members of the other coalition, with Burger, who held out until the end, ranked last by all the other justices. Under FB, the first majority coalition of five or more members to form would comprise Stewart and either the liberal coalition and one or more conservatives, or the conservative coalition and one or more liberals. The first BU coalition to form would then include all justices except Burger.

Once the other seven justices had reached agreement, however, the pressure was on Burger, who felt that the others had been "merciless" and that he had been "sandbagged" (Woodward and Armstrong, 1979, pp. 340-341). Ultimately he acquiesced, not wanting

[16] Stearns (2000, p. 236) classifies Blackmun as liberal, though he almost always voted with Burger, whom he classifies as conservative. In fact, Burger and Blackmun had the highest agreement level of any pair of justices in the 1970-74 period (83.5 percent), which is why they were called the "Minnesota twins."

to be a minority of one—but not without claiming authorship of the "committee" decision, which he made his own.[17]

We now turn to the 5-4 case (*Miller v. California*, 413 U.S. 15 (1973)). The Supreme Court has considered many obscenity cases over the past 40 years, and almost all its decisions have been divided and contentious. Instead of examining the build-up of coalitions on any single case, we jump to the final stage of this decision. With the Court deadlocked 4-4, it turned to Blackmun to cast the fifth and decisive vote: Between the two protagonists, "he [Blackmun] could make his new friend Brennan or his old friend the Chief [Burger] author of the majority opinion" (Woodward and Armstrong, 1979, p. 252).

Blackmun worried that Burger's broad definition of obscenity could lead to the banning of much worthwhile literature. When Blackmun threatened not to support Burger, Burger reluctantly agreed to incorporate a more limited definition of obscenity into his opinion. Blackmun then became Burger's fifth vote; subsequently, Brennan revised his opinion as a dissent (Woodward and Armstrong, 1979, p. 252).

In effect, Blackmun ranked Burger coalition members above Brennan coalition members, and they him, making this 5-member coalition both an FB and a BU majority coalition. But, of course, it did not become a winning coalition until Burger made a concession, illustrating that the rankings presumed in the FB and BU models may not be set in stone.

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[17] And how might this decision be described? "It was now virtually impossible to trace the turns and twists the opinion had taken: ideas articulated by Douglas and Powell, modified by Brennan, quickly sketched by the Chief [Burger]; a section substituted by White; a footnote dropped for Marshall; Blackmun's facts embroidered over the Chief's; Stewart's constant tinkering and his ultimatum" (Woodward and Armstrong, 1979, p. 344).

In fact, players change their minds, sometimes because of a change of heart and sometimes for strategic reasons (Burger in the two cases considered here). As an example of a more sincere switch, Brennan renounced all definitions of obscenity after the 1972 case, allying himself with Douglas's more liberal view and ending their 16 years of disagreement on this issue (Woodward and Armstrong, 1979, p. 253).

There are other voting bodies in which our models seem applicable, including the U.S. House of Representatives. Indeed, as in the Supreme Court, there is a bimodal distribution of majority sizes, based on the 12,688 roll call votes between 1955 through 1990:[18]

Percent majority:	50–60	60-70	70-80	80-90	90-100
Percent of roll calls:	26	19	14	11	30

Although the minimum occurs in the 80-90 percent range, not the middle 70-80 percent range, the two modes are the near-majority and near-unanimity ranges, consistent with the BU model.

In the final section we will assess the stability of the grand and smaller majority coalitions in light of both the FB and BU models. We will also suggest some empirical observations and data that might be useful in further testing the model.

8. Conclusions

BU seems most applicable to studying coalition formation in multimember courts and legislatures, in which small subsets of members coalesce and build up to a majority, all of whose members rank each other highest and are therefore stable. FB probably better describes the formation of a governing coalition in parliamentary democracies, wherein disconnected coalitions sometimes form. Because parties in such coalitions rank some parties outside the coalition higher than parties in it, these coalitions are at best semi-stable.[19]

Insofar as voters' preferences are single-peaked, the coalition governments that form are usually connected. Indeed, they are often described by such terms as "left-center" or "center-right." On occasion, however, the left and right do get together and form national-unity governments—sometimes in response to a crisis, like the threat of war—in which many members may be far from each other's favorite coalition partners.

Such semi-stable coalitions, which may be disconnected, tend not to last. According to Riker's (1962) size principle, some of their members grow disaffected and leave if there are insufficient resources to reward them in an oversized coalition.

Through manipulation, players can disrupt semi-stable coalitions by announcing false preferences. Not all these changes, however, may be purely opportunistic. For example, Jim Jeffords, a US Senator from Vermont, switched from the Republican party to become an independent in 2001, turning the Democratic party into the majority party in the Senate. He seems to have been motivated by a genuine belief that he could better serve Vermont and his country by changing his party affiliation. By contrast, we suggested that the preference changes that create bandwagons may not be so sincere.

To conclude, coalition-formation processes affect the size and stability of the coalitions they generate. If stability can be measured by durability, then our models may

[18] We are grateful to Jeffrey E. Cohen for calling our attention to these data, which were compiled by David W. Rohde for the Inter-University Consortium for Political and Social Research in January 1995.

provide insight into why parliamentary coalitions in a country like Italy are less durable than those in the Scandinavian countries, where government coalitions sometimes do not include even a simple majority of members.

The models might also enhance our understanding of the stability of coalitions in other arenas, including international relations. Some international alliances like NATO have been long-lasting, others ephemeral. Is the process that led to the former more BU-like, the latter more FB-like? Our models, we believe, provide tools for investigating such questions.

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[19] Because the significant players in parliamentary democracies are different-size parties, strategic considerations come into play that the FB and BU do not take account of (see note 1). Data on coalition governments in Western Europe can be found in Müller and Strom (2000).

Appendix

Proposition 7. *BU majority coalitions may appear—or not appear—at any level, up to the appearance of the grand coalition.*

Proof. Assume that there are $n > 3$ players and that $m = \lceil (n + 1)/2 \rceil$. The following algorithm positions player i at x_i on the real line. Players' preferences are then defined by Euclidean distance—that is, player i prefers player j over player k if the distance between x_i and x_j is less than the distance between x_j and x_k. The following algorithm constructs cardinally single-peaked preferences in which the only stable majority coalitions are of size k_h, $h = 1, 2, \ldots, t$, where $m \leq k_1 < k_2 \ldots < k_t = n$. Note that $t \geq 1$.

Assign $x_1 = 0$. For $1 < i \leq m$, let player i's position be defined recursively by $x_i = 2x_{i-1} + 1$. By construction, for $1 < i < m$, player i's single-peaked preferences are

i: $i - 1 \quad i - 2 \ \ldots \ 2 \quad 1 \quad i + 1 \quad i + 2 \ldots m - 1 \quad m \quad m + 1 \ldots n$.

Notice that player i most prefers player $i - 1$ as a coalition partner.

The construction considers two separate cases. First suppose $t = 1$ (i.e., only the grand coalition is stable and $k_1 = n$.) Let $x_{m+1} = x_m + 1/2$ and define x_i for $i > m + 1$ by $x_{m+k+1} = x_{m+k} + x_{m-k+1} - x_{m-k}$. Since the distance between x_{m+k+1} and x_{m+k} is the same as the distance between x_{m-k+1} and x_{m-k}, it follows that, for $m + 1 < j < n$, player j most prefers player $j + 1$.

Player m has the following preferences, depending on whether n is odd or even, respectively,

m: $m + 1 \quad m - 1 \quad m + 2 \quad m - 2 \ldots m + k \quad m - k \ldots n \quad 1$; or

m: $m + 1 \quad m - 1 \quad m + 2 \quad m - 2 \ldots m + k \quad m - k \ldots n \quad 2 \quad 1$.

Because preferences are single-peaked, any k-stable coalition for $k \geq m$ must contain the median voter(s). Hence, we can focus on player m's preferences.

Any k-stable majority coalition must comprise the first $k - 1$ players in player m's ranking. If $k - 1 = 2l$, then player m ranks player $m - l$ in position $k - 1$ of its ranking. So a k-stable coalition must contain players m and $m - l$. But, player $m - l$ most prefers player $m - l - 1$. Since player m does not rank player $m - l - 1$ among its $k - 1$ most-preferred coalition partners (because player $m - l$ is the least-preferred player in the coalition), the coalition is not k-stable. Similarly, if $k - 1 = 2l - 1$, then player m ranks player $m + l$ in position $k - 1$ of its ranking.

A k-stable coalition must contain players m and $m + l$. But, player $m + l$ most prefers player $m + l + 1$ as a coalition partner, which is not among player m's top $k - 1$ players. And no majority coalition of size $k < n$ is stable.

Now suppose $t > 1$, so $k_1 < n$. For $m \leq i < k_1 < n$, let j satisfy $i = m + j$. Then player i's position is given by $x_i = x_m$ for $j = 0$ and $x_i = x_{i-1} + \frac{1}{2^j}$ for $j \geq 1$. Similarly, for $k_1 < i \leq k_2$, let j satisfy $i = k_1 + j$. Define $x_i = 2x_m + 2$ for $j = 1$ and $x_i = x_{i-1} + \frac{1}{2^{j-1}}$ for $j > 1$. Finally, for $k_{s-1} < i \leq k_s \leq n$, let j satisfy $i = k_{s-1} + j$. Player i's position is given by $x_i = 2x_{s-1} + 2$ for $j = 1$ and $x_i = x_{i-1} + \frac{1}{2^{j-1}}$ for $j > 1$.

It is possible to describe player i's preferences for $i \geq m$, too. If $k_{s-1} < i \leq k_s$ where j satisfies $i = k_{s-1} + j$, then player i's ranking of coalition partners is

$\boldsymbol{i}$: $i + 1 \quad i + 2 \ldots k_s \quad i - 1 \quad i - 2 \ldots 2 \quad 1 \quad k_s + 1 \quad k_s + 2 \ldots n$.

Any majority stable coalition that contains player i must contain a player j where $1 < j < m$. But, by construction, player j most prefers player $j - 1$. This implies that all of the players less than i must be in the coalition. Since player i's most-preferred coalition partners are players $i + 1$, $i + 2$, $i + 3$, …, and k_s, only coalitions of the form $\{1, 2, \ldots, k_s\}$ are stable. The same argument holds for i where $m \leq i \leq k_h$ as well. Q.E.D.

Proposition 9. *The probability of a BU coalition, starting at k = m, decreases to a minimum at some intermediate value of k before increasing to 1 at k = n. More precisely, for each* $n \geq 3$, *there exists an integer* $k_0(n) = k_0$, *satisfying* $m \leq k_0 < n$, *such that* $P(n, k + 1) < P(n, k)$ *if* $m \leq k < k_0$, $P(n, k_0 + 1) \geq P(n, k_0)$, *and* $P(n, k + 1) > P(n, k)$ *if* $k > k_0$. *Moreover,* $k_0(n) > m$ *whenever* $n \geq 5$.

Proof. Because each player can rank all other players in $(n - 1)!$ ways, there are a total of $[(n - 1)!]^n$ preference profiles. Suppose a k-coalition is stable, where $2 \leq k \leq n$. Then for each member of the coalition, the first $k - 1$ players in its preference ranking must be the other members of the coalition. It follows that the number of preference rankings admitting a stable coalition of k members is

$$\binom{n}{k}[(k-1)!(n-k)!]^{k}[(n-1)!]^{n-k}. \tag{1}$$

To justify this formula, note that there are $\binom{n}{k}$ ways to choose the members of the stable k-coalition. For each of the k members of this coalition, the other members (which come highest in its preference ranking) can be arranged in $(k - 1)!$ ways; for each of the non-members of the coalition, which come lower in its preference ranking, can be arranged in $(n - k)!$ ways. Finally, there are $(n - 1)!$ ways to choose the preference rankings of each of the $n - k$ non-members of the coalition.

Formula (1) is not very useful for small values of k, because it double-counts instances when there are two or more disjoint stable coalitions with k members. But for $m \le k \le n$, formula (1) gives the number of preference rankings, out of $[(n - 1)!]^n$, in which a (unique) stable majority coalition exists (see Proposition 8).

It follows from the preceding argument that in a random society with n players, the probability that there is a BU coalition with k members, where $m \le k \le n$, is

$$P(n,k) = \frac{\binom{n}{k}[(k-1)!(n-k)!]^k[(n-1)!)]^{n-k}}{[(n-1)!]^n}$$

$$= \binom{n}{k}\left[\frac{(k-1)!(n-k)!}{(n-1)!}\right]^k = \frac{\binom{n}{k}}{\binom{n-1}{k-1}^k}. \qquad (2)$$

By construction, $0 < P(n, k) \le 1$. Also, it is easy to check that $P(n, n) = 1$.

If $k < n$, it follows from (2) that

$$\frac{P(n,k+1)}{P(n,k)} = \frac{\binom{n}{k+1}\binom{n-1}{k}^{-(k+1)}}{\binom{n}{k}\binom{n-1}{k-1}^{-k}}$$

$$= \frac{\binom{n}{k+1}}{\binom{n}{k}} \left[\frac{\binom{n-1}{k-1}}{\binom{n-1}{k}} \right]^k \frac{1}{\binom{n-1}{k}}$$

$$= \frac{n-k}{k+1} \left(\frac{k}{n-k} \right)^k \frac{1}{\binom{n-1}{k}}. \tag{3}$$

From (3) it follows that $P(n, k + 1) \geq P(n, k)$ if and only if

$$\binom{n-1}{k} \leq \frac{n-k}{k+1} \left(\frac{k}{n-k} \right)^k. \tag{4}$$

Moreover, $P(n, k + 1) > P(n, k)$ if and only if strict inequality holds in (4).

Now suppose that k satisfies $m \leq k < n$ and that $P(n, k + 1) \geq P(n, k)$. Therefore (4) holds. We show that $P(n, k + 2) > P(n, k + 1)$ by showing that strict inequality holds on the right side of (4) when $k + 1$ is substituted for k. Because $k + 1 > k > n/2$, it follows that

$$\binom{n-1}{k+1} < \binom{n-1}{k} \leq \frac{n-k}{k+1} \left(\frac{k}{n-k} \right)^k < \frac{n-k-1}{k+2} \left(\frac{k+1}{n-k-1} \right)^{k+1}. \tag{5}$$

The final equality of (5) is true because

$$\frac{k}{n-k} < \frac{k+1}{n-k-1}$$

and

$$\frac{n-k}{k+1} < \frac{n-k-1}{k+2}\left(\frac{k+1}{n-k-1}\right) = \frac{k+1}{k+2}. \qquad (6)$$

To verify the inequality of (6), note that it is equivalent to

$$2k^2 - (n-4)k - (2n-1) > 0,$$

which is easily shown to be true because $k > n/2$.

It is not difficult to show that when $n \geq 5$ and $k = m$, the right side of (4) is approximately equal to $e^{3/2} = 4.48$. Therefore, k_0, the minimum value of k for which (4) holds, exceeds m if and only if $n \geq 5$. Q.E.D.

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Table 1

Probabilities of Stable (BU) Coalitions (*P*, *Q*) and First-Forming Stable (BU_1) Coalitions (P_1, Q_1)

A. All preference profiles equiprobable

	k=2	*k=3*	*k=4*	*k=5*	*k=6*	*k=7*	*k=8*	*k=9*
P(3,k)	0.75	1						
$P_1(3,k)$	0.75	0.25						
P(4,k)		0.1481	1					
$P_1(4,k)$		0.1481	0.8519					
P(5,k)		0.0463	0.0195	1				
$P_1(5,k)$		0.0463	0.0166	0.9371				
P(7,k)			2.19×10^{-4}	2.77×10^{-5}	1.50×10^{-4}	1		
$P_1(7,k)$			2.19×10^{-4}	2.76×10^{-5}	1.50×10^{-4}	0.9996		
P(9,k)				7.50×10^{-8}	2.72×10^{-9}	2.67×10^{-9}	5.36×10^{-7}	1
$P_1(9,k)$				7.50×10^{-8}	2.72×10^{-9}	2.67×10^{-9}	5.36×10^{-7}	0.999999

B. All ordinally single-peaked preference profiles equiprobable

	k=2	*k=3*	*k=4*	*k=5*	*k=6*	*k=7*	*k=8*	*k=9*
Q(3,k)	1	1						
$Q_1(3,k)$	1	0						
Q(4,k)		0.4444	1					
$Q_1(4,k)$		0.4444	0.5556					
Q(5,k)		0.3333	0.1875	1				
$Q_1(5,k)$		0.3333	0.1042	0.5625				
Q(7,k)			0.0640	0.0640	0.0308	1		
$Q_1(7,k)$			0.0640	0.0591	0.0272	0.8497		
Q(9,k)				8.55×10^{-3}	2.75×10^{-3}	2.12×10^{-3}	4.81×10^{-3}	1
$Q_1(9,k)$				8.55×10^{-3}	2.54×10^{-3}	2.01×10^{-3}	4.65×10^{-3}	0.9822