Statistical Mechanics of Portfolios of Options

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ABSTRACT

The essential math-physics and associated numerical algorithms underlying a reasonable approach to trading a portfolio of options (PO) is outlined. A description is given of risk-slides, asset disbursement, dynamic balancing, and value indicators.

Keywords: Statistical Mechanics; Trading Financial Markets; Simulated Annealing

1. INTRODUCTION

The essential math-physics and associated numerical algorithms underlying a reasonable approach to trading a portfolio of options (PO) is given.

To perform these analyses, we use methods of nonlinear nonequilibrium multivariate statistical mechanics and associated numerical algorithms which have been tested in several systems, e.g., that typically arise in such diverse fields as finance [1-4], neuroscience [5-11], and combat simulations [12,13].

All aspects presented here have been tested with working code on real market data.

This paper stresses hedging a Δ-hedged option on an index with baskets of Δ-hedged options on its components. However, it is clear that the same methodology can be applied to any option portfolio management, simply by altering specific constraints imposed by the definition of an index.

2. BASIC FORMALISM

2.1. Index-Component Volatility Relationship

Consider an index B with N components S^i.

$$B = \sum_i^N \omega_i S^i \tag{1}$$

This underlying constraint is true in differential form,

$$dB = \sum_i^N \omega_i dS^i \tag{2}$$

Consider each of these $N + 1$ markets as defined as a stochastic process, consistent with the stochastic processes traders use to define their option processes.

$$dB = f^B\, dt + \hat{g}^B dw^B$$

$$dS^i = f^i\, dt + \hat{g}^i dw^i$$

$$< dw^i > = 0$$

$$< dw^i(t) dw^j(t') > = \rho_{ij} \delta(t - t')\, dt$$

$$\rho_{ii} = 1 \tag{3}$$

where the correlation matrix ρ_{ij} is calculated below. The variance of dB is calculated straightforwardly in terms of its correlated components,

$$< dB^2 > - < dB >^2 = (\hat{g}^B)^2$$

$$< dB^2 > - < dB >^2 = < (\sum_i \omega_i dS^i)^2 > - < \sum_i \omega_i dS^i >^2$$

$$(\hat{g}^B)^2 = \sum_i \sum_j \omega_i \omega_j \rho_{ij} \hat{g}^i \hat{g}^j \tag{4}$$

2.2. Specific Option Models

For the Black-Scholes (BS) model,

$$f^B = b^B B$$

$$\hat{g}^B = \sigma^B B$$

$$f^i = b^i S^i$$

$$\hat{g}^i = \sigma^i S^i$$

$$(\sigma^B B)^2 = \sum_i \sum_j \omega_i \omega_j \rho_{ij} \sigma^i \sigma^j S^i S^j \tag{5}$$

As another example, the Ornstein-Uhlenbeck model is defined with a constant diffusion term. In general, we can treat quite generally nonlinear models of underlyings and their derivatives [4,14].

2.3. Probability Distributions

To develop probability distributions, the correlated dw processes are decomposed into independent dz processes,

$$\hat{g}^i dw^i = \sum_m \hat{g}^i_m dz^m$$

$$< dz^m > = 0$$

$$< dz^m(t) dz^n(t') > = \delta(t - t') \delta^{mn} \, dt \tag{6}$$

The covariance matrix g^{ij} and correlation matrix ρ_{ij} are defined by

$$g^{ij} = \sum_m \hat{g}^i_m \hat{g}^j_m$$

$$\rho_{ij} = \frac{g^{ij}}{(g^{ii} g^{jj})^{1/2}} \tag{7}$$

The short-time conditional probability is developed as

$$P[S; t'|S; t] = \frac{1}{(2\pi dt)^{N/2} g^{1/2}} \exp(-L dt)$$

$$L = \frac{1}{2} \sum_i \sum_j (\frac{dS^i}{dt} - f^i) g_{ij} (\frac{dS^j}{dt} - f^j)$$

$$g = \det(g^{ij})$$

$$(g_{ij}) = (g^{ij})^{-1} \tag{8}$$

where L is the proper Lagrangian of the system.

The algebra above is developed in the prepoint Itô discretization, sufficient for this exposition. Details of the induced Riemannian algebra and deeper understanding of the metric g_{ij} are given in other references [1,15-18].

2.4. Implied Volatility Metric

The only practical way of getting the covariance matrix is from historical data. Since the options are taken on univariate underlyings, implied volatilities backed out from the option prices do not have these correlations. However, we build a covariance matrix for implied volatilities by scaling historical volatilities as they appear in bilinear form in g^{ij},

$$\hat{g}^{IV\,i} = s_i \,\hat{g}^i \tag{9}$$

This leads to new g_{ij}'s and g's. In this way we handle the covariances for different strikes regions.

2.5. Stochastic Volatility

Since we will want to examine scenarios that may change by standard deviations (StdDevs) of volatilities, we formulate a $2N + 2$ system, including stochastic volatilities.

$$d\sigma^i = \varepsilon dt + \xi dv^i$$

$$\{dS^i\} \rightarrow \{dM^i\} = \{dS^i, d\sigma^i\} \tag{10}$$

Note that the structure of the extended $2Nx2N$ symmetric covariance matrix is in terms of three NxN matrices,

$$g^{ij} = \begin{pmatrix} A & C \\ C^{\text{sym}} & B \end{pmatrix} = \begin{pmatrix} \text{CoVar}(dS\ dS) & \text{CoVar}(d\sigma\ dS) \\ \text{CoVar}(dS\ d\sigma) & \text{CoVar}(d\sigma\ d\sigma) \end{pmatrix} \tag{11}$$

Experience suggests that the $d\sigma$ distributions are much narrower than the dS distributions, so that we use for the mean of the $d\sigma$ the differenced usual historical or implied volatility of dS. We have examined this in the context of full 2-factor stochastic volatility models for some markets [19].

3. RISK SLIDE

3.1. Greeks

Consider a position Π containing an options and its underlying, perhaps in Δ-hedged proportions. The change in P/L associated with changes in time dt, underlying dS and volatility $d\sigma$ is given by a sum over the "Greeks," as derived from a Taylor expansion of the differenced position,

$$d\Pi = \frac{\partial\Pi}{\partial S}\,dS + \frac{1}{2}\frac{\partial^2\Pi}{\partial dS^2}\,dS^2 + \frac{\partial\Pi}{\partial\sigma}\,d\sigma + \frac{\partial\Pi}{\partial t}\,dt + \frac{\partial^2\Pi}{\partial S\,\partial\sigma}\,dS\,d\sigma + \frac{1}{2}\frac{\partial^2\Pi}{\partial\sigma^2}\,d\sigma^2 + \frac{1}{6}\frac{\partial^3\Pi}{\partial S^2\,\partial\sigma}\,dS^2\,d\sigma + \cdots$$

$$d\Pi = \Delta\,dS + \frac{1}{2}\Gamma\,dS^2 + \mathrm{K}\,d\sigma + \Theta\,dt + \Delta'\,dS\,d\sigma + \frac{1}{2}\mathrm{K}'\,d\sigma^2 + \frac{1}{6}\Gamma'\,dS^2\,d\sigma + \cdots \tag{12}$$

This equation illustrates how different Greeks are properly scaled to appropriate "dollar" values [20]. The Greeks and the covariance matrix are functions of strikes.

The Greeks are calculated using models selected by traders. They can draw from BS, OU or more general unvariable or multivariable models [4,14,19].

3.2. Standard Deviation Moves

The question might be posed "if the index moves 1 or 2 StdDev in its underlying and/or in its volatility, what will be the likely commensurate StdDev moves in its components?" At first glance this question appears to be ill-posed. The answer is not unique as there can be many combinations of moves in the components that will give rise to a given move in the index. Thus, we approach this inverse problem.

Consider a trajectory through all components. On any one component node we can assume that a move in StdDevs in units {-2, -1, 0, +1, +2} is made in dS and/or $d\sigma$. (In practice, we have found it better to use units of fractional StdDevs.) At each such node, we record the associated short-time distribution, calculated by giving dM^i the indicated movement. Associated with this trajectory is a move in the index dB using Eq. (1) and in its volatility $d\sigma^B$ using Eq. (4), using the same ρ_{ij} matrix Eq. (7) for each set of

StdDevs (dependent on the market model). Consider taking a lot of such trajectories. We coarse-grain into dB-$d\sigma^B$ bins the associated values of the movements of the index.

Note that these coarse-grained bins are not used to represent any histogram of the index with respect to dB-$d\sigma^B$ frequencies of occurrence, as each contributor in a given bin is to be weighted by the properly normalized path distributions. The appropriate distribution, Eq. (8), of an index moving to any bin in one folding time is already defined by the specific option model used by traders, which is used to develop Greeks, etc., e.g., the BS model.

3.3. Aggregation of Information

We gather and renormalize the node-probabilities in various ways. For example, the P/L, Eq. (12), associated with a specific movement in the index can be calculated from its Greeks. This can be compared to the likely movements in the components, by weighting by node-probabilities the component P/L's, Eq. (12), of only those paths that contribute to this specific move in the index. This information also can be made Greek-specific, e.g., comparing Γ's, etc.

3.4. Mini-Risk Slide

If a basket of components smaller than the full set of components is chosen to represent the index, e.g., using algorithms described below under Dynamic Balancing, then a mini-risk slide is developed using only nodes of the risk-slide paths corresponding to this basket.

After a basket is determined, a new coarse-grained set of index bins is easily developed using only this subset of components. Their associated node-probabilities are used to develop a mini-risk slide specific to this basket.

3.5. Disbursement of Assets

First, the money invested in a basket of components should be related to the money invested in the index B according to expected values,

$$B \ \mathrm{StdDev}_B = \mathrm{Basket} \ \mathrm{StdDev}_{\mathrm{Basket}} \tag{13}$$

This might be modified by the value indicator described below.

Second, the money invested in the basket of components should be dispersed according the expected values of P/L's of the components, with respect to the renormalized node-probability distributions of the paths developed by the risk slide above.

3.6. ASA MULTI_MIN Sampling of Paths

The task of gathering trajectories for these risk slides is formidable. For example, consider 5 states each for StdDev moves in dS^i and $d\sigma^i$, 25 states per component. For the DOW30, this would represent $25^{30} \approx 10^{42}$ trajectories; for the SP500, this would represent $25^{500} \approx 10^{700}$ trajectories!

We do not have to give up on this statistical mechanics approach. Rather, we can take advantage of this formulation. We can use the node-probability distributions to define a "cost function" to use in a maximum-likelihood importance-sampling approach to find a relatively few "good" paths, i.e., with not too small probabilities, to effectively sample the huge combinatoric space. The cost function is the effective action derived from Eq. (8),

$$A_{eff} = L dt + \frac{1}{2} \ln g + \frac{N}{2} \ln(2\pi dt) \tag{14}$$

Adaptive simulated annealing (ASA) is a powerful sampling algorithm for statistically finding the global minimum of a quite general nonlinear and/or stochastic system [21]. Among the over 100 OPTIONS available for tuning systems is MULTI_MIN. When turned on with a coarse resolution specified for the parameters to be optimized, a user-defined number of close local minima are returned. We take the coarse resolution to be simple integral values of the (fractional) StdDev moves. It seems even 100 trajectories for the DOW30 gives a good representation useful for a risk slide. The actual number of trajectories clearly is dependent on the market selected.

The ASA OPTION ASA_SAMPLE illustrates how generated-parameter and acceptance-cost probability distributions can saved to use the power of ASA importance-sampling for integrals. Here, we are concerned with establishing a few good paths through the system, and we use the market-model probability distributions for weights.

4. DYNAMIC BALANCING

In practice it is difficult, if not desirable, to hedge the full set of components against the index. For example, there are issue of liquidity, handling partial fills, not wanting to buy or sell particular components at an unfavorable time, wanting to take some extra advantage of particular good buys at a time, etc.

In practice, it may be most profitable to dynamically hedge a basket of a subset of all the components of an index, based on criteria of minimizing risk and maximizing profit. The following algorithm does this by "surfing" on a wobbly (stochastic) risk ribbon while presenting sorted and ranked alternatives of best buys and sales of possible changes in baskets.

4.1. Minimizing Risk — StdDev Cost Function

The parameters fit by ASA include the number of rounded lots of contracts bought or sold of the subset of components. The cost function has two main parts, the constraints and the goodness of fit function.

The goodness of fit is the difference between the variance of the subset of components and the variance of the full set of components the subset is trying to emulate.

There can be several kinds of constraints. To avoid multiple regions of similar volatility matches but with different estimates of the index value, the sum of the weighted lots of components is constrained to be within 10% of the index price. We can use a value indicator to fix a specific component to be included or not included in the basket optimization.

4.2. Maximizing Profit — ASA MULTI_MIN Alternatives

MULTI_MIN is turned on during the basket optimization. Thus several minima are offered that have a reasonable fit to the volatility of the full set of components, i.e., all having some reasonable risk control.

All current bid-ask prices are used for each basket in the set that is returned. These are sorted and ranked according to their current profitability if the basket were to be changed. Another sorting and ranking is performed of our value indicator, discussed in the next section, to offer a complementary guide as the best current way to dynamically balance the portfolio.

5. GENERIC VALUE INDICATOR

6. Bubble Indicator

To take advantage of market inefficiencies, trades are initiated when it seem that an indicator, e.g., price, volatility, volatility ratios, etc., are considered to be outliers and when it is expected that the indicator will mean revert back to some statistical norm. We call the excursion out of and back into this statistical norm a "bubble."

The first step is to use a stochastic model to determine the first and second moments of the data. We can use standard statistics which implicitly assumes a Gaussian normal model of the data, or we can use ASA to fit a relatively more general stochastic model to the data.

A window must be used to develop these moments. We can use traders' judgments of these windows, or use some autocorrelation analyses to try to find some representative half-life of the system.

These kinds of fits generally take in most of the data within a fraction of a StdDev of the data. Our bubbles lie outside this set of data, so we perform secondary analyses/fits on the outlier data to determine its first and second moments, histograms of heights, durations and areas of these bubbles to categorize them to be used as trading indicators.

Since the market generally behaves asymmetrically when going up versus down, we typically develop two sets of such bubble indicators.

We can weight these indicators by the projected relatively long-time probability distributions of the market variables, e.g., using our PATHTREE [14] or PATHINT codes [4] which have proven to be very robust and precise in other systems as well [7].

6.1. CMI Correlates

The above bubble analyses are performed on in-sample training and out-of-sample testing data. However, in real-time trading, is of course is to late to take advantage of a bubble if it already has reentered the statistical norm.

Therefore, we look for correlations of another indicator with a shorter lifetime than the bubble indicator, to use as a proxy for what is statistically expected of a given bubble as it leaves the statistical norm.

Good indicators to correlate with the duration or the area of bubbles are canonical momenta indicators (CMI). These have proven useful in other market studies [3,22,23].

CMI are derived from our Lagrangian L. For example, the CMI of a given S^i variable is

$$\mathrm{CMI}_i = \frac{\partial L}{\partial S^i} = \sum_j (dS^j/dt - f^j)\, g_{ij} \tag{15}$$

Even though the correlated cross terms are important, in practice it often is useful to collect the CMI separately for each market, to compare with bubbles separately for each market.

In real-time, since we would like to sort and rank the bubbles as value indicators of expected payoffs, we use the CMI as proxies for this purpose.

7. CONCLUSION

We have used methods of nonlinear nonequilibrium multivariate statistical mechanics to develop short-time probability distributions, together with associated numerical algorithms, e.g., adaptive simulated annealing (ASA), to give a detailed methodology for trading portfolios of options.

Risk-slides are developed to assess distributions of P/L's and of sets of specific Greeks across components that are correlated with an index, as a function of fractional StdDev moves in underlyings and volatilities. This approach also contributes to an algorithm for disbursement of component assets to be correlated with a disbursement on the index.

We develop an algorithm for real-time dynamic balancing to determine sets of reasonably optimal sub-baskets that satisfy risk constraints, while presenting alternative sorted and ranked buys and sells that maximize profits. This can be visualized as "surfing" a risk ribbon to benefit from profitable trades in the risk-management process.

We develop a generic value indicators, based on Training and Testing sets of "bubbles" of exiting and reentering statistical norms of indicators, e.g., prices, volatilities, ratios of other indicators, etc. Faster indicators, canonical momenta indicators (CMI) are correlated to these bubbles to give sets of sorted and ranked expected value indicators.

This methodology can be applied to any option portfolio management, simply by altering specific constraints imposed by the definition of an index.

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