THE UNIVERSITY OF CHICAGO

CONJUGACY CLASSES IN MAXIMAL PARABOLIC SUBGROUPS OF GENERAL LINEAR GROUPS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
AUGUST 2000

ABSTRACT

We compute conjugacy classes in maximal parabolic subgroups of the general linear group. This computation proceeds by reducing to a "matrix problem". Such problems involve finding normal forms for matrices under a specified set of row and column operations. We solve the relevant matrix problem in small dimensional cases. This gives us all conjugacy classes in maximal parabolic subgroups over a perfect field when one of the two blocks has dimension less than 6. In particular, this includes every maximal parabolic subgroup of $\mathrm{GL}_n(k)$, for $n < 12$ and k a perfect field. If our field is finite of size q, we also show that the number of conjugacy classes, and so the number of characters, of these groups is a polynomial in q with integral coefficients.

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PREFACE

A great deal of progress has been made recently towards describing the representation theory of reductive algebraic groups. For example, the study of representations of finite reductive groups was greatly advanced by the work of Deligne and Lusztig (1976) and has been an active field of research. Conjugacy classes in reductive groups have been investigated by Springer and Steinberg (1970). In comparison, little is known for solvable algebraic groups (Isaacs and Karagueuzian, 1998). Even less is known about groups which are neither reductive nor solvable.

The parabolic subgroups of the general linear group are among the simplest such "mixed" groups. Each is a semidirect product of the unipotent radical (which is a nilpotent normal subgroup) with a Levi complement (which is a reductive group). Representations of the Levi complement can be inflated to the parabolic—this is vital to the inductive step of classifications of representations of the general linear group. Drozd (1992) generalized these inflated representations to a much larger class of mixed groups and showed they are Zariski dense in the set of irreducible representations. Almost nothing is known about the other representations of the maximal parabolics. Richardson et al. (1992) have studied the conjugacy classes in parabolic subgroups with abelian unipotent radical which are contained in the unipotent radical. A series of papers has subsequently been written on this subject (Hille and Röhrle, 1999, 1997; Jürgens and Röhrle, 1998; Popov and Röhrle, 1997; Popov, 1997; Röhrle, 1996, 1997a,b, 1999). Some of these results use matrix problems similar to the ones discussed here.

In this thesis, we describe all the conjugacy classes in maximal parabolic subgroups of a general linear group over a perfect field when one of the two blocks has dimension less than 6. In particular, this includes every maximal parabolic subgroup of $\mathrm{GL}_n(k)$ for $n < 12$ and k a perfect field. We proceed by reducing to a "matrix problem". Such problems involve finding normal forms for matrices under a specified set of row and column operations. These problems have been extensively studied since they were described by Nazarova and Roĭter (1972).

We use standard group theoretic notation as in Alperin and Bell (1995). The equality symbol denotes natural isomorphism as well as strict equality.

I would like to thank my advisor, Jon Alperin. I am also grateful to Laszlo Babai, Cedric Bonnafé, Joseph Chuang, Matt Frank, George Glauberman, David Hemmer, Paul Li, Paul Sally, and Larry Wilson for their suggestions and support. This research was done under the support of a Fulbright graduate fellowship.

CHAPTER 1
INTRODUCTION

The aim of this thesis is to compute conjugacy classes in maximal parabolic subgroups of the general linear group. This can be considered as a first step towards a better understanding of the representation theory of parabolic subgroups of reductive algebraic groups.

Fix a field k; we usually require it to be perfect. In particular, our results hold for any field which is algebraically closed, characteristic zero, or finite. Most of our techniques are borrowed from the theory of finite dimensional algebras over a field. Many of the groups we deal with are *multiplicative groups* of k-algebras. For example, $\mathrm{GL}_n(k) = \mathrm{M}_n(k)^\times$, the multiplicative group of the full algebra of $n \times n$ matrices over k.

We define a parabolic subgroup P^λ to be the multiplicative subgroup of the block triangular algebra

$$\begin{pmatrix} \mathrm{M}_{\lambda_1}(k) & \mathrm{M}_{\lambda_1,\lambda_2}(k) & \cdots & \mathrm{M}_{\lambda_1,\lambda_s}(k) \\ 0 & \mathrm{M}_{\lambda_2}(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{M}_{\lambda_{s-1},\lambda_s}(k) \\ 0 & \cdots & 0 & \mathrm{M}_{\lambda_s}(k) \end{pmatrix}.$$

Here $\mathrm{M}_{m,n}(k)$ is the set of $m \times n$ matrices over k and λ is a *composition* of n, that is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of natural numbers whose sum is $n = |\lambda|$. A few words need to be said about this definition. Firstly note that when we write a matrix with sets for its entries, we just mean the set of matrices with entries in the given sets. We often use algebras and groups consisting of matrices with different kinds of entry in different positions. Although an element of P^λ is defined as a matrix of matrices, called a *block matrix*, we identify it with a matrix over k in the obvious way.

Recall that P^λ is a semidirect product $U \rtimes L$; that is, $UL = P^\lambda$, $U \cap L = 1$ and U is normal. Here U is the *unipotent radical*

$$\begin{pmatrix} I_{\lambda_1} & \mathrm{M}_{\lambda_1,\lambda_2}(k) & \cdots & \mathrm{M}_{\lambda_1,\lambda_s}(k) \\ 0 & I_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{M}_{\lambda_{s-1},\lambda_s}(k) \\ 0 & \cdots & 0 & I_{\lambda_s} \end{pmatrix},$$

and L is the *Levi complement*

$$\begin{pmatrix} \mathrm{GL}_{\lambda_1}(k) & 0 & \cdots & 0 \\ 0 & \mathrm{GL}_{\lambda_2}(k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{GL}_{\lambda_s}(k) \end{pmatrix} = \mathrm{GL}_{\lambda_1}(k) \oplus \cdots \oplus \mathrm{GL}_{\lambda_s}(k)\,.$$

Further U is solvable and L is reductive.

In the current work, we are primarily concerned with maximal parabolics; that is, those of the form $P^{(m,n)}$. It seems likely however that the techniques developed here will also be useful for more general parabolics. The unipotent radical of a maximal parabolic is abelian and can be identified with $\mathrm{M}_{m,n}(k)$ since

$$\begin{pmatrix} I_m & v \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & w \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & v+w \\ 0 & I_n \end{pmatrix}.$$

The Levi subgroup acts on the unipotent radical in the natural manner:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 1 & AvB^{-1} \\ 0 & 1 \end{pmatrix}.$$

The following result allows us to describe the conjugacy classes in a semidirect product over an abelian groups.

Lemma 2.1.1. *Let the group G be a semidirect product $U \rtimes L$ with U abelian. Then, for every h in L, the conjugacy classes in G intersecting Uh are in one-to-one corre-*

spondence with the orbits of $C_L(h)$ *on* $C^U(h) = U/[U, h]$.

This lemma shows that the conjugacy classes in $P^{(m,n)} = U \rtimes L$ can be found in four steps:

1. Find a set $\{h\}$ of conjugacy class representatives for L.

2. Compute the centralizer $C_L(h)$ for each h.

3. Compute the cocentralizer $C^U(h) = U/[U, h]$ for each h.

4. Find a set of representatives for the orbits of $C_L(h)$ acting on $C^U(h)$.

The first step reduces immediately to finding conjugacy class representatives for $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$. We want these representatives to be as close to diagonal as possible. When all the eigenvalues are rational (ie. in k), we can use the Jordan normal form. If we have an irrational eigenvalue, we use the generalized Jordan normal form described in Section 2.2. Instead of having Jordan blocks, we have blocks of the form

$$J_l(A) = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ I_d & A & 0 & \cdots & 0 \\ 0 & I_d & A & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & I_d & A \end{pmatrix}$$

where A is the companion matrix of a separable, irreducible polynomial and l is the number of times A appears. It turns out that we can assume, without loss of generality, that the matrices in $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ both have one and the same (generalized) eigenvalue. So we can confine our attentions to matrices of the form

$$J_\lambda(A) = \bigoplus_{i=1}^{s} J_{\lambda_i}(A),$$

where λ is a *partition*—a composition whose parts are in decreasing order. We can also show that the centralizers and cocentralizers are always the same as in the rational case, except that we have to work over a separable extension of k, rather than k itself. We just consider the matrices with rational eigenvalue α for the remainder of this introduction.

Computing centralizers in L also reduces to computing them in $\mathrm{GL}_n(k)$. First we find $C_{\mathrm{M}_n(k)}(J_\lambda(\alpha))$. This algebra tends to be unwieldy, so we show it is isomorphic to $k[x]_\lambda$, which is a quotient of a matrix algebra over $k[x]$. This can be handled more easily because the dimension of the matrices are smaller, even though the entries are from larger rings. The representative h of L is of the form $J_\mu(\alpha) \oplus J_\nu(\alpha)$, where $|\mu| = m$ and $|\nu| = n$. Hence the group $C_L(h)$ will be isomorphic to $k[x]_\mu{}^\times \oplus k[x]_\nu{}^\times$.

Next we find a similar description of the cocentralizer as a module over this centralizer. Here we use the fact that U is identified with $\mathrm{M}_{m,n}(k)$, which is in turn naturally isomorphic to $k^m \otimes (k^n)^t$, where k^m is the m-dimensional column space and $(k^n)^t$ is the n-dimensional row space. We can use this description of U to show that $C^U(h)$ is isomorphic to a module, $k[x]_{\mu\times\nu}$, which has a particularly nice structure as a set of matrices with entries in different k-algebras.

Finally we wish to find representatives for the orbits of the centralizer on the cocentralizer. We show that $k[x]_\mu{}^\times \oplus k[x]_\nu{}^\times \cong C_L(h)$ is generated by certain matrices analogous to the elementary matrices. This implies that the orbits of this group on $k[x]_{\mu\times\nu} \cong C^U(h)$ are the same as the orbits under a specified set of row and column operations. A problem which involves finding representatives for matrices under the action of a given set of row and column operations is called a *matrix problem*. Such problems have been studied for many years by the Kiev school and they are closely linked to the representation theory of algebras—a good reference on them is Gabriel and Roiter (1997). There is a classification of matrix problems by type: finite type problems have finitely many orbits whose representatives can be independent of the field; while for infinite type, the number of orbits depends on the field and is infinite whenever the field is. Infinite type problems can further be divided into tame type, where an explicit solution is known; and wild type, which are equivalent to the classical unsolved problem of finding a normal form for a pair of noncommuting matrices. The Brauer-Thrall conjecture, proved by Nazarova, L. A.

and Roĭter, A. V. (1973), shows that every matrix problem is either finite, tame or wild.

We prove that our matrix problem is finite type if, and only if, one of the partitions is small:

Theorem 3.1.1. *The matrix problem* $k[x]_{(4,2)\times(4,2)}$ *is infinite type.*

Theorem 3.1.2. *The matrix problem* $k[x]_{\mu\times\nu}$ *is finite type for* ν *arbitrary and* μ *of the form* $(2^{m_2}, 1^{m_1})$, $(r, 1^{m_1})$ *or* $(3, 2)$. *In particular,* $k[x]_{\mu\times\nu}$ *is finite type whenever* $|\mu| < 6$.

This immediately gives us the following result on conjugacy classes.

Corollary 3.1.3. *Computing the conjugacy classes in* $P^{(m,n)}$ *over a perfect field reduces to matrix problems of finite type if, and only if, either* $m < 6$ *or* $n < 6$. *In particular, computing conjugacy classes in the maximal parabolics of the general linear group over a perfect field reduces to finite type problems if, and only if, the dimension is less than 12.*

If the field is finite of size q, then the number of conjugacy classes, and so the number of characters, of these groups is a polynomial in q with integral coefficients.

As an application of this result we recompute the conjugacy classes in the well known *affine general linear group*

$$\mathrm{AGL}_n(k) = \begin{pmatrix} 1 & (k^n)^t \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}.$$

Theorem 3.2.1. *A set of conjugacy class representatives for* $\mathrm{AGL}_n(k)$ *is given by* *the matrices*

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline & N & 0 \\ & 0 & E \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|cccccc} 1 & 0 & 0 & \cdots & e & \cdots & 0 \\ \hline & N & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & E_1 & & & & 0 \\ & \vdots & & \ddots & & & \vdots \\ & 0 & & & E_i & & 0 \\ & \vdots & & & & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & \cdots & E_m \end{array}\right),$$

where $E = \bigoplus_{i=1}^{r} E_i$, $E_i = J_i(1)^{\oplus l_i}$, N *does not have 1 as an eigenvalue and is in normal form, and* $e = (1, 0, 0, \dots)$.

CHAPTER 2
CENTRALIZERS AND COCENTRALIZERS

2.1 Conjugacy classes in semidirect products over abelian groups

We prove a lemma which describes the conjugacy classes in a semidirect product with abelian normal subgroup. This is analogous to the description of the characters proved by Clifford theory (Curtis and Reiner, 1990, section 11B).

Lemma 2.1.1. *Let the group G be a semidirect product $U \rtimes L$ with U abelian. Then, for every h in L, the conjugacy classes in G intersecting Uh are in one-to-one correspondence with the orbits of $C_L(h)$ on $C^U(h) = U/[U,h]$.*

Proof. Let u, u_1, u_2 and v be elements of U, let h and k be elements of L. Then $(vk)(uh)(vk)^{-1} = v \cdot kuk^{-1} \cdot khk^{-1}v^{-1}$, which is equal to $kuk^{-1} \cdot v \cdot khk^{-1}v^{-1}$ since v and kuk^{-1} are in U which is abelian. This can be rearranged to the product of $kuk^{-1}[v, khk^{-1}] \in U$ and $khk^{-1} \in L$. So vk conjugates u_1h to u_2h if, and only if, k centralizes h and $ku_1k^{-1}[v,h] = u_2$. □

In the remainder of this chapter, we apply this lemma to the maximal parabolic subgroup

$$G = P^{(m,n)} = \begin{pmatrix} \mathrm{GL}_m(k) & \mathrm{M}_{m,n}(k) \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}$$

with *unipotent radical*

$$U = \begin{pmatrix} I_m & \mathrm{M}_{m,n}(k) \\ 0 & I_n \end{pmatrix}$$

and *Levi complement*

$$L = \begin{pmatrix} \mathrm{GL}_m(k) & 0 \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}.$$

In Section 2.2 we describe the generalized Jordan normal form, which gives us a set of conjugacy class representatives for L. Then, for every such representative h, we compute the centralizer $C_L(h)$ in Sections 2.3 and 2.4 and the cocentralizer $C^U(h)$ in Section 2.5. Along the way we show that finding representatives of the orbits of $C_L(h)$ on $C^U(h)$ is equivalent to a matrix problem. Note that Sections 2.2, 2.3 and 2.5 each have two subsections: in the first we consider matrices with rational eigenvalues, in the second we show that for an irrational separable eigenvalue we get essentially the same thing, but over the extension of k with the eigenvalue adjoined. If you are only interested in algebraic closed fields, you need only read the first subsection in each section.

2.2 Generalized Jordan normal form

We need a set of conjugacy class representatives for the Levi complement L, so that we can apply the lemma of the previous section. Since this group is just $\mathrm{GL}_m(k) \oplus \mathrm{GL}_n(k)$, it suffices to give representatives of the similarity classes of invertible matrices. For our purposes, these representatives should be rational (ie. defined over k) but also as close to diagonal as possible. The generalized Jordan normal form has these properties. A proof that any matrix over a perfect field can be conjugated to a matrix in this form can be found in Mal'cev (1963).

Recall that the *direct sum* of the matrices A and B is

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

with a similar definition applying for any finite number of matrices. Corresponding to every direct sum decomposition of a matrix, there is a decomposition of the underlying vector space into a direct sum of subspaces invariant under that matrix. An element of $\mathrm{M}_n(k)$ which is not similar to a direct sum of smaller square matrices over k is called *indecomposable*. Every square matrix is a direct sum of indecomposables.

Fix an $n \times n$ matrix A. Given p, a monic irreducible polynomial over k, the subspace

$$V_p = \{v \in k^n : p(A)^m v = 0 \text{ for some natural number } m\}$$

is easily seen to be A-invariant. If this subspace is nonzero, we say p is a *generalized eigenvalue* of A and V_p is the corresponding *generalized eigenspace*. In fact, k^n is a direct sum of the generalized eigenspaces, so we get a corresponding decomposition of A. In particular, an indecomposable matrix has a unique generalized eigenvalue whose generalized eigenspace is the entire underlying vector space.

Rational case

Suppose every generalized eigenvalue of A is of the form $p(t) = t - \alpha$, for some α in k. Then each such α is also an eigenvalue. So A is similar to $\bigoplus_\alpha A_\alpha$ where A_α has unique eigenvalue α. The indecomposables with eigenvalue α are just the Jordan blocks $J_m(\alpha)$. Hence A_α is similar to $J_{\lambda_\alpha}(\alpha)$ for some partition λ_α and A is similar to $\bigoplus_\alpha J_{\lambda_\alpha}(\alpha)$ where α runs over the eigenvalues of A and $n = \sum_\alpha |\lambda_\alpha|$. Of course, this is just the ordinary Jordan normal form.

Irrational case

Now suppose the generalized eigenvalues of A are arbitrary monic, separable, irreducible polynomials.

Let p be such a polynomial and write $K = k(\alpha)$, where α is a root of p in the algebraic closure $\bar{k}$. Then K is a separable field extension of k and multiplication by α induces a k-linear transformation $K \to K$. The matrix of this transformation with respect to the k-basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is just the companion matrix

$$C_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{d-2} & a_{d-1} \end{pmatrix}.$$

This is an indecomposable matrix with generalized eigenvalue p. More generally, every indecomposable matrix with generalized eigenvalue p is similar to the matrix $J_m(C_p)$ for some natural number m.

So the matrix A is similar to $\bigoplus_p J_{\lambda_p}(C_p)$, where p runs over the generalized eigenvalues of A and the λ_p are partitions with $n = \sum_p |\lambda_p| \deg(p)$. This is the *generalized Jordan normal form* of A.

2.3 Centralizers in general linear groups

We describe the centralizers in general linear groups. Our description is explicit provided that all the generalized eigenvalues of our matrix are separable over k. We first compute the centralizer in $M = \mathrm{M}_n(k)$, then use this to find the centralizer in $G = \mathrm{GL}_n(k)$. We give a description of the centralizer suitable for our subsequent calculations, but this is a classical result, which can be found in (Jacobson, 1985, Section 3.11).

Let $A = \bigoplus_p J_{\lambda_p}(C_p)$ be an element of G in generalized Jordan normal form. Corresponding to this decomposition of A is a decomposition of k^n into a direct sum of generalized eigenspaces V_p. Further, a matrix B that centralizes A also centralizes $p(A)$, and so V_p is invariant under B. Hence $C_M(A) = \bigoplus_p C_{\mathrm{M}_{n_p(k)}}(J_{\lambda_p}(C_p))$ where $n_p = |\lambda_p| \deg(p)$. We may now assume, without loss of generality, that A has a single generalized eigenvalue p with corresponding partition $\lambda = \lambda_p$.

Rational case

First consider $p(t) = t - \alpha$, ie. the matrix has rational eigenvalue α. Now $A = J_\lambda(\alpha) = \alpha I_n + J_\lambda(0)$ and αI_n is in the center of M, so $C_M(A) = C_M(J_\lambda(0))$. We write J_λ for $J_\lambda(0)$ and $k[x]_n$ for $k[x]/(x^n)$. With $\lambda = (n)$ there is an isomorphism $C_M(J_n) \to k[x]_n$ taking J_n to x. The natural action of $C_M(J_n)$ on k^n is equivalent to the regular action of $k[x]_n$ on itself. We generalize this to an arbitrary partition.

Take a matrix B centralizing J_λ and write it in block form

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1s} \\ \vdots & \ddots & \vdots \\ B_{s1} & \cdots & B_{ss} \end{pmatrix},$$

where B_{ij} is a $\lambda_i \times \lambda_j$ matrix. Then $BJ_\lambda = J_\lambda B$ implies $B_{ij}J_{\lambda_j} = J_{\lambda_i}B_{ij}$ for all i and j. If we write $B_{ij} = (b_{l,m})$, this becomes $b_{l,m+1} = b_{l-1,m}$ and $b_{1\lambda_j} = b_{l,\lambda_j} = b_{\lambda_i,m} = 0$ for $l = 2, \ldots, \lambda_i$ and $m = 1, \ldots, \lambda_j - 1$. Hence B_{ij} is

$$\left(\begin{matrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{\lambda_i} & \cdots & b_1 & b_0 \end{matrix} \quad 0 \right) \text{ or } \begin{pmatrix} & & 0 & \\ b_{\lambda_j-\lambda_i} & 0 & \cdots & 0 \\ b_{\lambda_j-\lambda_i+1} & b_{\lambda_j-\lambda_i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{\lambda_j} & \cdots & b_{\lambda_j-\lambda_i+1} & b_{\lambda_j-\lambda_i} \end{pmatrix}$$

for $\lambda_i \leq \lambda_j$ or $\lambda_i \geq \lambda_j$ respectively. The appearance of the full matrix B is illustrated by Figure 2.1 for $\lambda = (5, 3, 3, 2)$, where each line represents entries which are equal and blank spaces represent zero entries. Define $X^a_{c\times d}$ to be the $c \times d$ matrix whose (i, j)-entry is 1 if $i = j + a$ and 0 otherwise. Then B_{ij} can be written as $\sum_{a=0}^{\lambda_i} b_a X^a_{\lambda_i\times\lambda_j}$ or $\sum_{a=\lambda_j-\lambda_i}^{\lambda_j} b_a X^a_{\lambda_i\times\lambda_j}$ respectively. It is easily checked that $X^a_{c\times d}X^b_{d\times e} = X^{a+b}_{c\times e}$ for

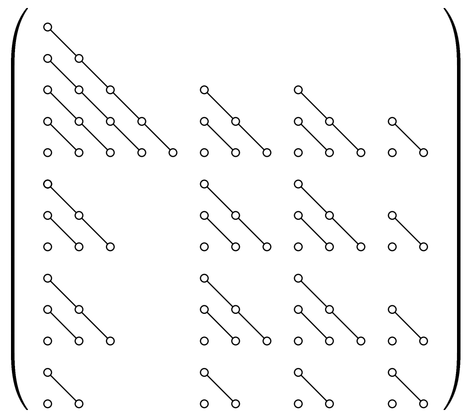


Figure 2.1: An element of the centralizer

any nonnegative integers a, b, c, d, e. This identity gives us an algebra homomorphism

$$\begin{pmatrix} k[x] & x^{\lambda_1-\lambda_2}k[x] & \cdots & x^{\lambda_1-\lambda_s}k[x] \\ k[x] & k[x] & \cdots & x^{\lambda_2-\lambda_s}k[x] \\ \vdots & \vdots & \ddots & \vdots \\ k[x] & k[x] & \cdots & k[x] \end{pmatrix} \to C_M(A)$$

which takes x^a in the (i,j)-entry to $X^a_{\lambda_i\times\lambda_j}$ in the (i,j)-block. The matrices $X^a_{c\times d}$ are linearly independent for $a = 0, \ldots, \min(c,d)$ and zero for $a > \min(c,d)$. So this homomorphism is surjective and its kernel is

$$\begin{pmatrix} (x^{\lambda_1}) & (x^{\lambda_1}) & \cdots & (x^{\lambda_1}) \\ (x^{\lambda_2}) & (x^{\lambda_2}) & \cdots & (x^{\lambda_2}) \\ \vdots & \vdots & \ddots & \vdots \\ (x^{\lambda_s}) & (x^{\lambda_s}) & \cdots & (x^{\lambda_s}) \end{pmatrix}.$$

Hence $C_M(J_\lambda)$ is isomorphic to the quotient algebra

$$k[x]_\lambda = \begin{pmatrix} k[x]_{\lambda_1} & x^{\lambda_1-\lambda_2}k[x]_{\lambda_1} & \cdots & x^{\lambda_1-\lambda_s}k[x]_{\lambda_1} \\ k[x]_{\lambda_2} & k[x]_{\lambda_2} & \cdots & x^{\lambda_2-\lambda_s}k[x]_{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ k[x]_{\lambda_s} & k[x]_{\lambda_s} & \cdots & k[x]_{\lambda_s} \end{pmatrix}.$$

Note that elements of $k[x]_\lambda$ can be multiplied naively: just multiply the matrices normally, then remove any monomial whose degree is too large for its position in the matrix.

Next we consider the natural action of this algebra. The left action of $C_M(J_\lambda)$ on k^n is identical to its action on its own first column. This is easily seen to be isomorphic to the action of $k[x]_\lambda$ on its first column which is

$$k[x]_{\lambda\times-} = \begin{pmatrix} k[x]_{\lambda_1} \\ \vdots \\ k[x]_{\lambda_s} \end{pmatrix}.$$

We also need the right action on row vectors. There is an isomorphism $D : k[x]_\lambda \to {k[x]_\lambda}^t$ given by

$$\begin{pmatrix} a_{11} & x^{\lambda_1-\lambda_2}a_{12} & \cdots & x^{\lambda_1-\lambda_s}a_{1s} \\ a_{21} & a_{22} & \cdots & x^{\lambda_2-\lambda_s}a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ x^{\lambda_1-\lambda_2}a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x^{\lambda_1-\lambda_s}a_{s1} & x^{\lambda_2-\lambda_s}a_{s2} & \cdots & a_{ss} \end{pmatrix}.$$

We consider $k[x]_\lambda$ to act naturally on $k[x]_{-\times\lambda} = (k[x]_{\lambda\times-})^t$ via this isomorphism.

We now turn to the multiplicative group of $k[x]_\lambda$, which is isomorphic to $C_G(A) = C_M(A)^\times$. Now $B \in k[x]_\lambda$ can be written

$$B = B_0 + (xI)B_1 + (xI)^2B_2 + \cdots + (xI)^{\lambda_s-1}B_{\lambda_s-1},$$

where the entries of each B_i are in k and xI is the matrix with x in each diagonal entry and zero elsewhere. But $(xI)^{\lambda_s} = 0$, so xI is nilpotent and B is invertible exactly when B_0 is. Writing λ as $(r^{l_r}, \ldots, 2^{l_2}, 1^{l_1})$, where l_j is the number of times the part j occurs, we see that B_0 is block lower triangular of the form

$$B_0 = \begin{pmatrix} B_{rr} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ B_{1r} & \cdots & B_{11} \end{pmatrix},$$

with B_{ij} an $l_i \times l_j$ matrix. Hence B is invertible if, and only if, all the matrices B_{ii} are invertible, and we have described ${k[x]_\lambda}^\times$.

Irrational case

Finally we tackle the case where p is nonlinear and separable. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_d$ be the distinct roots of p in the algebraic closure $\bar{k}$. We take A to be $J_\lambda(C_p)$. Recall that C_p is the k-matrix of multiplication by α on $K = k(\alpha)$ with respect to the basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$. Hence $C_{\mathrm{M}_d(k)}(C_p)$ is isomorphic to the centralizer of α in $\mathrm{End}_k(K)$, which is just $\mathrm{End}_K(K) = K$. On the other hand, the Jordan normal form of C_p is $D_p = \mathrm{diag}(\alpha_1, \ldots, \alpha_d)$, so $tD_pt^{-1} = C_p$ for some t in $GL_d(\bar{k})$. The centralizer of D_p in $\mathrm{M}_d(\bar{k})$ is the algebra of diagonal matrices $\bar{k}^{\oplus d}$. Hence

$$C_{\mathrm{M}_d(\bar{k})}(C_p) = t(C_{\mathrm{M}_d(\bar{k})}(D_p))t^{-1} = t(\bar{k}^{\oplus d})t^{-1},$$

so $C_{\mathrm{M}_d(k)}(C_p) \cong K$ is the set of elements of $t(\bar{k}^{\oplus d})t^{-1}$ defined over k.

Turning now to $J_\lambda(C_p)$, we see it is conjugated to $J_\lambda(D_p)$ by $t^{\oplus n}$. Further, $J_\lambda(D_p)$ is conjugated to $J_\lambda(\alpha_1) \oplus \cdots \oplus J_\lambda(\alpha_d)$ by the obvious permutation of basis elements. From the rational case, we know that the centralizer in $\mathrm{M}_n(\bar{k})$ of this last matrix is isomorphic to ${\bar{k}[x]_\lambda}^{\oplus d}$. Reversing these conjugations we get $C_{\mathrm{M}_n(\bar{k})}(J_\lambda(D_p)) \cong (\bar{k}^{\oplus d})[x]_\lambda$ and $C_{\mathrm{M}_n(\bar{k})}(J_\lambda(C_p)) \cong \left(t(\bar{k}^{\oplus d})t^{-1}\right)[x]_\lambda$. So $C_M(J_\lambda(C))$ is just the set of elements of this algebra defined over k, which is $K[x]_\lambda$. The natural action and multiplicative group can now be computed as in the rational case.

An example should make this process clearer. Suppose $k = \mathbb{Q}$, $p(t) = t^2 - 3$, and $\lambda = (2)$. Then $K = \mathbb{Q}(\sqrt{3})$, $\alpha_1 = \sqrt{3}$ and $\alpha_2 = -\sqrt{3}$. Hence $J_\lambda(C_p)$, $J_\lambda(D_p)$, and

$J_\lambda(\alpha_1) \oplus J_\lambda(\alpha_2)$ are

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right), \left(\begin{array}{cc|cc} \sqrt{3} & 0 & 1 & 0 \\ 0 & -\sqrt{3} & 0 & 1 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right), \text{ and } \left(\begin{array}{cc|cc} \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \hline 0 & 0 & -\sqrt{3} & 1 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right)$$

respectively. The centralizers of $J_\lambda(\alpha_1) \oplus J_\lambda(\alpha_2)$ and $J_\lambda(D_p)$ consist of matrices of the form

$$\left(\begin{array}{cc|cc} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ \hline 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{array}\right) \text{ and } \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & c & 0 & d \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \end{array}\right)$$

respectively. Finally $C_M(J_\lambda(C_p)) \cong K[x]_\lambda = K[x]_2$.

2.4 Generators for the centralizers

In order to find the orbits of the centralizers on the cocentralizers, we need a generating set for the centralizers. The generators we use are analogous to the elementary matrices of linear algebra—thus finding the orbits becomes a matrix problem.

Using the notation of the previous section, $C_G(A)$ is isomorphic to the group ${K[x]_\lambda}^\times$, which we write in block form as

$$\begin{pmatrix} \mathrm{M}_{l_r}(K[x]_r)^\times & \cdots & \mathrm{M}_{l_r l_2}(x^{r-2}K[x]_r) & \mathrm{M}_{l_r l_1}(x^{r-1}K[x]_r) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{M}_{l_2 l_r}(K[x]_2) & \cdots & \mathrm{M}_{l_2}(K[x]_2)^\times & \mathrm{M}_{l_2 l_1}(xK[x]_2) \\ \mathrm{M}_{l_1 l_r}(K) & \cdots & \mathrm{M}_{l_1 l_2}(K) & \mathrm{M}_{l_1}(K)^\times \end{pmatrix}$$

with $\mathrm{M}_l(K[x]_r)^\times = \mathrm{GL}_l(K) + \mathrm{M}_l(xK[x]_r)$.

Define the following matrices in ${K[x]_\lambda}^\times$:

- $M_{i,a}(l)$, for $i = 1, \dots, r$, $a \in {k[x]_{l_i}}^\times$, and $l = 1, \dots, l_i$: diagonal entries all 1, except for the (l,l)-entry in the (i,i)-block which is equal to a; off-diagonal

Matrix	Row operation	Column operation
$M_{i,a}(l)$	$R_{i,l} \to a \cdot R_{i,l}$	$C_{i,l} \to C_{i,l} \cdot a$
$E_i(l,m)$	$R_{i,l} \leftrightarrow R_{i,m}$	$C_{i,l} \leftrightarrow C_{i,m}$
$A_{i\leq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + ax^{i-j} \cdot R_{j,m}$	$C_{j,m} \to C_{j,m} + C_{i,l} \cdot ax^{i-j}$
$A_{i\geq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + a \cdot R_{j,m}$	$C_{j,m} \to C_{j,m} + C_{i,l} \cdot a$

Table 2.1: Regular action of $K[x]_\lambda{}^\times$

entries all 0.

- $E_i(l,m)$, for $i = 1,\ldots,r$, and $l,m = 1,\ldots,l_i$: diagonal entries all 1 and off diagonal entries all 0, except in the (i,i)-block where the (l,l) and (m,m)-entries are 0 and the (l,m) and (m,l)-entries are 1.

- $A_{i\leq j,a}(l,m)$, for $i,j = 1,\ldots,r$ with $i \leq j$, $a \in k[x]_i$, $l = 1,\ldots,l_i$ and $m = 1,\ldots,l_j$: diagonal entries all 1; off diagonal entries all zero, except in the (i,j)-block where the (l,m)-entry is $x^{i-j}a$.

- $A_{i\geq j,a}(l,m)$, for $i,j = 1,\ldots,r$ with $i \geq j$, $a \in k[x]_j$, $l = 1,\ldots,l_i$ and $m = 1,\ldots,l_j$: diagonal entries all 1; off diagonal entries all zero, except in the (i,j)-block where the (l,m)-entry is a.

Denote the lth row in the ith block by $R_{i,l}$, and the lth column in the ith block by $C_{i,l}$. Then these matrices act on $K[x]_\lambda{}^\times$ as in Table 2.1.

In order to prove that these matrices generate $K[x]_\lambda{}^\times$, it suffices to show that we can reduce any matrix in this group to the identity using these row and column operations. We proceed by induction on the number of parts of λ (ie. the dimension of our matrices). The result is clear if λ has one part. Now take a matrix

$$B = B_0 + (xI)B_1 + (xI)^2 B_2 + \cdots + (xI)^{r-1} B_{r-1}$$

in $K[x]_\lambda{}^\times$ and write

$$B_0 = \begin{pmatrix} B_{rr} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ B_{1r} & \cdots & B_{11} \end{pmatrix}$$

Matrix	Row operation	Column operation
$M_{i,a}(l)$	$R_{i,l} \to a \cdot R_{i,l}$	$C_{i,l} \to C_{i,l} \cdot a$
$E_i(l,m)$	$R_{i,l} \leftrightarrow R_{i,m}$	$C_{i,l} \leftrightarrow C_{i,m}$
$A_{i\leq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + ax^{i-j} \cdot R_{j,m}$	$C_{i,l} \to C_{i,l} + C_{j,m} \cdot ax^{i-j}$
$A_{i\geq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + a \cdot R_{j,m}$	$C_{i,l} \to C_{i,l} + C_{j,m} \cdot a$

Table 2.2: Natural action of $K[x]_\lambda{}^\times$

as is Section 2.3. Since B_0 is invertible, we know B_{rr} is invertible, and so we can use row and column operations within the first block to get the (1,1)-entry of B to be one. Now, by adding multiples of the top row to the other rows, we can make every other entry in the first column zero. We can also add multiples of the left-most column of B to a column in the ith block, as long as we also multiply by x^{r-i}. This is not a problem since an entry in the ith block of the top row must be a multiple of x^{r-i} anyway. We can now ignore the first row and column and reduce the rest of the matrix to the identity by induction. Hence we are done.

In subsequent sections we study the natural action of these matrices on the spaces $K[x]_{\lambda\times-}$ and $K[x]_{-\times\lambda}$. These actions are slightly different from the regular action, because the right action is via the isomorphism D of Section 2.3. The row and column operations for the natural action are in Table 2.2.

2.5 Cocentralizers

Now that we have the centralizer and its generators in terms of algebras, we find a similar description for the cocentralizer and the action of the generators on it.

Let $G = P^{(m,n)} = U \rtimes L$. The Levi complement is $L = \mathrm{GL}_m(k) \oplus \mathrm{GL}_n(k)$ and we can identify the unipotent radical U with the additive group of $\mathrm{M}_{m,n}(k)$. Note that $\mathrm{M}_{m,n}(k) = k^m \otimes (k^n)^t$ where we are, as always, tensoring over k. Let $h = A \oplus B$, where $A \in \mathrm{GL}_m(k)$ and $B \in \mathrm{GL}_n(k)$ are both in generalized Jordan normal form. The action of L on U is given by $A \oplus B \cdot v = AvB^{-1}$. We wish to describe $C^U(h) = U/[U,h]$ as a $C_L(h)$-module over k. Using transfer of structure and the fact that $I \oplus B^{-1}$ is in the center of $C_L(h)$, this module is isomorphic to

$(I \oplus B^{-1}) \cdot C^U(h) = U/((1 \oplus B^{-1}) \cdot [U, A \oplus B])$. Finally,

$$\begin{aligned}(I \oplus B^{-1}) \cdot [U, A \oplus B] &= \{(I \oplus B^{-1}) \cdot (v - A \oplus B \cdot v) : v \in U\} \\ &= \{vB - Av : v \in U\}.\end{aligned}$$

We know that $C_L(h)$ is a direct sum of $C_{\mathrm{GL}_{m(k)}}(A)$ and $C_{\mathrm{GL}_{n(k)}}(B)$, each of which is a direct sum of centralizers corresponding to the generalized eigenvalues of A and B. So a matrix v in $(I \oplus B^{-1})[U, A \oplus B]$ can be broken up into blocks corresponding to these centralizers and we can treat each block separately. Hence we may assume, without loss of generality, that A and B each have a single generalized eigenvalue.

Take $A = J_\mu(C_p)$ and $B = J_\nu(C_q)$ where p and q are monic, separable, irreducible polynomials. Let $K = k(\alpha)$ and $K' = k(\beta)$ with $\alpha, \beta \in \bar{k}$ the roots of p and q respectively. We identify $C_L(A \oplus B) = C_{\mathrm{GL}_{m(k)}}(A) \oplus C_{\mathrm{GL}_{n(k)}}(B)$ with $K[x]_\mu{}^\times \oplus K'[y]_\nu{}^\times$. This allows us to identify $U = k^m \otimes (k^n)^t$ with

$$K[x]_{\mu\times -} \otimes K'[y]_{-\times\nu} = \begin{pmatrix} K[x]_{\mu_1} \\ \vdots \\ K[x]_{\mu_s} \end{pmatrix} \otimes \begin{pmatrix} K'[y]_{\nu_1} & \cdots & K'[y]_{\nu_t} \end{pmatrix} = \begin{pmatrix} R'_{11} & \cdots & R'_{1t} \\ \vdots & \ddots & \vdots \\ R'_{s1} & \cdots & R'_{st} \end{pmatrix}$$

where

$$R'_{ij} = K[x]/(x^{\mu_i}) \otimes K'[y]/(y^{\nu_j}) = K \otimes K'[x, y]/(x^{\mu_i}, y^{\nu_j}).$$

Now the action of A on k^m corresponds to the action of $(\alpha+x)I_m$ on $K[x]_{\mu\times -}$ and the action of B on $(k^n)^t$ corresponds to $(\beta+y)I_n$ on $K'[y]_{-\times\nu}$. Hence $(I\oplus B^{-1})\cdot[U, A\oplus B]$ is identified with the set of elements of the form

$$v(\beta + y)I_m - (\alpha + x)I_n v = (\beta - \alpha + y - x)v$$

for $v \in K[x]_{\mu\times -} \otimes K'[y]_{-\times\nu}$. Hence the the (i, j)-entry of $C^U(h) \cong U/(I \oplus B^{-1}) \cdot [U, A \oplus B]$ is identified with

$$R_{ij} = R'_{ij}/(\beta - \alpha + y - x) = K \otimes K'[x, y]/(x^{\mu_i}, y^{\nu_j}, \beta - \alpha + y - x).$$

Rational case

Suppose that α and β are both in k. Then $K = K' = k$ and

$$R = R_{ij} = k[x,y]/(x^{\mu_i}, y^{\nu_j}, \beta - \alpha + y - x),$$

If $\alpha \neq \beta$, then $\operatorname{rad} R$ contains x, y and $\beta - \alpha = x - y$. So the head of R, $R/\operatorname{rad} R$, maps onto $k/(\beta - \alpha) = 0$ and hence $R = 0$.

So we can assume $\alpha = \beta$. Then $x = y$ in R, so $R = k[x]/(x^{\mu_i}, x^{\nu_j}) = k[x]_{l_{ij}}$ where l_{ij} is the minimum of μ_i and ν_j. Hence $C^U(h)$ becomes

$$k[x]_{\mu\times\nu} = \begin{pmatrix} k[x]_{l_{11}} & k[x]_{l_{12}} & \cdots & k[x]_{l_{1s}} \\ k[x]_{l_{21}} & k[x]_{l_{12}} & \cdots & k[x]_{l_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ k[x]_{l_{r1}} & k[x]_{l_{r2}} & \cdots & k[x]_{l_{rs}} \end{pmatrix}, \quad l_{ij} = \min(\mu_i, \nu_j).$$

Since x and y are identified, we have $C_L(h) \cong k[x]_\mu{}^\times \oplus k[y]_\nu{}^\times = k[x]_\mu{}^\times \oplus k[x]_\nu{}^\times$ acting on $C^U(h) \cong k[x]_{\mu\times\nu}$.

Irrational case

Now consider arbitrary monic, separable, irreducible polynomials p and q. Since $K' = k[u]/(q(u))$, we have

$$\begin{aligned} R = R_{ij} &= K[x,y,u]/(q(u), x^{\mu_i}, y^{\nu_j}, u - \alpha + y - x) \\ &= K[x,y]/(q(x - y + \alpha), x^{\mu_i}, y^{\nu_j}). \end{aligned}$$

Over the field K, $q(u) = (u-\alpha)^\varepsilon f(u)$, where ε is 1 or 0 depending on whether p and q are equal or unequal. In either case $f(\alpha) \neq 0$. So we have $q(x - y + \alpha) = (x-y)^\varepsilon f(x - y + \alpha)$ and $\operatorname{rad}((x-y)^\varepsilon, f(x-y+\alpha)) = K[x,y]$ as it contains $x - y$ and so also contains $f(\alpha)$, which is a unit. Hence $((x-y)^\varepsilon, f(x-y+\alpha)) = K[x,y]$

and, by the Chinese Remainder theorem (Lang, 1993, Section III.2),

$$R = K[x,y]/((x-y)^{\varepsilon}, x^{\mu_i}, y^{\nu_j}) \oplus K[x,y]/(f(x-y+\alpha), x^{\mu_i}, y^{\nu_j}).$$

The second summand is trivial since its radical contains x and y, so its head maps onto $K/(f(\alpha)) = 0$. The first summand is trivial for $\varepsilon = 0$ and is $K[x]_{l_{ij}}$ for $\varepsilon = 1$. Hence $C^U(h)$ is trivial for $p \neq q$ and is isomorphic to $K[x]_{\mu\times\nu}$ for $p = q$. Once again $C_L(h)$ can be identified with $K[x]_\mu{}^\times \oplus K[x]_\nu{}^\times$.

So we have reduced our problem to finding the orbits of $K[x]_\mu{}^\times \oplus K[x]_\nu{}^\times$ on $K[x]_{\mu\times\nu}$, for appropriate fields K. Further, this action is given by the row and column operations of Table 2.2. So we have reduced to a matrix problem, which we also denote $K[x]_{\mu\times\nu}$. In the following chapter we will solve this matrix problem for small dimensions.

CHAPTER 3

SOLUTIONS IN SMALL DIMENSIONAL CASES

3.1 Solving the matrix problem for small dimensions

We solve the matrix problem $k[x]_{\mu\times\nu}$, described in the previous chapter, for an arbitrary field k and either $|\mu|$ or $|\nu|$ less than 6. In particular this gives us the conjugacy classes for maximal parabolics of the general linear group of dimension less than 12 over a perfect field.

Let $\mu = (r^{m_r}, \ldots, 2^{m_2}, 1^{m_1})$ and $\nu = (s^{n_s}, \ldots, 2^{n_2}, 1^{n_1})$ be a pair of partitions with $m = |\mu|$ and $n = |\nu|$. We wish to find a normal form for matrices in

$$k[x]_{\mu\times\nu} = \begin{pmatrix} \mathrm{M}_{m_r n_s}\left(k[x]_{\min(r,s)}\right) & \cdots & \mathrm{M}_{m_r n_2}(k[x]_2) & \mathrm{M}_{m_r n_1}(k) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{M}_{m_2 n_s}(k[x]_2) & \cdots & \mathrm{M}_{m_2 n_2}(k[x]_2) & \mathrm{M}_{m_2 n_1}(k) \\ \mathrm{M}_{m_1 n_s}(k) & \cdots & \mathrm{M}_{m_1 n_2}(k) & \mathrm{M}_{m_1 n_1}(k) \end{pmatrix}$$

under the row and column operations of Table 2.2. I find it useful to visualize such a matrix as a three dimensional array of elements of k, with rows and columns as usual, and levels corresponding to the powers of x. This array is not rectangular since the number of levels depends on which row and column you are in. Figure 3.1 illustrates such an array for $\mu = (6, 5^2, 4^2, 3, 2)$ and $\nu = (5^2, 4, 2^2, 1)$. So, for example, multiplying a row by $1 + x^a$ takes every level in that row and adds its entries i levels higher up in the same row. Note that to add a column to another column i blocks to the left, we also have to move i levels up. We don't have this complication when adding to a column on the right, although we cannot add to a lower level. Similar we need to move to a higher level when adding a row to another row above it.

We now prove that our matrix problem can be infinite type when $m = n = 6$.

Theorem 3.1.1. *The matrix problem $k[x]_{(4,2)\times(4,2)}$ is infinite type.*

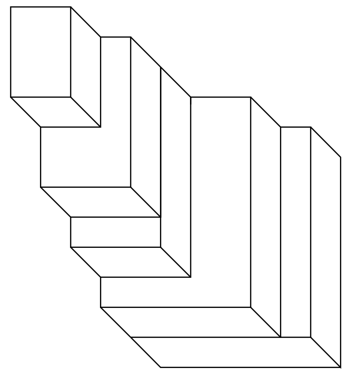


Figure 3.1: A three dimensional array

Proof. Consider matrices in $k[x]_{(4,2)\times(4,2)}$ of the form

$$\begin{pmatrix} \alpha x^2 + \cdots & \beta x + \cdots \\ \gamma x + \cdots & \delta + \cdots \end{pmatrix}$$

for $\alpha, \beta, \gamma, \delta$ in $k^\times$. It is easily checked that every allowable row or column operation leads to another matrix of the same form and preserves the value of $\alpha\beta^{-1}\gamma^{-1}\delta$. Hence there are at least as many orbits as elements of $k^\times$, and the problem is infinite type. □

Next we prove our main theorem, showing that all smaller matrix problems are finite type.

Theorem 3.1.2. *The matrix problem $k[x]_{\mu\times\nu}$ is finite type for ν arbitrary and μ of the form $(2^{m_2}, 1^{m_1})$, $(r, 1^{m_1})$ or $(3, 2)$. In particular, $k[x]_{\mu\times\nu}$ is finite type whenever $|\mu| < 6$.*

Proof. Our basic approach is to solve the 0th level using the permissable row and column operations, then to solve the 1st level using only those operations which preserve the 0th level, and so on. It is a general property of finite type matrix problems that solutions can be found with every entry either 0 or 1. We call positions with a 1 entry *pivots*. These pivots can be used to “kill” other positions (ie. make them 0 with a row or column operation).

The proof is in four cases:

1. First we consider $\mu = (2^{m_2}, 1^{m_1})$. The solution for the 0th level is shown in the cutaway diagram of Figure 3.2(a). In this diagram $\nu = (5^{m_5}, \dots, 1^{m_1})$ but the general case is easily seen to be similar. The blocks are divided by solid lines. Each square containing a diagonal line is an identity matrix (of course, they are not all actually the same size). Now we can use the pivots in the 0th level to kill everything in the 1st level, except for the shaded blocks

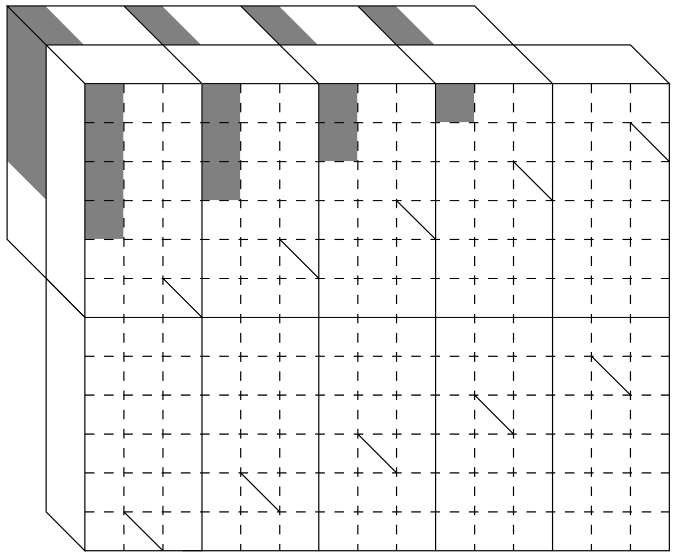
   The shaded blocks of the 1st level are redrawn in Figure 3.2(b). We can add columns to blocks on the left, but not on the right. Also we can add rows to blocks below but not above, because, when adding to a block below, the damage done by one pivot can be repaired by a column operation from another pivot. This level can now be solved as shown.

2. Next we consider $\mu = (r)$, which is shown in Figure 3.3. Find the first nonzero block starting in the bottom left as shown. We can use row operations to put a pivot at the left hand end of this block and then kill the other entries indicated by the arrows. Now ignore all the entries marked with an arrow or a zero, and repeat the same process with what remains.

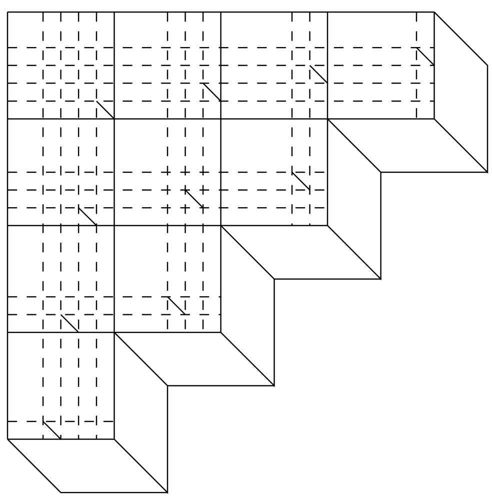
3. The case $\mu = (r, 1)$ is illustrated in Figure 3.4. We start by solving the 2nd row: find the first nonzero block, make a pivot in that block and kill the rest of the row. Then, ignoring the shaded part, we solve the rest of the 1st row as with $\mu = (r)$. We can now use a column multiplication to ensure that the shaded positions contains a single 1, followed by a row multiplication to repair any damage this does to the pivot in the second row.

   The solution for $\mu = (r, 1^{m_1})$ is easily seen to be similar, except that there can be more than one shaded column.

4. Finally we turn to $\mu = (3, 2)$. First we solve the 0th level as in Figure 3.5(a), and remove the two pivotal columns. Now, in the second level, all column operations are allowed, and row multiplication is allowed, because the damage it does to the pivots can be repaired by a column multiplication. However, the rows cannot be added to each other. This level is solved as in Figure 3.5(b).



(a) Level 0



(b) Level 1

Figure 3.2: $\mu = (2^{m_2}, 1^{m_1})$

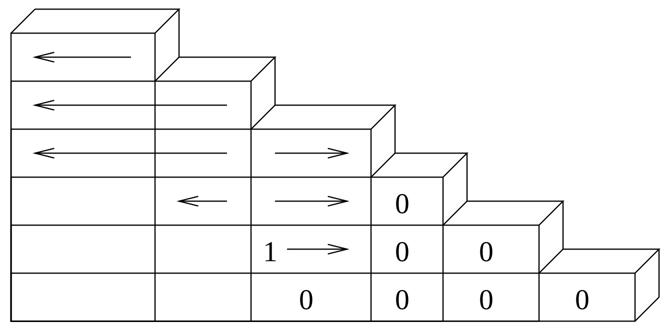


Figure 3.3: $\mu = (r)$

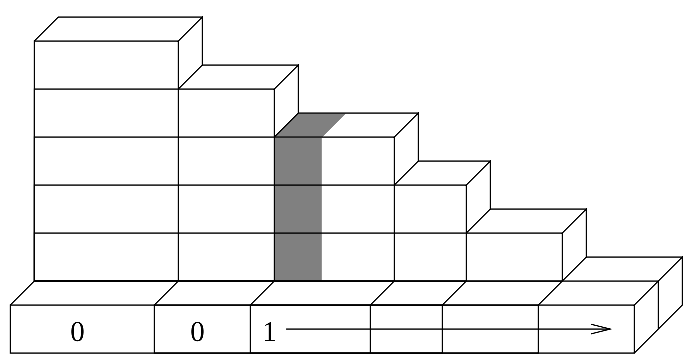
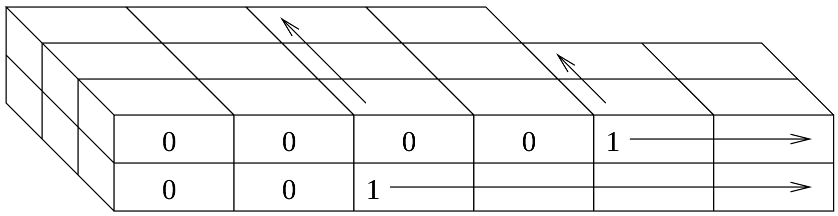
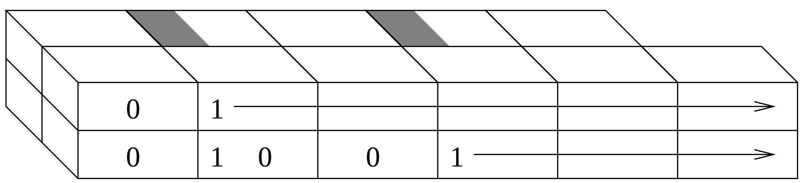


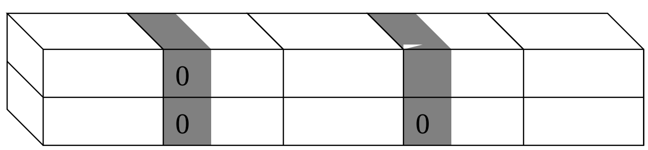
Figure 3.4: $\mu = (r, 1)$



(a) Level 0



(b) Level 1



(c) Level 2

Figure 3.5: $\mu = (3, 2)$

> For level 2 we get the same row and column operations as in level 1, except that the shaded columns cannot be added to other columns. However this does not cause a problem, since all but one entry in these columns has already been killed by a pivot on level 1.

The final claim follows because all partitions of a number less than 6 are of one of these three forms. □

Corollary 3.1.3. *Computing the conjugacy classes in $P^{(m,n)}$ over a perfect field reduces to matrix problems of finite type if, and only if, either $m < 6$ or $n < 6$. In particular, computing conjugacy classes in the maximal parabolics of the general linear group over a perfect field reduces to finite type problems if, and only if, the dimension is less than 12.*

Proof. The previous theorem, together with the results of Chapter 2, show that we get finite type problems if $m < 6$. By symmetry, this is also true for $n < 6$. By Theorem 3.1.1, $P^{(6,6)}$ involves a problem of infinite type and it follows easily that $P^{(m,n)}$ does whenever $m, n \geq 6$. □

When our field is finite we get the following result.

Corollary 3.1.4. *Suppose that k is finite of size q and either $m < 6$ or $n < 6$. Then the number of conjugacy classes, and therefore the number of irreducible characters, of $P^{(m,n)}$ is a polynomial in q with integral coefficients.*

Proof. The number of solutions of the relevant matrix problems is independent of q. Hence this result follows immediately from the well known fact that the number of characters of $\mathrm{GL}_n(q)$ is a polynomial in q with integral coefficients. □

Note that the proof of the Theorem 3.1.2 also provides a procedure for solving these finite type problems, so this section gives an implicit description of all conjugacy classes in the parabolic subgroups mentioned in Corollary 3.1.3. For example, suppose the original eigenvalue is α, $\mu = \nu = (4, 2)$, and our orbit representative in $k[x]_{\mu \times \nu}$ is

$$\begin{pmatrix} \beta x^2 & x \\ x & 1 \end{pmatrix}.$$

Then our conjugacy class representative in $P^{(6,6)}$ is

$$\left(\begin{array}{cccccc|cccccc}
\alpha & & & & & & & & & & & \\
1 & \alpha & & & & & & & & & & \\
 & 1 & \alpha & & & & \beta & & & & & \\
 & & 1 & \alpha & & & & \beta & & & 1 & \\
 & & & & \alpha & & & & & & 1 & \\
 & & & & 1 & \alpha & 1 & & & & & 1 \\
\hline
 & & & & & & \alpha & & & & & \\
 & & & & & & 1 & \alpha & & & & \\
 & & & & & & & 1 & \alpha & & & \\
 & & & & & & & & 1 & \alpha & & \\
 & & & & & & & & & & \alpha & \\
 & & & & & & & & & & 1 & \alpha
\end{array}\right),$$

where blank entries are zero.

3.2 The affine general linear group

We apply the results of the previous section to the affine general linear groups (Solomon, 1971, 1972). The representation theory of these well known groups is computed in Zelevinsky (1981). The affine general linear group of degree n over the field k, $\mathrm{AGL}_n(k)$, is the semidirect product of $\mathrm{GL}_n(k)$ and the row space $(k^n)^t$. It can be realized as the subgroup

$$\begin{pmatrix} 1 & (k^n)^t \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}$$

of $\mathrm{GL}_{n+1}(k)$. The generalized Jordan normal form provides a set of conjugacy class representatives for $\mathrm{GL}_n(k)$. Each can be written in the form $N \oplus E$ where N has no eigenvalues equal to 1 and E has eigenvalue 1. There is a partition

$\lambda = (r^{l_r}, \dots, 2^{l_2}, 1^{l_1})$ so that

$$E = J_\lambda(1) = \bigoplus_{i=1}^{r} E_i$$

where $E_i = J_i(1)^{\oplus l_i}$.

Theorem 3.2.1. *A set of conjugacy class representatives for* $\mathrm{AGL}_n(k)$ *is given by the matrices*

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline & N & 0 \\ & 0 & E \end{array}\right) \quad \textit{and} \quad \left(\begin{array}{c|ccccccc} 1 & 0 & 0 & \cdots & e & \cdots & 0 \\ \hline & N & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & E_1 & & & & 0 \\ & \vdots & & \ddots & & & \vdots \\ & 0 & & & E_i & & 0 \\ & \vdots & & & & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & \cdots & E_m \end{array}\right),$$

where $e = (1, 0, 0, \dots)$.

Proof. Let $Z = kI_{n+1}$ be the center of $\mathrm{GL}_{n+1}(k)$. Then $P^{(1,n)} = Z \cdot \mathrm{AGL}_n(k)$ and so the conjugacy classes in $\mathrm{AGL}_n(k)$ are just the noncentral conjugacy classes in $P^{(1,n)}$ intersected with $\mathrm{AGL}_n(k)$. Hence we need the matrices given by the proof of Theorem 3.1.2 with $\mu = (1)$ and 1 in the first summand of $L = \mathrm{GL}_1(k) \oplus \mathrm{GL}_n(k)$. The result is now immediate. $\square$

Let k be a finite field. We denote by c_n the number of conjugacy classes in $\mathrm{GL}_n(k)$ and use the convention that $\mathrm{GL}_0(k)$ is the trivial group. For $d = 0, 1, \dots, n$, we consider the conjugacy class representatives of $\mathrm{AGL}_n(k)$ with an e above a Jordan block of size d. Then $A = N \oplus E$ is an arbitrary conjugacy class representative of $\mathrm{GL}_n(k)$, except that it must have a least one Jordan block of size d and eigenvalue 1. If you remove one such block of size d from A, you get an arbitrary conjugacy class representative of $\mathrm{GL}_{n-d}(k)$, of which there are c_{n-d}. So the total number of conjugacy class representatives of $\mathrm{AGL}_n(k)$ is

$$\sum_{d=0}^{n} c_{n-d} = c_n + c_{n-1} + \cdots + c_0.$$

This agrees with the count of the number of irreducible characters of $\mathrm{AGL}_n(k)$ gotten by Zelevinsky (1981).

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