Receding Horizon Optimal Control for Some Stochastic Hybrid Systems

Christos G. Cassandras* and **Reetabrata Mookherjee***
Dept. of Manufacturing Engineering
and Center for Information and Systems Engineering
Boston University
Brookline, MA 02446
cgc@bu.edu, rbm@bu.edu

TECHNICAL REPORT

Abstract

We consider optimal control problems for a class of hybrid systems with switches dependent on an external event process. In the case where all event times in this process are fully known, the solution to such problems was obtained in prior work. When event times are uncertain or unknown, we have proposed a Receding Horizon (RH) control scheme in which only some future event information is available within a time window of length T and have obtained several properties of this scheme. In this paper, we provide a full set of properties for this scheme, including the fact that the error due to lack of future event information is monotonically decreasing under certain conditions and may be zero for segments of the sample path, depending on the window length T. This enables the use of a controller based on rough estimates of future events with limited loss of optimality properties.

Keywords: Hybrid Systems, Discrete Event Systems, Receding Horizon, Optimal Control

1 Introduction

Hybrid Systems are characterized by the combination of *time-driven* and *event-driven* dynamics. A simple way to think of a hybrid system is as one characterized by a set of operating "modes", each one evolving according to time-driven dynamics described by differential (or difference) equations. The system switches between modes through discrete events which may be controlled or uncontrolled. Controlling the switching times, when possible, and choosing among

*Supported in part by the National Science Foundation under Grant EEC-00-88073, by AFOSR under contract F49620-01-0056, and by ARO under under grant DAAD19-01-0610.

several feasible modes, whenever such choices are available, gives rise to a rich class of optimal control problems. This has motivated efforts to extend classical optimal control formulations [1],[2],[3] and to apply dynamic programming techniques [4],[5] to hybrid systems. While in principle this is possible, the computational complexity involved becomes prohibitive. Therefore, it is often natural to hierarchically decompose a system into a lower-level component representing physical processes characterized by time-driven dynamics and a higher-level component controlling discrete events related to these physical processes (see [6],[7]). The explicit solution of the lower and higher-level problems depends on the specifics of the time-driven and event-driven dynamics involved (see [6],[7],[8]).

To complicate matters, some of the most interesting problems one encounters in dealing with hybrid systems involve some form of uncertainty, which generally calls for stochastic modeling and solution techniques. In this paper, we attempt to deal with this issue for a class of optimal control problems where the event-driven dynamics may be captured through 'max-plus' equations. In such cases, switching times are controllable, but they are dependent upon an external event process in which the associated event times $\{a_1, \ldots, a_N\}$ are generally unknown. If this sequence is fully deterministic, optimization problems formulated as in [9],[10] can be efficiently solved through the "Forward Algorithm" presented in [11]. If it is not, then one approach is to model it as a stochastic process as in [12] where the structure of an optimal policy can be determined, but explicit calculations are difficult.

In [13], we take a different approach in the way we view uncertainties in $\{a_1, \ldots, a_N\}$. In past work, it was assumed that all future events following any time t when a control decision needs to be made are known. Thus, if we associate with t an "information window" $[t, t+T]$, it was assumed that $T = \infty$. A natural next step is to consider $T < \infty$ and within a window $[t, t+T]$ assume that event times (if any are present) are deterministically known. Event times outside $[t, t+T]$ can only be described probabilistically, possibly estimated, or there may be no information at all regarding events beyond time $t+T$. In short, the optimal control problem we now tackle is defined over a "receding horizon" determined by the length T of the window at the controller's disposal. The nature of this scheme lends itself to what we term a *Receding Horizon* (RH) controller (commonly associated with optimal control problems for which feedback solutions are extremely hard or impossible to obtain, e.g., [14], and it is usually encountered in model predictive control). Our work explores the extent to which this way of dealing with uncertainty is effective by analyzing the relationship between the optimal and the RH controller's cost/performance and the role that T plays. In [13], we initiated this study by establishing a number of properties of the RH controller under the assumption that the optimal sample path does not contain certain events termed "critical" (as defined in [10]). In [15], we developed some further properties when critical events are allowed. In this paper, we complete this line of work by developing some further properties and unifying them all for any arbitrary optimal sample path. A key result we can now establish is a complete characterization of the conditions under which the *error* introduced by the RH controller relative to the optimal one ismonotonically decreasing and may become zero for parts of the sample path.

The paper is organized as follows. Section 2 reviews the framework we use for the class of hybrid systems and related optimal control problems considered. In Section 3 we review the proposed RH control scheme. In Section 4 we present several properties of the RH controls and analyze the difference between the RH and the optimal controls as a function of the length of the receding horizon window.

2 Optimal Control Problem and Solution

In the hybrid systems we consider, the state of the system consists of *temporal* and *physical* components. The temporal components keep track of the time information for events that may cause switches in the operating mode of the system. Let $i = 1, 2, \ldots$ index these events. We denote the *physical state* of the system after the ith event by $z_i(t)$ with dynamics:

$$\dot{z}_i = g_i(z_i, u_i, t), \quad z_i(x_{i-1}) = z_i^0, \quad t \in [x_{i-1}, x_i) \tag{1}$$

where u_i is the control applied over an interval $[x_{i-1}, x_i)$ defined by two event occurrences at times x_{i-1} and x_i. In what follows, we shall write u_i to denote a function $u_i(t)$ defined over $[x_{i-1}, x_i)$; similarly for $z_i(t)$. We shall assume that $u_i(t)$ is allowed to be piecewise continuous and is in general an n-dimensional vector.

In the case of a *single* event process in the system, the event-driven dynamics characterizing the *temporal states* x_i are given by

$$x_i = x_{i-1} + s_i(z_i, u_i) \tag{2}$$

for $i = 1, 2, \ldots$, where $s_i(\cdot)$ is the amount of time between switches, which generally depends on the physical state z_i and control u_i over $[x_{i-1}, x_i)$. This is a simple linear relationship in x_i and when the dynamics in (1) are also linear, one obtains the class of switched linear systems (e.g., see [8]). In this case, the complexity is concentrated in determining the optimal amount of time spent in mode i, given by $s_i(z_i, u_i)$, whereas the event driven dynamics yielding the switching time sequence $\{x_1, x_2, \ldots\}$ are extremely simple.

In the case of *multiple* asynchronous event processes in the system, indexed by $j = 1, \ldots, M$, we need to introduce a *Timed Automaton* which determines which of the M events triggers the next switch and at what precise time. The exact structure of a timed automaton is described in [16]. We omit the details and simply represent the event-driven dynamics in the form

$$x_i = x_{i-1} + s_i(y_{i,1}, \ldots, y_{i,M}, z_i, u_i) \tag{3}$$

where $y_{i,1}, \ldots, y_{i,M}$ are the event clocks of the timed automaton (through which the triggering event and its occurrence time for the next switch are determined) after the $(i-1)$th switch. Looking at (1) and (3), note also that the choice of control u_i affects both the physical state z_i and the temporal state x_i. Thus, the switches at times $x_1, x_2, \ldots$ are generally *not* exogenous events that dictate changes in the state dynamics, but rather temporal states intricately connected to the control of the system; this is one of the crucial elements of a "hybrid" system.

The class of problems we will now concentrate on involves event-driven switching time dynamics described by

$$x_i = \max(x_{i-1}, a_i) + s_i(u_i) \tag{4}$$

where $\{a_i\}$, $i = 1, \ldots, N$, is a given sequence of event times corresponding to an asynchronous event process operating independently of the physical processes $\{z_i(t),\ t \in [x_{i-1}, x_i)\}$. This "max-plus" recursive equation is the well known Lindley equation in queueing theory [16]. In comparing (4) to (2), note that the key difference is the presence of the max function, which introduces a nondifferentiable component into the solution of the overall problem.

A typical application of this setting arises in a manufacturing system workstation, where the mode switches correspond to jobs that we index by $i = 1, \ldots, N$. A job is associated with a *temporal* state evolving according to (4) where x_i is the departure time of the ith job when the server processes one job at a time on a first-come first-served nonpreemptive basis. Jobs arriving when the server is busy wait in an infinite-capacity queue and $\{a_1, \ldots, a_N\}$ is a sequence of job arrival times. The processing time of the ith job (which we will denote by C_i) is $s_i(u_i)$. A job is also associated with a *physical* state evolving according to (1) and describing changes the ith job undergoes while in process in such quantities as the temperature, size, weight or some other measure of the "quality" of the job. Thus, the ith "mode" of a workcenter in this context corresponds to the processing of the ith job. The interaction of the time-driven and event-driven dynamics leads to a natural trade-off between temporal requirements on job completion times and physical requirements on the quality of the completed jobs. One can, therefore, formulate optimization problems as in [9], [10] where the control variables are either the processing times of jobs or they affect the time-driven dynamics that ultimately control processing times (and, hence, the mode switching times). By the nature of the event-driven dynamics, these problems are inherently non-convex and non-differentiable. Moreover, their dimension (number of independent variables) is identical to the number of jobs, which can easily be in the hundreds or thousands; the resulting complexity defies general-purpose algorithms (like dynamic programming). Recently, however, structural properties of the problem were exploited to decompose the entire optimal control problem into a set of smaller convex optimization subproblems with linear constraints. This has led to algorithms whose complexity (measured in the number of convex constrained optimization problems required to solve) was reduced from exponential in N (the number of jobs processed) to linear in N. In particular, the "Forward Algorithm" presented in [11] identifies the unique optimal controls and has a complexity of precisely N.

Let us briefly review the optimization problem introduced in [10] and solved through the Forward Algorithm developed in [11]. When the processing of each job stops as soon as a given "quality level" in its physical state z_i is reached and the control is the amount of processing time, i.e., $s_i(u_i) = u_i$ in (4), the problem has been shown to become

$$\min_{u_1,\ldots,u_N} \sum_{i=1}^{N} [\theta_i(u_i) + \psi_i(x_i)] \tag{5}$$

subject to (4), with control variables u_i assumed to be scalar and not time-dependent. Thus, u_i is the processing time of the ith job, chosen at the beginning of its processing cycle, i.e., at time $\max(x_{i-1}, a_i)$. The cost function $\theta_i(u_i)$ penalizes poor physical quality, in the sense that less processing time monotonically decreases quality and, hence, increases a cost $\theta_i(u_i)$, while $\psi_i(x_i)$ imposes a cost on the departure time x_i. As in [10], we make the following assumptions:

Assumption A1. For each $i = 1, \ldots, N$, $\theta_i(\cdot)$ is strictly convex, twice continuously differentiable and monotonically decreasing with $\lim_{u_i \to 0^+} \theta_i(u_i) = -\lim_{u_i \to 0^+} \frac{d\theta_i}{du_i} = \infty$ and $\lim_{u_i \to \infty} \theta_i(u_i) = \lim_{u_i \to \infty} \frac{d\theta_i}{du_i} = 0$.

Assumption A2. For each $i = 1 = 1, \ldots, N$, $\psi_i(\cdot)$ is strictly convex, twice continuously differentiable, and its minimum is obtained at a finite point δ_i.

As an example $\theta_i(u_i) = \frac{1}{u_i}$ and $\psi_i(x_i) = (x_i - d_i)^2$ satisfy the assumptions above. In this case, d_i is the specified deadline of job C_i, therefore $(x_i - d_i)$ measures earliness and tardiness of the job.

A typical sample path of this system can be partitioned into "busy" and "idle" periods. During a busy period the server is processing a job; after completing the processing of a job, if no other job is available in the queue, an idle period starts and is terminated upon arrival of the next job. Formally, we review the definitions made in [10]: An *idle* period is a time interval (x_k, a_{k+1}) such that $x_k < a_{k+1}$ for any $k = 1, ..., N-1$. A *Busy Period* (BP) is a set of contiguous jobs $\{k, ..., n\}$, $1 \leq k \leq n \leq N$ such that: (i) $x_{k-1} < a_k$, (ii) $x_n < a_{n+1}$, and (iii) $x_i \geq a_{i+1}$ for every $i = k, ..., n-1$. A *busy-period structure* is a partition of the jobs $1, ..., N$ into busy periods. A job C_i is *critical* if it departs at the arrival time of the next job C_{i+1}, i.e. $x_i = a_{i+1}$. A contiguous job subset $\{k, \ldots, n\}$, $1 \leq k \leq n \leq N$, is said to be a *block* if: (i) $x_{k-1} \leq a_k$ and $x_n \leq a$, and (ii) the subset contains no critical jobs.

Obtaining an explicit solution to problem (5) is tantamount to identifying the BP structure of the optimal state trajectory and then solving a nonlinear optimization problem within each BP. Let us denote a BP that starts at a_k and ends at x_n by the job indices (k, n). Then, we define the problem $Q(k, n)$:

$$Q(k,n): \min_{u_k, \ldots, u_n} \left\{ \sum_{i=k}^{n} \{\theta_i(u_i) + \psi_i(a_k + \sum_{j=k}^{i} u_j)\} : u_i \geq 0 \right\} \tag{6}$$

$$\text{s.t.} \quad a_k + \sum_{j=k}^{i} u_j \geq a_{i+1}, \quad i = k, \ldots, n-1$$

Note that we have set $\psi_i(x_i) = \psi_i(a_k + \sum_{j=k}^{i} u_j)$ since, within a BP, $x_{j+1} = x_j + u_{j+1}$ for all $i = k, \ldots, n-1$. The constraint represents the requirement $x_i \geq a_{i+1}$ for any job $i = k, \ldots, n-1$ belonging to the BP. Since the cost functional is continuously differentiable and strictly convex, the problem $Q(k, n)$ is also a convex optimization problem with linear constraints and has a unique solution at a finite point. The solution of $Q(k, n)$ is denoted by $u_j^*(k, n)$ for $j = k, \ldots, n$, and the corresponding departure times are $x_j^*(k, n)$.

The optimal solution in (5) is denoted by u_i^*, $i = 1, \ldots, N$ and the corresponding departure times are x_i^*. It was shown in [10] that the optimal solution is unique. The Forward Algorithm for obtaining this solution is based on the fact (proved in [11]) that $x_i^* = x_i^*(k, n)$ for k, n such that $x_n^*(k, n) \leq a_{n+1}$: Letting $k = n = 1$, we first solve the linearly constrained convex optimization problem $Q(k, n)$ and obtain the control $u_j^*(k, n)$, $j = k, \ldots, n$ and departure times $x_j^*(k, n)$, $j = k, \ldots, n$. Then, the structure of BPs is identified by checking if $x_n^*(k, n) \leq a_{n+1}$. If $\{k, \cdots, n\}$ is identified as a single BP, the optimal control is given by $u_j^* = u_j^*(k, n)$, $j = k, \ldots, n$.

Then, the process is repeated for a new BP starting at a_{n+1}. This algorithm requires N total steps.

3 Receding Horizon Control

Throughout the discussion above we have assumed the sequence $\{a_1, \ldots, a_N\}$ to be deterministic, i.e., a schedule of all external events is known in advance. In what follows, we shall assume that knowledge of the future at time t is limited to a "window" $[t, t+T]$ for some given T. It is then natural to solve a sequence of problems of the form (5) replacing $(u_1, \ldots, u_N)$ by some $(u_i, \ldots, u_{i+r})$ where C_i is the next job whose processing time needs to be assigned and C_{i+r} is the last job known to arrive at some time $a_{i+r} \le t+T$. The window is updated at every decision instant, i.e., upon departure of a job.

We distinguish the optimal controls u_i^* and corresponding departure times x_i^*, $i = 1, \ldots, N$, from those obtained through a control scheme limited in future knowledge by denoting the latter by $\tilde{u}_i$ and $\tilde{x}_i$ respectively. We shall also use the index t to represent the last job processed under such a controller, so that the current information window is $[\tilde{x}_t, \tilde{x}_t + T]$ and the *Receding Horizon* (RH) for the overall problem is $\tilde{x}_t + T$. We will also assume that any arrival time information provided at $\tilde{x}_t$ is "perfect" in the sense that both the optimal and the RH controller make decisions based on the same $\{a_i\}$ such that $\tilde{x}_t < a_i \le \tilde{x}_t + T$ (of course, the optimal controller has the entire $\{a_i\}$ sequence available).

Assuming the current decision time to be $\tilde{x}_t$, the index of the last job contained within $[\tilde{x}_t, \tilde{x}_t + T]$ is given by

$$l = \arg\max_{r \ge t} \{a_r : a_r \le \tilde{x}_t + T\} \tag{7}$$

where we note that $l = t$ indicates that there are no arrival events in $[\tilde{x}_t, \tilde{x}_t + T]$.

Similar to $Q(k, n)$ in (6), let us now consider a problem $\tilde{Q}(t+1, n)$ defined for a sample path generated by the RH controller when a decision time for job C_{t+1} comes up:

$$\tilde{Q}(t+1, n) : \min_{\tilde{u}_{t+1}, \ldots, \tilde{u}_n} \left\{ \sum_{i=t+1}^{n} \{\theta_i(\tilde{u}_i) + \psi_i[\max(\tilde{x}_t, a_{t+1}) + \sum_{j=t+1}^{i} \tilde{u}_j]\} : \tilde{u}_i \ge 0 \right\} \tag{8}$$

$$\text{s.t.} \quad \max(\tilde{x}_t, a_{t+1}) + \sum_{j=t+1}^{i} \tilde{u}_j \ge a_{i+1}, \quad i = t+1, \ldots, n-1$$

Setting $k = t+1$ in (6), the only difference between $Q(t+1, n)$ and $\tilde{Q}(t+1, n)$ lies in replacing a_{t+1} by $\max(\tilde{x}_t, a_{t+1})$. This is due to the fact that $Q(t+1, n)$ is *always* solved with the knowledge that C_{t+1} starts a BP. However, in the RH control scheme C_{t+1} does not necessarily start a BP. In particular, at time $\tilde{x}_t$ the controller can determine whether $a_{t+1} \le \tilde{x}_t$, in which case C_{t+1} cannot start a BP. When this is true, $\tilde{Q}(t+1, n)$ above is solved with $\max(\tilde{x}_t, a_{t+1}) = \tilde{x}_t$ for $n = t+1, \ldots, l$ as if $\tilde{x}_t$ were initiating a BP and there are only $l - t$ jobs left to process. Since in this scheme all controls $\tilde{u}_j$, $j \le t$ are already fixed at the time that $\tilde{u}_{t+1}$ is evaluated, our goal is to determine the "optimal" controls for the remainder of a BP or $\tilde{x}_t + T$, whichever comes first. The solution of $\tilde{Q}(t+1, n)$ is denoted by $\tilde{u}_i(t+1, n)$, $i = t+1, \ldots, n-1$.

In view of the above, let us specify the RH controller operation at time $\tilde{x}_t$ assuming that $l > t$. We begin by setting $a_{l+1} = \infty$, since the controller ignores any future job arrival beyond C_l. Then: (i) Solve $\tilde{Q}(t+1, n)$ starting with $n = t+1$, (ii) If $\tilde{x}_n(t+1, n) \leq a_{n+1}$, then set $\tilde{u}_{t+1} = \tilde{u}_{t+1}(t+1, n)$ and $\tilde{x}_{t+1} = \max(\tilde{x}_t, a_{t+1}) + \tilde{u}_{t+1}$. Otherwise, repeat (i)-(ii) with $n = t+2, \ldots, l$ until $\tilde{x}_n(t+1, n) < a_{n+1}$ for some $n \leq l$.

In the case where $l = t$, there is no decision to be made at $\tilde{x}_t$, since there is no further information on any arriving job after C_t. The RH controller moves the decision time to $\tilde{x}_t + T$ and repeats the process until $a_{t+1} \leq \tilde{x}_t + mT$ for some $m = 1, 2, \ldots$, at which point the decision time becomes a_{t+1} and $\tilde{Q}(t+1, n)$ is solved with $\max(\tilde{x}_t, a_{t+1}) = a_{t+1}$.

Note that the function of the RH controller is to ultimately determine a *single* control $\tilde{u}_{t+1}$ for the next job requiring a processing time assigned to it. This is done by taking into account all past and future information in the form of $a_{t+1}, \ldots, a_l$, where l clearly depends on T through (7). The determination of $\tilde{u}_{t+1}$ is made by solving an optimal control problem of the form (5) using the Forward Algorithm, except that we act as if the sample path starts at time $\tilde{x}_t$ (or a_{t+1}) and the complete sequence $\{a_{t+1}, \ldots, a_N\}$ is replaced by one with $a_i = \infty$ for all $i > l$. The process repeats at time $\tilde{x}_{t+1} = \max(\tilde{x}_t, a_{t+1}) + \tilde{u}_{t+1}$ with a new RH window $[\tilde{x}_{t+1}, \tilde{x}_{t+1} + T]$. As already pointed out, it is of course possible that a window $[\tilde{x}_t, \tilde{x}_t + T]$ contains no arrival time information, including a_{t+1}.

4 Receding Horizon (RH) Controller Properties

Our analysis of the RH controller properties, compared to the behavior obtained under the optimal control, is organized in two parts. First, we consider the last *block* of any optimal path BP; if the BP does not contain any critical jobs, then, clearly, the last block coincides with an entire such BP. Next, we consider all other blocks (if any) in a BP. In both cases, we establish properties of the RH controlled system for jobs indexed by t such that $k - 1 \leq t < n$. Before doing so, however, we begin with two simple properties of the RH controller, expressed as Lemmas 4.1 and 4.2 below, which apply to all blocks in an optimal path BP. The first one states that if the RH controller is applied before an optimal path block starts and the RH window contains all arrival times in this block (see Fig. 1 illustrating this situation), then the blocks in both sample paths are identical.

Lemma 4.1 *Let (k, n) define a block on some optimal sample path BP and let $\tilde{x}_{k-1}$ be the current decision time on the RH sample path with $\tilde{x}_{k-1} + T \geq a_n$. If (k, n) is the first block in the BP assume that $\tilde{x}_{k-1} \leq a_k$, otherwise assume $\tilde{x}_{k-1} = a_k$. Then*

$$\tilde{x}_i = x_i^* \quad \text{and} \quad \tilde{u}_i = u_i^* \quad \text{for all } i = k, \ldots, n$$

Proof. See Appendix ∎

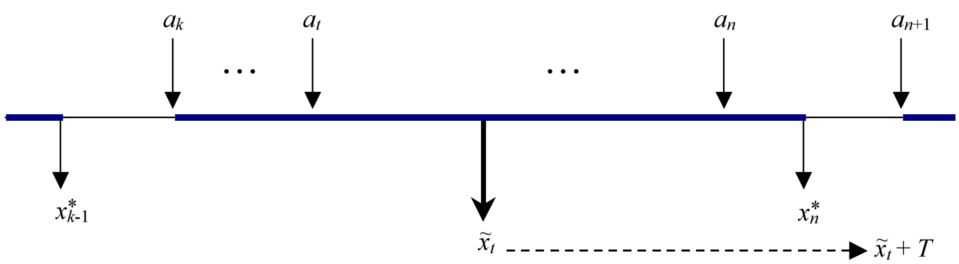

Figure 1: A sample path example with $\tilde{x}_t + T \geq a_n$

The second lemma qualifies only the RH control $\tilde{u}_k$ for the *first* job of an optimal path block, asserting that it is lower bounded by the optimal control u_k^* provided that a RH Block also starts with C_k.

Lemma 4.2 *Let (k, n) define a block on an optimal sample path. Let $\tilde{x}_{k-1}$ be the current decision time on the RH sample path and suppose $\tilde{x}_{k-1} \leq a_k$ and the RH window $[\tilde{x}_{k-1}, \tilde{x}_{k-1}+T]$ is such that $a_l \leq \tilde{x}_{k-1} + T < a_{n+1}$ where $l = \arg\max_{i>k-1}\{a_i : a_i \leq \tilde{x}_{k-1} + T\}$. Then*

$$\tilde{x}_k \geq x_k^* \quad \textit{and} \quad \tilde{u}_k \geq u_k^*$$

Proof. See Appendix ■

It can be inferred from the above two lemmas that under the stated conditions C_n is also critical in the RH path. Since we know C_n is the last job of an optimal path block, we have $x_n^* = a_{n+1}$ and, therefore, $\tilde{x}_n = x_n^* = a_{n+1}$. In addition, these lemmas can be readily extended to RH windows that include multiple blocks. Let (k, n) be an optimal path BP with C_{B_1} and C_{B_2} being the first two critical jobs in this BP. Suppose, C_k starts a new BP in the RH path as well (i.e., $\tilde{x}_{k-1} < a_k$) and $\tilde{x}_{k-1} + T \geq a_{B_2+1}$. Then, Lemmas 4.1 and 4.2 are applicable for both blocks $\{C_k, \ldots, C_{B_1}\}$ and $\{C_{B_1+1}, \ldots, C_{B_2}\}$, i.e., $\tilde{x}_i = x_i^*$ for all $i = k, \ldots, B_2$. In general, if at time $\tilde{x}_{k-1}$ the RH window includes the arrival times of the first b blocks of an optimal path BP, then these lemmas apply to all b blocks.

4.1 Analysis of the Last Block in Optimal Path BPs

Let us consider an optimal path BP containing at least one critical job. We make a simple observation that relates the last block of such a BP to a BP that has no critical jobs. In particular, recall the definitions used in [10]:

$$\xi_i^{*+} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{n(i)} \frac{d\psi_j}{dx_j^*}$$

where $n(i)$ is the next job after i that ends a BP, and

$$\xi_i^{*-} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{m(i)} \frac{d\psi_j}{dx_j^*}$$

where $m(i)$ is the next job which is critical after C_i within the BP containing C_i, if such a job exists; otherwise $m(i) = n(i)$. For a BP that contains no critical jobs, it follows from Theorem 3.1 and Lemma 5.1 in [10] that all i in this BP satisfy the optimality condition

$$\xi_i^{*+} = \xi_i^{*-} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{n(i)} \frac{d\psi_j}{dx_j^*} = 0$$

Similarly, for any job i in the last block of a BP we have $m(i) = n(i)$ and the exact same optimality condition is satisfied. The implication of this observation is that all properties for an optimal path BP without any critical job are identical to those of the last block of any optimal path BP. For the remainder of this subsection, we will assume that (k, n) forms the *last block* of any optimal path BP. Our first result is that the RH controlled system will also not have any critical jobs over the range $k, \ldots, n-1$; regarding the nth job, it is possible that it becomes critical, i.e., $\tilde{x}_n = a_{n+1}$. Based on this fact, we establish the following two key properties of the RH controller: (i) $\tilde{x}_i \geq x_i^*$, $k \leq i \leq n$, and (ii) The error $\varepsilon_i = \tilde{x}_i - x_i^*$ is monotonically decreasing when the condition $\tilde{x}_t + T \geq a_n$ holds (as seen in Fig. 1). The implication of these properties is that the RH controller often incurs no error upon completion of some BPs and this error remains bounded and starts monotonically decreasing under certain conditions.

In order to establish the main results mentioned above we need the following two auxiliary technical lemmas that will facilitate our proofs.

Lemma 4.3 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on the RH sample path that ends with the job $C_{\tilde{n}}$. Suppose this BP contains at least one critical job and B indexes the last such critical job with $t+1 \leq B < \min\{n, \tilde{n}\}$. For any $i \in \{t+1, \ldots, B-1\}$, if $x_j^* > \tilde{x}_j$ for all $j = i+1, \ldots, B$, then $x_i^* > \tilde{x}_i$.*

Proof. See Appendix ■

Lemma 4.3 can easily be extended to the situation where there is no critical job in the RH BP that contains job C_{t+1}. In this case, we may set $B = \min\{n, \tilde{n}\}$ and obtain the following.

Corollary 4.1 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on some BP of the receding horizon sample path that ends with the job $C_{\tilde{n}}$ and contains no critical jobs. For any $i \in \{t+1, \ldots, B-1\}$, where $B = \min\{n, \tilde{n}\}$, if $x_j^* > \tilde{x}_j$ for all $j = i+1, \ldots, B$, then $x_i^* > \tilde{x}_i$.*

Proof. See Appendix ■

Lemma 4.4 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on some BP of the RH sample path that ends with job $C_{\tilde{n}}$. For any $i \in \{t+1, \ldots, \min(n, \tilde{n})\}$, if $x_i^* > \tilde{x}_i$ and none of the jobs $C_{i+1}, \ldots, C_{\tilde{n}-1}$ are critical in the RH sample path then $x_j^* > \tilde{x}_j$ for all $j = i+1, \ldots, \min(n, \tilde{n})$.*

Proof. See Appendix ■

At this point we need to establish one more result regarding a BP in the RH path relative to a block (k, n) in the optimal path of some BP. Specifically, if the current point in time is $\tilde{x}_t$ with $k-1 \leq t < n$ and $\tilde{x}_t + T \geq a_n$, then the RH BP that contains C_{t+1} ends at or after job C_n.

Lemma 4.5 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on the RH sample path that ends with job $C_{\tilde{n}}$. If $\tilde{x}_t + T \geq a_n$, then*

$$\tilde{n} \geq n$$

Proof. See Appendix ■

Now we are in a position to state and prove a key property of jobs in the RH path contained within the last BP block of an optimal path, i.e., that none of these jobs can be critical in the RH path (whereas, it is possible for C_n to become critical in the RH path).

Lemma 4.6 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on some BP of the RH sample path that ends with job $C_{\tilde{n}}$. Then, none of the jobs $C_{t+1}, \ldots, C_{\min\{\tilde{n}, n-1\}}$ can be critical in the RH path.*

Proof. See Appendix ■

It is easily seen from this lemma that the only possibility for a critical job in the RH path arises for indices n corresponding to the end of the last BP block (k, n) of an optimal path. In this case, it is possible that $\tilde{x}_n = a_{n+1}$.

At this point we are able to establish one of the main results of this section, i.e., the fact that the RH controller is such that $\tilde{x}_i \geq x_i^*$ for all $i = 1, 2, \ldots$ To do so, we prove two theorems corresponding to the two cases that may arise when the current decision time on the RH path is $\tilde{x}_t$ and t is contained in some optimal path BP (k, n): (i) $\tilde{x}_t + T \geq a_n$ (Theorem 4.1), and (ii) $\tilde{x}_t + T < a_n$ (Theorem 4.2). We first establish one additional auxiliary lemma which is useful in case (i).

Lemma 4.7 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on some BP of the RH sample path that ends with the job $C_{\tilde{n}}$ and assume that $\tilde{x}_t + T \geq a_n$. If $n = \tilde{n}$ and there exists some $i \in \{t+1, \ldots, n\}$ for which $\tilde{x}_i < x_i^*$, then $\tilde{u}_{i+1} \geq u_{i+1}^*$.*

Proof. See Appendix ■

Theorem 4.1 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on the RH sample path that ends with the job $C_{\tilde{n}}$ and assume that $\tilde{x}_t + T \geq a_n$. Then,*

$$\tilde{x}_i \geq x_i^* \quad \textit{for all } i = t+1, \ldots, n$$

Proof. Assume that there exists some $i \in \{t+1, \ldots, n\}$ such that $\tilde{x}_i < x_i^*$ while $\tilde{x}_{i-1} \geq x_{i-1}^*$(note that the last inequality is always satisfied for some i: for $t = k-1$, if $\tilde{x}_{k-1} < a_k$ then, by Lemma 4.1 we have $\tilde{x}_k = x_k^*$, otherwise we have $\tilde{x}_{k-1} \geq a_k = x_{k-1}^*$). Next, we will establish a contradiction. First, it follows from these two inequalities that $u_i^* > \tilde{u}_i$. In addition, since $\tilde{x}_t + T \geq a_n$ we can apply Lemma 4.5, therefore $\tilde{n} \geq n$. We now consider the following two possible cases.

Case 1: $n = \tilde{n}$. From Lemma 4.6, none of the jobs $C_{t+1}, \ldots, C_{n-1}$ can be critical. Therefore, the optimality conditions for controls $\tilde{u}_i$ and u_i^* (using Theorem 3.1 in [10]) give

$$\frac{d\theta_i}{d\tilde{u}_i} + \sum_{j=i}^{n} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_i}{du_i^*} + \sum_{j=i}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

Since $u_i^* > \tilde{u}_i$, Assumption **A1** implies that $\frac{d\theta_i}{du_i^*} > \frac{d\theta_i}{d\tilde{u}_i}$, and it follows that

$$\sum_{j=i}^{n} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=i}^{n} \frac{d\psi_j}{dx_j^*} \tag{9}$$

Similarly, the optimality conditions for controls $\tilde{u}_{i+1}$ and $u_{i+1}^*$are

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{n} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

Since $n = \tilde{n}$ and we have assumed that $\tilde{x}_i < x_i^*$, Lemma 4.7 implies that $\tilde{u}_{i+1} \geq u_{i+1}^*$. Therefore, by Assumption **A1**, $\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} \geq \frac{d\theta_{i+1}}{du_{i+1}^*}$ and the equation above gives

$$\sum_{j=i+1}^{n} \frac{d\psi_j}{d\tilde{x}_j} \leq \sum_{j=i+1}^{n} \frac{d\psi_j}{dx_j^*}$$

Moreover, since $\tilde{x}_i < x_i^*$, Assumption **A2** implies that $\frac{d\psi_i}{d\tilde{x}_i} < \frac{d\psi_i}{dx_i^*}$, so we can write

$$\sum_{j=i}^{n} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=i}^{n} \frac{d\psi_j}{dx_j^*}$$

which clearly contradicts (9) and completes the proof for *Case 1*.

Case 2: $\tilde{n} > n$. In this case, $x_n^* < a_{n+1}$, but $\tilde{x}_n \geq a_{n+1}$, which implies that $\tilde{x}_n > x_n^*$. On the other hand, we have $\tilde{x}_i < x_i^*$ for some $i < n$. Therefore, there exists at least one index $l \leq n$

such that $\tilde{x}_j < x_j^*$ for all $j \in \{i, \ldots, l-1\}$ and $\tilde{x}_l \geq x_l^*$. By Assumption **A2** applied to all $j \in \{i, \ldots, l-1\}$, we have

$$\sum_{j=i}^{l-1} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=i}^{l-1} \frac{d\psi_j}{dx_j^*} \tag{10}$$

In addition, when $\tilde{n} > n$ there may be one or more critical jobs in the RH BP, but, recalling Lemma 4.6 and the Remark following it, the next possible one after C_i is C_n. Thus, C_i and C_l are situated in the same block of the RH sample path. This allows us to use Lemma 5.2 in [10], which asserts that $\tilde{\xi}_i^+ = \tilde{\xi}_l^+$, i.e.,

$$\frac{d\theta_l}{d\tilde{u}_l} + \sum_{j=l}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_i}{d\tilde{u}_i} + \sum_{j=i}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j}$$

and cancelling common terms we obtain

$$\frac{d\theta_l}{d\tilde{u}_l} = \frac{d\theta_i}{d\tilde{u}_i} + \sum_{j=i}^{l-1} \frac{d\psi_j}{d\tilde{x}_j} \tag{11}$$

On the other hand, the optimality conditions of the controls u_i^* and u_l^* give

$$\frac{d\theta_l}{du_l^*} + \sum_{j=l}^{n} \frac{d\psi_j}{dx_j^*} = \frac{d\theta_i}{du_i^*} + \sum_{j=i}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

and cancelling common terms we get

$$\frac{d\theta_l}{du_l^*} = \frac{d\theta_i}{du_i^*} + \sum_{j=i}^{l-1} \frac{d\psi_j}{dx_j^*} \tag{12}$$

Moreover, since $u_i^* > \tilde{u}_i$, it follows from Assumption **A1** that $\frac{d\theta_i}{d\tilde{u}_i} < \frac{d\theta_i}{du_i^*}$. In view of this inequality and (10), it follows from (11) and (12) that

$$\frac{d\theta_l}{d\tilde{u}_l} < \frac{d\theta_l}{du_l^*}$$

which, by Assumption **A1**, implies that $\tilde{u}_l < u_l^*$. Since $\tilde{x}_{l-1} < x_{l-1}^*$, this further implies that $\tilde{x}_l < x_l^*$ which contradicts the fact that l was defined so that $\tilde{x}_l \geq x_l^*$. This completes the proof for *Case 2*. ■

The second theorem states a similar result, but we now consider the case where at the current decision time $\tilde{x}_t$ the RH window is not large enough to include the arrival time of C_n.

Theorem 4.2 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k-1 \leq t < n$, be the current decision time on the RH sample path that ends with the job $C_{\tilde{n}}$ and assume that the receding horizon window $[\tilde{x}_t, \tilde{x}_t + T]$ is such that $a_l \leq \tilde{x}_t + T < a_n$ where $l = \arg\max_{i>t}\{a_i : a_i \leq \tilde{x}_t + T\}$. Then,*

$$\tilde{x}_i \geq x_i^* \quad \text{for all } i = t+1, \ldots, l$$

Proof. Clearly, the RH path BP seen at decision time $\tilde{x}_t$ is such that $\tilde{n} \leq l < n$. It follows from Lemma 4.6 that none of the jobs $C_{t+1}, \ldots, C_{\tilde{n}}$ are critical in the RH path. We now prove the result using an inductive argument, starting with $\tilde{x}_k \geq x_k^*$ (corresponding to $t = k-1$). We consider two possible cases.

Case 1: C_k starts a new BP in the RH path, i.e., $\tilde{x}_{k-1} < a_k$. In this case, we can use Lemma 4.2 to immediately obtain $\tilde{x}_k \geq x_k^*$.

Case 2: C_k does not start a new BP in the RH path. This implies that $\tilde{x}_{k-1} \geq a_k$, while on the optimal path we have $x_{k-1}^* \leq a_k$ ($x_{k-1}^* < a_k$ if (k, n) is the BP without any critical job on the optimal path, and $x_{k-1}^* = a_k$ if (k, n) isn't the first Block of optimal paht BP). It follows that $\tilde{x}_{k-1} \geq x_{k-1}^*$. Now assume that $\tilde{x}_k < x_k^*$ and we shall establish a contradiction. Since none of the jobs $C_{t+1}, \ldots, C_{\tilde{n}}$ are critical in the RH path, we may apply Lemma 4.4 (setting $i = k$) to get $x_j^* > \tilde{x}_j$ for all $j = k+1, \ldots, \tilde{n}$.. Thus, we get

$$\tilde{x}_j < x_j^* \text{ for all } j = k, \ldots, \tilde{n} \tag{13}$$

Using Assumption **A2** for all inequalities above gives

$$\sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{dx_j^*} \tag{14}$$

Next, the optimality condition for the controls u_k^* and $\tilde{u}_k$ (using Theorem 3.1 in [10]) requires that

$$\frac{d\theta_k}{d\tilde{u}_k} + \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_k}{du_k^*} + \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*} = 0 \tag{15}$$

Since $\tilde{x}_{k-1} \geq x_{k-1}^*$ and we have assumed $\tilde{x}_k < x_k^*$, it follows that $u_k^* > \tilde{u}_k$. By Assumption **A1**, we have $\frac{d\theta_k}{du_k^*} > \frac{d\theta_k}{d\tilde{u}_k}$. Thus, (15) implies

$$\sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*}$$

Taking into account (14), the above inequality becomes

$$\sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} < 0 \tag{16}$$

The optimality condition for $u_{\tilde{n}+1}^*$is

$$\frac{d\theta_{i+1}}{du_{\tilde{n}+1}^*} + \sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

and in view of (16), we get

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} > 0$$

contradicting the property that $\theta_{n+1}(\cdot)$ is monotonically decreasing under Assumption **A1**. This completes the proof of $\tilde{x}_k \geq x_k^*$.

Next, we assume the result holds for $t = r < \tilde{n}$, i.e.,

$$\tilde{x}_{r+1} \geq x_{r+1}^* \tag{17}$$

We then need to prove the same inequality applies for $t = r+1$, i.e., $\tilde{x}_{r+2} \geq x_{r+2}^*$, to complete the induction argument. Again we assume the converse of this relationship, i.e. $\tilde{x}_{r+2} < x_{r+2}^*$, and will show a contradiction.

The optimalities of the controls $\tilde{u}_{r+2}$ and u_{r+2}^* require that

$$\frac{d\theta_{r+2}}{d\tilde{u}_{r+2}} + \sum_{j=r+2}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{r+2}}{du_{r+2}^*} + \sum_{j=r+2}^{n} \frac{d\psi_j}{dx_j^*} = 0 \tag{18}$$

Recalling (17) and $\tilde{x}_{r+2} < x_{r+2}^*$ we get $\tilde{u}_{r+2} < u_{r+2}^*$, which, by Assumption **A1**, implies that $\frac{d\theta_{r+2}}{d\tilde{u}_{r+2}} < \frac{d\theta_{r+2}}{du_{r+2}^*}$. Therefore, (18) implies that

$$\sum_{j=r+2}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=r+2}^{n} \frac{d\psi_j}{dx_j^*} \tag{19}$$

Moreover, applying Lemma 4.4 once again, we have $x_j^* > \tilde{x}_j$ for all $j = r+2, \ldots, \tilde{n}$. Thus, using Assumption **A2** for all these inequalities gives

$$\sum_{j=r+2}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=r+2}^{\tilde{n}} \frac{d\psi_j}{dx_j^*} \tag{20}$$

Combining (20) with (19) we obtain

$$\sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} < 0 \tag{21}$$

The optimality condition for the control $u_{\tilde{n}+1}^*$is that

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} + \sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

and, in view of (21), we get

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} > 0 \tag{22}$$

contradicting the property that $\theta_{n+1}(\cdot)$ is monotonically decreasing under Assumption **A1**. This shows that $\tilde{x}_{r+2} \geq x_{r+2}^*$ and completes the inductive argument. Consequently, we have $\tilde{x}_i \geq x_i^*$ for all $i = t+1, \ldots, \tilde{n}$.

Moreover, it is now easy to see that $\tilde{n} = l$. Suppose that $\tilde{n} < l$. Since we have just shown that $\tilde{x}_{\tilde{n}} \geq x^*_{\tilde{n}}$ and we know that on the optimal path $x^*_{\tilde{n}} > a_{\tilde{n}+1}$, it follows that $\tilde{x}_{\tilde{n}} > a_{\tilde{n}+1}$, which contradicts the fact that $C_{\tilde{n}}$ ends a BP. ■

Combining Theorems 4.1 and 4.2, it is clear that the inequality $\tilde{x}_i \geq x^*_i$ applies for all $i = k, \ldots, n$ in the last block of any optimal path BP. If $t = k - 1$ and the condition $\tilde{x}_t + T \geq a_n$ holds, then by Theorem 4.1 $\tilde{x}_i \geq x^*_i$ for all $i = k, \ldots, n$. If, on the other hand, $\tilde{x}_t + T < a_n$ and the RH window is such that $\tilde{x}_t + T \geq a_l$, then Theorem 4.2 applies and we have $\tilde{x}_i \geq x^*_i$ for all $i = k, \ldots, l$. Subsequently, for $t \geq k$, either Theorem 4.2 continues to apply so that $\tilde{x}_i \geq x^*_i$ for all $i = k+1, \ldots, l'$ with $l' \geq l$ or, at some point, $\tilde{x}_t + T \geq a_n$ will be satisfied and we get $\tilde{x}_i \geq x^*_i$ for all $i = k, \ldots, n$.

It is also clear that the RH path cannot have any critical jobs for $i = k, \ldots, n-1$, since $\tilde{x}_i \geq x^*_i > a_{i+1}$. However, for $i = n$ we have $x^*_n < a_{n+1}$; therefore, it is possible to have $\tilde{x}_n = a_{n+1} > x^*_n$.

We can now summarize the following properties of the BP structure on the RH sample path, relative to the optimal sample path:

Property 1: Suppose (k, n) is the last block of some optimal path BP. Then the RH BP that contains job C_k cannot end before the optimal path BP ends with C_n. This follows directly from Theorems 4.1 and 4.2 since $\tilde{x}_i \geq x^*_i$ for all $i = k, \ldots, n$.

Property 2: Suppose (k, n) and $(n + 1, m)$ are two consecutive BPs on the optimal path containing no critical jobs. If a BP on the RH path also starts with C_k, then, by Property 1 above, $\tilde{x}_n \geq x^*_n$. If $\tilde{x}_n \geq a_{n+1}$, then this RH BP cannot end before C_{n+m}, i.e., the two BPs on the optimal path merge into one. If $\tilde{x}_{n+m} \geq a_{n+m+1}$, then the following BP on the optimal path is also incorporated into the RH BP and so on. This merging process stops when an idle period on the optimal path is sufficiently long to satisfy $\tilde{x}_i < a_{i+1}$ for some i.

Property 3: Suppose (k, n) is the last block of some optimal path BP. Then, all jobs $C_k, \ldots, C_{n-1}$ will also be non-critical jobs on the RH path. This was established in Lemma 4.6.

4.2 Analysis of All Other Blocks in Optimal Path BPs

We now consider any block of an optimal path BP, which, if it is not the last one in the BP, ends with a critical event. As we will see, in the presence of a critical job, some of the properties of the previous section are lost. We proceed by modifying Assumption **A2** to consider functions $\psi_i(\cdot)$ that are monotonically increasing. This is in fact a reasonable condition to impose from a practical standpoint, as discussed in [17]. Thus, we will replace Assumption **A2** by the following:

Assumption A3. For each $i = 1 = 1, \ldots, N$, $\psi_i(\cdot)$ is strictly convex, twice continuously differentiable, and monotonically increasing.

As an example $\psi_i(x_i) = (x_i - a_i)^2$ satisfies Assumption **A3** above. In this case, $(x_i - a_i)$ is the *system time* of job C_i.

As will become clear, this assumption is needed to establish some additional properties of the RH controller. In the remainder of this subsection, unless otherwise specified, we will assume (k, n) defines an optimal path BP (not a block as before) with at least one critical job.

Lemma 4.8 *Let (k, n) define an optimal path BP with at least one critical job. Let $\tilde{x}_t$, $k - 1 \leq t < n$, be the current decision time on some BP of the RH sample path that ends with $C_{\tilde{n}}$. If $l = \arg\max_{i \geq t-1}\{i : a_i \leq \tilde{x}_t + T\}$, then*

$$\tilde{n} \geq n \quad \text{if } l \geq n \quad \text{and} \quad \tilde{n} = l \quad \text{if } l < n$$

Proof. See Appendix ■

The next lemma considers the case where $\tilde{x}_t + T \geq a_n$ and asserts that in the RH path none of the jobs which are not critical in the optimal path can become critical, with the possible exception of the last job in the optimal path BP.

Lemma 4.9 *Let (k, n) define an optimal path BP with at least one critical job. Let $\tilde{x}_t$, $k - 1 \leq t < n$, be the current decision time on some BP of the RH sample path and assume that $\tilde{x}_t + T \geq a_n$. Then, none of the non-critical jobs of the optimal path BP, can be critical in the RH path with the exception of C_n.*

Proof. See Appendix ■

The next theorem is significant because it allows us to establish the fact that $\tilde{x}_i$ upper bounds x_i^* under certain conditions.

Theorem 4.3 *Let (k, n) define an optimal path BP with at least one critical job. Let $\tilde{x}_t$, $k - 1 \leq t < n$, be the current decision time on some BP of the RH sample path and assume that $\tilde{x}_t + T \geq a_n$. If there are r critical jobs indexed by $B_1 < \ldots < B_r$ in the optimal path between C_{t+1} and C_n and none of these jobs is critical in the RH path, then*

$$\begin{aligned} \tilde{x}_i &\geq x_i^* \text{ for all } i = t+1, \ldots, n \\ \text{and } \tilde{x}_i &> x_i^* \text{ for all } i = B_1, \ldots, B_r \end{aligned}$$

Proof. If $C_{\tilde{n}}$ is the last job of the RH path that contains C_{t+1}, then from Lemma 4.8 , $\tilde{n} \geq n$. Further, note that this situation is exactly identical to *Case 1* in the proof of of Lemma 4.9 (see Appendix). Repeating the exact same argument leading to (71) we obtain

$$\tilde{x}_i \geq x_i^* \text{ for all } i = \{t+1, \ldots, B_1 - 1), \ldots, \{B_{r-1} + 1, \ldots, B_r - 1\}, \{B_r + 1, \ldots, n\}$$

In addition, since $\tilde{n} \geq n$ and $C_{B_1}, \ldots, C_{B_r}$ are critical in the optimal path but non-critical in RH path, we have

$$\tilde{x}_i > x_i^* = a_{i+1} \text{ for all } i = B_1, \ldots, B_r$$

which completes the proof. ■

4.3 Error Reducing Properties of the RH Controller

Let us define

$$\varepsilon_i = \tilde{x}_i - x_i^* \tag{23}$$

to be the *error* in the ith departure time resulting from applying the RH controller instead of the optimal controller. In what follows, we shall establish that the RH controller is characterized by an *error reducing property* under certain conditions. To begin with, returning to Theorems 4.1 and 4.2, it is immediately obvious that $\varepsilon_i \geq 0$ for all $i = k, \ldots, n$, as long as we consider an optimal path BP (k, n) with no critical jobs. Moreover, recalling Lemma 4.1, if the current decision time on the RH path is $\tilde{x}_t$, $t < k$, and $\tilde{x}_t + T \geq a_n$, then $\varepsilon_i = 0$ for all $i = k, \ldots, n$. In other words, if the RH window contains an entire optimal path BP, then the RH controller yields the optimal controls. On the other hand, if the optimal path does contain one or more critical jobs, it is no longer true that $\varepsilon_i \geq 0$; in fact, the error can become negative (note that Theorem 4.3 only holds for $\tilde{x}_t + T \geq a_n$). However, as we show next, there are segments of a RH path such that $\varepsilon_{i+1} \leq \varepsilon_i$ for several i in a row, and there are conditions under which we can in fact get $\varepsilon_i = 0$ for several i in a row.

Let us first concentrate again on the last block in an optimal path BP (or an entire BP with no critical jobs). We will show that in such a block the above error reducing property holds when T is sufficiently large. This property rests upon the following lemma.

Lemma 4.10 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$, $k \leq t < n$, be the current decision time on some RH sample path and assume that $\tilde{x}_t + T \geq a_n$. Then, under Assumptions **A1**, **A2**,*

$$u_i^* \geq \tilde{u}_i \quad \text{for all } i = t+1, \ldots, n$$

Proof. See Appendix ■

Combining Lemma 4.2 with the result above gives us a good insight into the error made by the RH controller as the window rolls forward in time. Specifically, when an optimal path BP (k, n) starts with some C_k and the RH controller is at $\tilde{x}_t < a_k$ with a window which is not large enough to include a_n, then $\tilde{u}_k \geq u_k^*$. As we move forward in time, as long as $\tilde{x}_t + T < a_n$, RH controls may continue to satisfy this inequality, i.e., $\tilde{u}_{k+1} \geq u_{k+1}^*, \ldots$, which increases the errors $\varepsilon_{k+1}, \ldots$ At some point, however, if $\tilde{x}_t + T \geq a_n$, then the lemma above asserts that this inequality is reversed for the remaining $C_{t+1}, \ldots, C_n$ and this in turn reduces the errors. This property is formally established in the following.

Theorem 4.4 *Let (k, n) define the last block on some optimal sample path BP. Let $\tilde{x}_t$ be the current decision time on some BP of the RH sample path and assume that $\tilde{x}_t + T \geq a_n$. Then, under Assumptions **A1**, **A2**,*

(i) If $t < k$ and C_k also starts a BP in the RH path: $\varepsilon_i = 0$ for all $i = k, \ldots, n$

(ii) If $k \leq t < n$:

$$\varepsilon_{i+1} \leq \varepsilon_i \quad \text{for all } i = t+1, \ldots, n$$

and if $\varepsilon_j = 0$ *for some* $j \in \{t+1, \ldots, n\}$*, then* $\varepsilon_i = 0$ *for all* $i \in \{j+1, \ldots, n\}$*.*

Proof. The first part is a direct consequence of Lemma 4.1. The second part follows from Lemma 4.10, where we established that under $\tilde{x}_t + T \geq a_n$ we have $\tilde{u}_{i+1} \leq u^*_{i+1}$ for $i = t, \ldots, n-1$, hence $\tilde{x}_{i+1} - \tilde{x}_i \leq x^*_{i+1} - x^*_i$. Using the definition of the error in (23), it follows that

$$\varepsilon_{i+1} \leq \varepsilon_i$$

The remainder of part (ii) part of the lemma easily follows from Lemma 4.10. Since $\tilde{u}_{j+1} \leq u^*_{j+1}$, either $\tilde{u}_{j+1} < u^*_{j+1}$ or $\tilde{u}_{j+1} = u^*_{j+1}$. Since $\varepsilon_j = \tilde{x}_j - x^*_j = 0$, the latter equality implies that $\varepsilon_{j+1} = 0$. It remains to show that the inequality $\tilde{u}_{j+1} < u^*_{j+1}$ is infeasible. Since $\varepsilon_j = 0$ implies that $\tilde{x}_j = x^*_j$, then $\tilde{u}_{j+1} < u^*_{j+1}$ implies that $\tilde{x}_{j+1} < x^*_{j+1}$which contradicts Theorem 4.1. Thus, $\varepsilon_{j+1} = 0$ and a similar argument applies for all remaining $\varepsilon_{j+2}, \ldots, \varepsilon_n$. ■

Regarding all other blocks, we can use the properties derived in the last two sections and, under Assumption **A3**, we can establish a similar error reducing property.

Theorem 4.5 *Let* (k, n) *define an optimal path BP with at least one critical job. Let* $\tilde{x}_t$*,* $k - 1 \leq t < n$*, be the current decision time on some BP of the RH sample path and assume that* $\tilde{x}_t + T \geq a_n$*. If there are* r *critical jobs indexed by* $B_1 < \ldots < B_r$ *in the optimal path between* C_{t+1} *and* C_n *and none of these jobs is critical in the RH path, then, under Assumptions* ***A1****,* ***A3****,*

$$\varepsilon_{i+1} \leq \varepsilon_i \text{ for all } i = t+2, \ldots, n$$

Proof. Let us assume that the RH BP that contains C_{t+1} ends at $C_{\tilde{n}}$. Since $\tilde{x}_t + T \geq a_n$, $l \geq n$, we have $\min(n, l) = n$, therefore Lemma 4.8 implies $\tilde{n} \geq n$. Also, assume $C_{\tilde{B}}$ is the next RH critical job after C_{t+1} in the same RH BP. As none of the jobs $C_{B_1}, \ldots, C_{B_r}$ is critical in the RH path, then using Lemma 4.9 none of the jobs $C_{t+1}, \ldots, C_{n-1}$ is critical in the RH path, which implies $\tilde{B} \geq n$.

Now considering the optimality condition of $\tilde{u}_{t+1}$(using Theorem 3.1 and Lemma 5.1 in [10]), we have

$$\tilde{\xi}^-_{t+1} = \frac{d\theta_{t+1}}{d\tilde{u}_{t+1}} + \sum_{j=t+1}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} \leq 0$$

while the optimality condition of u^*_{t+1} gives

$$\xi^+_{t+1} = \frac{d\theta_{t+1}}{du^*_{t+1}} + \sum_{j=t+1}^{n} \frac{d\psi_j}{dx^*_j} \geq 0$$

Combining these two inequalities yields

$$\frac{d\theta_{t+1}}{d\tilde{u}_{t+1}} + \sum_{j=t+1}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} \leq \frac{d\theta_{t+1}}{du^*_{t+1}} + \sum_{j=t+1}^{n} \frac{d\psi_j}{dx^*_j} \tag{24}$$

However, since $\tilde{x}_t + T \geq a_n$ and none of the jobs $C_{B_1}, \ldots, C_{B_r}$ is critical in the RH path, it follows from Theorem 4.3 that

$$\tilde{x}_i \geq x_i^* \text{ for all } i = t+1, \ldots, n$$

which, under Assumption **A3**, gives

$$\sum_{j=t+1}^{n} \frac{d\psi_j}{dx_j^*} \leq \sum_{j=t+1}^{n} \frac{d\psi_j}{d\tilde{x}_j} \tag{25}$$

Comparing (24) with (25) we get

$$\frac{d\theta_{t+1}}{d\tilde{u}_{t+1}} + \sum_{j=n+1}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} \leq \frac{d\theta_{t+1}}{du_{t+1}^*}$$

Using the monotonically increasing property of $\psi_i(\cdot)$ under Assumption **A3**,$\sum_{j=n+1}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} \geq 0$, therefore

$$\frac{d\theta_{t+1}}{d\tilde{u}_{t+1}} \leq \frac{d\theta_{t+1}}{du_{t+1}^*}$$

which, under Assumption **A1**, yields $u_{t+1}^* \geq \tilde{u}_{t+1}$. This is a recursive relationship, and repeating the same process for the remaining jobs $C_{t+2}, \ldots, C_n$ we get

$$u_i^* \geq \tilde{u}_i \text{ for all } i = t+1, \ldots, n \tag{26}$$

Further, since $\tilde{x}_i \geq x_i^*$ for all $i = t+1, \ldots, n$, it follows that (26) for all $i = t+2, \ldots, n$ gives $x_i^* - x_{i-1}^* \geq \tilde{x}_i - \tilde{x}_{i-1}$ i.e. $\tilde{x}_{i-1} - x_{i-1}^* \geq \tilde{x}_i - x_i^*$. Then, using the definition of *error*, we get

$$\varepsilon_{i+1} \leq \varepsilon_i \text{ for all } i = t+2, \ldots, n$$

completing the proof. ■

An interesting observation resulting from the theorem above is that as long as no critical jobs are observed on an unfolding RH sample path (under Assumptions **A1**, **A3**), we can conclude that the error reducing property holds whenever T satisfies $\tilde{x}_t + T \geq a_n$.

Appendix

Proof of Lemma 4.1. The result is a consequence of Theorem 3 in *Cho et. al.* [11] which states that the optimal controls of the block formed by jobs $C_k, \ldots, C_n$ only depend on the arrivals times $a_k, \ldots, a_{n+1}$. If (k, n) is the first block in a BP, then the condition $\tilde{x}_{k-1} \leq a_k$ confirms that C_k starts a BP in both the RH and the optimal sample paths. Moreover, since $\tilde{x}_{k-1} + T \geq a_{n+1}$, the RH controller has knowledge of all $a_k, \ldots, a_{n+1}$ at time $\tilde{x}_{k-1}$. Therefore, from Theorem 3 of [11] $u_i^* = \tilde{u}_i = u_i(k, n)$ and

$$x_i^* = \tilde{x}_i \quad = a_k + \sum_{j=k}^{i} u_j^* \quad \text{for all } i = k, \ldots, n$$

which completes the proof. On the other hand, if (k, n) is not the first block in a BP, then the condition $\tilde{x}_{k-1} = a_k$ ensures that C_k starts a block in both the RH and the optimal sample paths and the rest of the argument is the same as above. ■

Proof of Lemma 4.2. Since (k, n) is an optimal path block, then the jobs $C_k, \ldots, C_{n-1}$ are non-critical in the optimal path. Let us consider a fictitious problem of the form $Q(k, n)$ in (6) with arrival times $a_k, \ldots, a_n$, $a_{n+1} = \infty$, and corresponding optimal departure times $x_i(k, n)$ for all $i = k, \ldots, n$. Clearly, the solution of this fictitious problem contains no critical jobs. It follows from Theorem 5.4 of [10] that the optimal path departure times x_i^* lower-bound those obtained from $Q(k, n)$, i.e.,

$$x_i(k, n) \geq x_i^* \text{ for all } i = k, \ldots, n \tag{27}$$

Next, we show that none of the jobs $C_k, \ldots, C_l$ in the RH path can emerge as critical under the given condition. Thus, assume that the RH BP evaluated at $\tilde{x}_{k-1}$ ends at $C_{\tilde{n}}$, $\tilde{n} \leq l$, and that there exists some $C_{\tilde{B}}$, $k \leq \tilde{B} < \tilde{n} < n$, which happens to be critical. Further, construct a fictitious RH path BP that starts with the same C_k and a window size T' such that $\tilde{x}_{k-1} + T' \geq a_{n+1}$ with $\tilde{x}_{k-1} \leq a_k$. We denote the departure times in this fictitious RH path by $\tilde{x}'_i$ for all $i = k, \ldots, n$. Applying Lemma 4.1 we have

$$\tilde{x}'_i = x_i^* \text{ for all } i = k, \ldots, n \tag{28}$$

Invoking Theorem 3 of [11] (as in the proof of Lemma 4.1) and considering the fact that both the original RH path block (with window size T) and the fictitious RH path block (with window size $T' > T$) start with job C_k, we can immediately write

$$\tilde{x}_i = \tilde{x}'_i = \tilde{x}_i(k, \tilde{B}) \quad \text{for all } i = k, \ldots, \tilde{B}$$

Again, since $C_{\tilde{B}}$ is a critical job in the RH path, $\tilde{x}_{\tilde{B}} = \tilde{x}'_{\tilde{B}} = a_{\tilde{B}+1}$. Comparing this with (28) implies that $x^*_{\tilde{B}} = \tilde{x}'_{\tilde{B}} = a_{\tilde{B}+1}$, i.e., $C_{\tilde{B}}$ is critical in the optimal path BP with $\tilde{B} < n$. However, we know that C_n is the last job of the current block in the optimal path with $n \geq l > \tilde{B}$. Thus, $\tilde{B}$ being critical in the optimal path contradicts our assumption. Therefore we conclude that none of the jobs $C_k, \ldots, C_l$ can be critical. Given this fact, we can apply Theorem 5.2 of [10] for the RH BP $(k, \tilde{n})$ and the fictitious one (k, n) considered earlier, where $\tilde{n} \leq n$ and neither one of these BPs contains critical jobs. Thus, we obtain

$$\tilde{x}_i \geq x_i(k, n) \text{ for all } i = k, \ldots, \tilde{n} \tag{29}$$

Then, combining (27) and (29) we get

$$\tilde{x}_i \geq x_i^* \text{ for all } i = k, \ldots, \tilde{n}$$

Finally, since $\tilde{x}_k \geq x_k^*$ we get $a_k + \tilde{u}_k \geq a_k + u_k^*$, i.e., $\tilde{u}_k \geq u_k^*$ which completes the proof. ■

Proof of Lemma 4.3. We prove this lemma by assuming that $x_i^* \leq \tilde{x}_i$ and showing a contradiction. Since there is at least one critical job in the RH path BP between any job $i \in \{t+1, \ldots, B-1\}$ and the end of the BP, then, using Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for the control $\tilde{u}_{i+1}$ requires that

$$\tilde{\xi}^+_{i+1} = \frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > 0$$

Since none of the jobs $C_k, \dots, C_n$ are critical in the optimal sample path , the optimality condition for u^*_{i+1} is

$$\xi^{*+}_{i+1} = \frac{d\theta_{i+1}}{du^*_{i+1}} + \sum_{j=i+1}^{n} \frac{d\psi_j}{dx^*_j} = 0$$

and we get

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \frac{d\theta_{i+1}}{du^*_{i+1}} + \sum_{j=i+1}^{n} \frac{d\psi_j}{dx^*_j} = 0 \tag{30}$$

By assumption, $x^*_{i+1} > \tilde{x}_{i+1}$, therefore $\max(x^*_i, a_{i+1}) + u^*_{i+1} > \max(\tilde{x}_i, a_{i+1}) + \tilde{u}_{i+1} \geq \tilde{x}_i + \tilde{u}_{i+1}$. Moreover, since there are no critical jobs in the optimal path BP, $x^*_i > a_{i+1}$, and we have $x^*_i + u^*_{i+1} > \tilde{x}_i + \tilde{u}_{i+1}$. Under the assumption $x^*_i \leq \tilde{x}_i$, it follows that $u^*_{i+1} > \tilde{u}_{i+1}$. Then, by Assumption **A1**,

$$\frac{d\theta_{i+1}}{du^*_{i+1}} > \frac{d\theta_{i+1}}{d\tilde{u}_{i+1}}$$

so that (30) implies

$$\sum_{j=i+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=i+1}^{n} \frac{d\psi_j}{dx^*_j} \tag{31}$$

By assumption, $x^*_j > \tilde{x}_j$ for all $j = i+1, \dots, B$, therefore, using Assumption **A2**,

$$\sum_{j=i+1}^{B} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=i+1}^{B} \frac{d\psi_j}{dx^*_j} \tag{32}$$

and since $B < \min\{n, \tilde{n}\}$, it follows from (31) that

$$\sum_{j=B+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=B+1}^{n} \frac{d\psi_j}{dx^*_j} \tag{33}$$

Since none of the jobs $C_{B+1}, \dots, C_{\tilde{n}}$ and $C_{B+1}, \dots, C_n$ are critical in the RH and optimal path respectively, the optimality condition for the controls u^*_{B+1} and $\tilde{u}_{B+1}$ (again, using Theorem 3.1 in [10]) gives

$$\frac{d\theta_{B+1}}{d\tilde{u}_{B+1}} + \sum_{j=B+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{B+1}}{du^*_{B+1}} + \sum_{j=B+1}^{n} \frac{d\psi_j}{dx^*_j} = 0$$

In view of (33), this yields

$$\frac{d\theta_{B+1}}{d\tilde{u}_{B+1}} < \frac{d\theta_{B+1}}{du^*_{B+1}}$$

Therefore, by Assumption **A1**, $\tilde{u}_{B+1} < u^*_{B+1}$. Since, by assumption, $x^*_B > \tilde{x}_B$, it follows that

$$x^*_{B+1} > \tilde{x}_{B+1}$$

We can use this inequality and recursively repeat the process above starting with (32) and replacing B with $B+1$ to obtain

$$x^*_i > \tilde{x}_i \quad \text{for all } i = B+1, \dots, \min\{n, \tilde{n}\} \tag{34}$$

and using Assumption **A2** for all i above results in

$$\sum_{j=B+1}^{\min\{n,\tilde{n}\}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=B+1}^{\min\{n,\tilde{n}\}} \frac{d\psi_j}{dx_j^*} \tag{35}$$

There are now two cases to consider:

Case 1: If $\tilde{n} > n$, (34) becomes $x_i^* > \tilde{x}_i$ for all $i = B+1, \ldots, n$. But since C_n is the last job of the BP (since it is the last Block) in the optimal path, we have $x_n^* < a_{n+1}$, and it follows that $\tilde{x}_n < x_n^* < a_{n+1}$. This implies that C_n is the last job of the current RH BP, which contradicts the condition $\tilde{n} > n$. This case is, therefore, infeasible.

Case 2: If $\tilde{n} \leq n$, then (35) becomes

$$\sum_{j=B+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=B+1}^{\tilde{n}} \frac{d\psi_j}{dx_j^*} \tag{36}$$

In view of (33), this implies

$$\sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} < 0 \tag{37}$$

Using the optimality condition for the control $u_{\tilde{n}+1}^*$:

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} + \sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

along with (37) we get

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} > 0$$

This is a contradiction of the property of $\theta_{\tilde{n}+1}(\cdot)$ to be monotonically decreasing under Assumption **A1** and the proof is complete. ■

Proof of Corollary 4.1. In the proof of Lemma 4.3, the optimality condition $\tilde{\xi}_j^+ > 0$ can be replaced by $\tilde{\xi}_j^+ \geq 0$ which does not affect any of the remaining arguments. ■

Proof of Lemma 4.4. Consider $j = i+1$ and assume $x_{i+1}^* \leq \tilde{x}_{i+1}$. We shall then establish a contradiction. Since job C_{i+1} is non-critical in the RH as well as the optimal path, using Theorem 3.1 in [10], the optimality conditions for the controls u_{i+1}^* and $\tilde{u}_{i+1}$ give

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{n} \frac{d\psi_j}{dx_j^*} = 0 \tag{38}$$

Under the assumption $x_{i+1}^* \leq \tilde{x}_{i+1}$, we have $\max(x_i^*, a_{i+1}) + u_{i+1}^* \leq \max(\tilde{x}_i, a_{i+1}) + \tilde{u}_{i+1}$. Since there are no critical jobs in the optimal path BP, $x_i^* > a_{i+1}$. On the other hand, for any job i

within the RH BP we have $\tilde{x}_i \geq a_{i+1}$. Thus, $x_i^* + u_{i+1}^* \leq \tilde{x}_i + \tilde{u}_{i+1}$. By assumption, $x_i^* > \tilde{x}_i$, therefore $u_{i+1}^* < \tilde{u}_{i+1}$. Then, by Assumption **A1**,

$$\frac{d\theta_{i+1}}{du_{i+1}^*} < \frac{d\theta_{i+1}}{d\tilde{u}_{i+1}}$$

so that (38) implies

$$\sum_{j=i+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=i+1}^{n} \frac{d\psi_j}{dx_j^*}$$

In addition, since $x_{i+1}^* \leq \tilde{x}_{i+1}$, using Assumption **A2** implies that

$$\sum_{j=i+2}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < \sum_{j=i+2}^{n} \frac{d\psi_j}{dx_j^*} \tag{39}$$

Now since job C_{i+2} is also non-critical in the RH as well as the optimal path, the optimality conditions for the controls u_{i+2}^* and $\tilde{u}_{i+2}$ (using Theorem 3.1 in [10]) give

$$\frac{d\theta_{i+2}}{d\tilde{u}_{i+2}} + \sum_{j=i+2}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{i+2}}{du_{i+2}^*} + \sum_{j=i+2}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

and using (39) we obtain

$$\frac{d\theta_{i+2}}{d\tilde{u}_{i+2}} > \frac{d\theta_{i+2}}{du_{i+2}^*}$$

Therefore, by Assumption **A1**, $\tilde{u}_{i+2} > u_{i+2}^*$. Since we are assuming $x_{i+1}^* \leq \tilde{x}_{i+1}$, it follows that

$$\tilde{x}_{i+2} > x_{i+2}^*$$

We can now use this inequality and recursively repeat the process above starting with (39) and replacing $i+2$ with $i+3$ to obtain

$$\tilde{x}_j > x_j^* \text{ for all } j = i+2, \ldots, \min(n, \tilde{n}) \tag{40}$$

There are now two cases to consider:

Case 1: If $\tilde{n} < n$, (40) becomes $\tilde{x}_j > x_j^*$ for all $j = i+2, \ldots, \tilde{n}$. But since $C_{\tilde{n}}$ is the last job of the BP in the RH path, we have $\tilde{x}_{\tilde{n}} < a_{\tilde{n}+1}$, and it follows that $x_{\tilde{n}}^* < \tilde{x}_{\tilde{n}} < a_{\tilde{n}+1}$. This implies that $C_{\tilde{n}}$ is the last job of the optimal path BP, which contradicts the condition $\tilde{n} < n$. This case is, therefore, infeasible.

Case 2: If $n \leq \tilde{n}$, then (40) becomes $\tilde{x}_j > x_j^*$ for all $j = i+2, \ldots, n$. Using Assumption **A2**, we get

$$\sum_{j=i+2}^{n} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=i+2}^{n} \frac{d\psi_j}{dx_j^*} \tag{41}$$

In view of (39), this implies

$$\sum_{j=n+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} < 0 \tag{42}$$

Using the optimality condition for $\tilde{u}_{n+1}$ (again, using Theorem 3.1 in [10]), since none of the jobs $C_{n+1}, \ldots, C_{\tilde{n}-1}$ are critical in the RH sample path:

$$\frac{d\theta_{n+1}}{d\tilde{u}_{n+1}} + \sum_{j=n+1}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} = 0$$

along with (42), we obtain

$$\frac{d\theta_{n+1}}{d\tilde{u}_{n+1}} > 0 \tag{43}$$

This is a contradiction of the property of $\theta_{n+1}(\cdot)$ to be monotonically decreasing under Assumption **A1**. Therefore, assuming $x^*_{i+1} \leq \tilde{x}_{i+1}$ has led to a contradiction, thus establishing the fact that $x^*_{i+1} > \tilde{x}_{i+1}$.

The same argument can now be repeated by replacing i by $i+1$ to prove that $x^*_{i+2} > \tilde{x}_{i+2}$ and the process repeats for all $j = i+1, ..., \min(n, \tilde{n})$. This completes the proof. ■

Proof of Lemma 4.5. We shall assume that $\tilde{n} < n$ and establish a contradiction. Since $C_{\tilde{n}}$ is the last job in the RH path we have $\tilde{x}_{\tilde{n}} < a_{\tilde{n}+1}$. However, the optimal path Block (as well as BP) does not end at $C_{\tilde{n}}$, therefore $x^*_{\tilde{n}} > a_{\tilde{n}+1}$. It follows that $x^*_{\tilde{n}} > \tilde{x}_{\tilde{n}}$. Set $B \equiv \min\{\tilde{n}, n\} = \tilde{n}$ and since $C_{\tilde{n}}$ is clearly not critical we may apply Corollary 4.1 with $i = B-1$ to obtain $x^*_{\tilde{n}-1} > \tilde{x}_{\tilde{n}-1}$. Next, consider job $C_{\tilde{n}-1}$. If it is not critical, we can again apply Corollary 4.1 with $i = \tilde{n}-2$, otherwise we apply Lemma 4.3 with $i = \tilde{n}-2$ and $B = \tilde{n}-1$ to obtain $x^*_{\tilde{n}-2} > \tilde{x}_{\tilde{n}-2}$. Following the same process recursively we establish that

$$x^*_i > \tilde{x}_i \text{ for all } i = t+1, ..., \tilde{n} \tag{44}$$

for any $t = k-1, \ldots, n-1$. Setting $t = k-1$ above gives

$$x^*_i > \tilde{x}_i \text{ for all } i = k, \ldots, \tilde{n} \tag{45}$$

There are now two cases to consider, as follows.

Case 1: If C_k starts a new RH BP, i.e., $\tilde{x}_{k-1} < a_k$, then we can invoke Lemma 4.1 which asserts that $\tilde{x}_k = x^*_k$. This directly contradicts the inequality $\tilde{x}_k < x^*_k$ that we obtain from (45) by setting $i = k$. We must conclude that $\tilde{n} \geq n$.

Case 2: If C_k does not start a new RH BP, we have $\tilde{x}_{k-1} \geq a_k$, while on the optimal path we have $x^*_{k-1} < a_k$.. Therefore, $\tilde{x}_{k-1} > x^*_{k-1}$. In addition, we have $x^*_k > \tilde{x}_k$ from (45) and it follows that $\tilde{u}_k < u^*_k$.

Using Theorem 3.1 and lemma 5.1 in [10], the optimality condition for $\tilde{u}_k$ requires that

$$\tilde{\xi}^+_k = \frac{d\theta_k}{d\tilde{u}_k} + \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} \geq 0$$

while the optimality for control u^*_k (recalling that there are no critical jobs in the optimal path BP) gives

$$\frac{d\theta_k}{du^*_k} + \sum_{j=k}^{n} \frac{d\psi_j}{dx^*_j} = 0$$

Combining these two relationships we get

$$\frac{d\theta_k}{d\tilde{u}_k} + \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} \geq \frac{d\theta_k}{du_k^*} + \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

Under the condition $\tilde{u}_k < u_k^*$ established above, we can use Assumption **A1** to get

$$\sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*}$$

In addition, using (45) and Assumption **A2** for all $i = k, \ldots, \tilde{n}$, we get

$$\sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} < \sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{dx_i^*}$$

Comparing the last two inequalities we obtain

$$\sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} < 0$$

Using the optimality condition of the control $u_{\tilde{n}+1}^*$ on the optimal path:

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} + \sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

it follows that

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} > 0 \tag{46}$$

which contradicts the property that $\theta_{n+1}(\cdot)$ is monotonically decreasing under Assumption **A1**. We must, therefore, conclude that $\tilde{n} \geq n$, which completes the proof. ■

Proof of Lemma 4.6. We shall assume that there exists at least one critical job and establish a contradiction. Let B index the last critical job (if there are more than one) with $t + 1 \leq B < \min\{\tilde{n}, n\}$. Since C_B is a critical job in the RH path but not the optimal path, we have $x_B^* > \tilde{x}_B = a_{B+1}$. Therefore we can apply Lemma 4.3 with $i = B - 1$ to obtain $x_{B-1}^* > \tilde{x}_{B-1}$. Repeating this process with $i = B - 2, \ldots, t + 1$ we then get

$$x_j^* > \tilde{x}_j \text{ for all } j = t + 1, \ldots, B \tag{47}$$

We now proceed by considering two mutually exclusive cases, as follows.

Case 1: $\tilde{x}_t + T \geq a_n$. In this case, Lemma 4.5 implies that $\tilde{n} \geq n$. If C_k starts a new RH BP, i.e., $\tilde{x}_k < a_{k+1}$, then we can invoke Lemma 4.1 stating that $\tilde{x}_k = x_k^*$ which directly contradicts (47) with $t = k-1$. If, on the other hand, C_k does not start a new RH BP, then $\tilde{x}_{k-1} \geq a_k$, while $x_{k-1}^* \leq a_k$ since C_k starts a Block in the optimal path ($x_{k-1}^* < a_k$ if (k, n) is the BP without any critical job on the optimal path, and $x^*_{k-1} = a_k$ if (k, n) isn't the first Block of optimal paht BP). It follows that $\tilde{x}_{k-1} \geq x^*_{k-1}$. In addition, setting $t = k - 1$ in (47), we obtain

$$x^*_i > \tilde{x}_i \text{ for all } i = k, \ldots, B \tag{48}$$

Since none of the jobs following C_B is critical in the RH BP and we have $x^*_B > \tilde{x}_B$, we can apply Lemma 4.4 with $i = B$ and $\min\{n, \tilde{n}\} = n$ to get

$$x^*_j > \tilde{x}_j \text{ for all } j = B + 1, \ldots, n \tag{49}$$

Combining (48) and (49) yields

$$x^*_i > \tilde{x}_i \text{ for all } i = k, \ldots, n \tag{50}$$

Now since C_n is the last job of the optimal path BP, we have $x^*_n < a_{n+1}$, whereas setting $i = n$ in (50) gives $x^*_n > \tilde{x}_n$, therefore, $\tilde{x}_n < x^*_n < a_{n+1}$ which contradicts the condition $n \leq \tilde{n}$. It follows that under this case C_B cannot be a critical job in the RH path.

Case 2: $\tilde{x}_t + T < a_n$. In this case, if C_k starts a new busy period in the RH path, i.e., $\tilde{x}_{k-1} < a_k$, then we can invoke Lemma 4.2 stating that $\tilde{x}_k \geq x^*_k$, which directly contradicts (47) with $t = k - 1$. Thus, we only need to consider the case where C_k does not start a new RH BP. Then, $\tilde{x}_{k-1} \geq a_k$, while $x^*_{k-1} \leq a_k$ since C_k starts a Block in the optimal path ($x^*_{k-1} < a_k$ if (k, n) is the BP without any critical job on the optimal path, and $x^*_{k-1} = a_k$ if (k, n) isn't the first Block of optimal paht BP). It follows that $\tilde{x}_{k-1} \geq x^*_{k-1}$. In addition, the fact that $\tilde{x}_t + T < a_n$ implies that the current RH BP must end with some $C_{\tilde{n}}$ such that $\tilde{n} < n$.

Setting $t = k - 1$ in (47), we have

$$x^*_i > \tilde{x}_i \text{ for all } i = k, \ldots, B \tag{51}$$

Moreover, since none of the jobs $C_{B+1}, \ldots, C_{\tilde{n}}$ are critical in the RH path and $x^*_B > \tilde{x}_B$, we can apply Lemma 4.4 with $i = B$ and $\min\{n, \tilde{n}\} = \tilde{n}$ to get

$$x^*_i > \tilde{x}_i \text{ for all } i = B + 1, ..., \tilde{n} \tag{52}$$

Combining (51) and (52) gives

$$x^*_i > \tilde{x}_i \text{ for all } i = k, \ldots, \tilde{n}$$

Thus, using Assumption **A2**, we have

$$\sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} < \sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{dx^*_i} \tag{53}$$

Using Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for the control $\tilde{u}_k$ requires that

$$\tilde{\xi}^+_k = \frac{d\theta_k}{d\tilde{u}_k} + \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} \geq 0$$

whereas the optimality of u_k^* (recalling that there are no critical jobs in the optimal path BP) is

$$\frac{d\theta_k}{du_k^*} + \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

Combining these two conditions implies

$$\frac{d\theta_k}{d\tilde{u}_k} + \sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} \geq \frac{d\theta_k}{du_k^*} + \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*}$$

Now, recalling that $\tilde{x}_{k-1} \geq x_{k-1}^*$ and $\tilde{x}_k < x_k^*$, we get $\tilde{u}_k < u_k^*$. By Assumption **A1**, $\frac{d\theta_k}{d\tilde{u}_k} < \frac{d\theta_k}{du_k^*}$ so that the inequality above reduces to

$$\sum_{j=k}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=k}^{n} \frac{d\psi_j}{dx_j^*}$$

and, in light of (53), we further obtain

$$\sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} < 0 \tag{54}$$

The optimality condition for the control $u_{\tilde{n}+1}^*$is

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} + \sum_{j=\tilde{n}+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

which, using (54), implies

$$\frac{d\theta_{\tilde{n}+1}}{du_{\tilde{n}+1}^*} > 0$$

contradicting the property that $\theta_{n+1}(\cdot)$ is monotonically decreasing under Assumption **A1**.

Therefore, we have shown that assuming there exists at least one critical job in the RH BP leads to a contradiction and the presumed last critical job, indexed by B, cannot be critical. Repeating the exact same process for any other presumed last critical job B', with $t+1 \leq B' < B$, leads to a similar contradiction and the proof is complete. ■

Proof of Lemma 4.7. First, using Lemma 4.6 with $n = \tilde{n}$, none of the jobs $C_{t+1}, \ldots, C_{n-1}$ can be critical in the RH sample path. Next, we will assume that when $\tilde{x}_i < x_i^*$ for some $i \in \{t+1, \ldots, n\}$ then $\tilde{u}_{i+1} < u_{i+1}^*$, which implies $\tilde{x}_{i+1} < x_{i+1}^*$. We will show that this leads to a contradiction.

The optimality condition for $\tilde{u}_{i+1}$ and u_{i+1}^* (using Theorem 3.1 in [10]) requires that

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{n} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{n} \frac{d\psi_j}{dx_j^*} = 0$$

Since $\tilde{u}_{i+1} < u^*_{i+1}$, Assumption **A1** implies that $\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} < \frac{d\theta_i}{du^*_i}$, therefore

$$\sum_{j=i+1}^{n} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=i+1}^{n} \frac{d\psi_j}{dx^*_j}$$

In addition, $\tilde{x}_{i+1} < x^*_{i+1}$ and Assumption **A2** implies that $\frac{d\psi_{i+1}}{d\tilde{x}_{i+1}} < \frac{d\psi_{i+1}}{dx^*_{i+1}}$, therefore the above inequality becomes

$$\sum_{j=i+2}^{n} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=i+2}^{n} \frac{d\psi_j}{dx^*_j} \tag{55}$$

Further, the optimality conditions for the controls $\tilde{u}_{i+2}$ and u^*_{i+2} give

$$\frac{d\theta_{i+2}}{d\tilde{u}_{i+2}} + \sum_{j=i+2}^{n} \frac{d\psi_j}{d\tilde{x}_j} = \frac{d\theta_{i+2}}{du^*_{i+2}} + \sum_{j=i+2}^{n} \frac{d\psi_j}{dx^*_j} = 0$$

and, in view of (55), we get

$$\frac{d\theta_{i+2}}{d\tilde{u}_{i+2}} < \frac{d\theta_{i+2}}{du^*_{i+2}}$$

which, by Assumption **A1**, implies that $u^*_{i+2} > \tilde{u}_{i+2}$. Recalling that $x^*_{i+1} > \tilde{x}_{i+1}$ it follows that $x^*_{i+2} > \tilde{x}_{i+2}$. Thus, we obtain a recursive relationship for all $i+1, \ldots, n$ from which it follows that

$$u^*_n > \tilde{u}_n \quad \text{and} \quad x^*_n > \tilde{x}_n$$

Therefore, using assumptions **A1** and **A2**,

$$\frac{d\theta_n}{du^*_n} > \frac{d\theta_n}{d\tilde{u}_n} \quad \text{and} \quad \frac{d\psi_n}{dx^*_n} > \frac{d\psi_n}{d\tilde{x}_n} \tag{56}$$

Finally, writing the optimality conditions for the controls u^*_n and $\tilde{u}_n$ we get

$$\frac{d\theta_n}{d\tilde{u}_n} + \frac{d\psi_n}{d\tilde{x}_n} = \frac{d\theta_n}{du^*_n} + \frac{d\psi_n}{dx^*_n} = 0$$

which clearly contradicts (56), and the proof is complete. ■

Proof of Lemma 4.8. If C_{t+1} is in the last block of an optimal path BP, then from Theorem **??** we have $\tilde{x}_i \geq x^*_i$ for all $i = t+1, ..., \min(n, l)$. Since the current optimal path BP ends at C_n, we have $x^*_i \geq a_{i+1}$ for all $i = t+1, ..., n-1$ and it follows that $\tilde{x}_i \geq a_{i+1}$ for all $i = t+1, ..., \min(n-1, l)$. Thus the RH path BP cannot end before $C_{\min\{n,l\}}$. On the other hand, if C_{t+1} is not in the last block of an optimal path BP, then let us assume that the RH path BP ends with $C_{\tilde{n}}$ such that $\tilde{n} < \min(n, l)$ and we shall establish a contradiction which completes the proof.

Since $C_{\tilde{n}}$ is the last job in the RH path BP, we have $a_{\tilde{n}+1} > \tilde{x}_{\tilde{n}}$. But since $C_{\tilde{n}}$ is not the last job in the optimal path BP, we have $a_{\tilde{n}+1} \leq x^*_{\tilde{n}}$, therefore

$$x^*_{\tilde{n}} > \tilde{x}_{\tilde{n}} \tag{57}$$

Now using Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for $\tilde{u}_{\tilde{n}}$ requires that

$$\frac{d\theta_{\tilde{n}}}{d\tilde{u}_{\tilde{n}}} + \frac{d\psi_{\tilde{n}}}{d\tilde{x}_{\tilde{n}}} = 0 \tag{58}$$

whereas on the optimal path the optimality condition for $u^*_{\tilde{n}}$ requires that

$$\xi^{*-}_{\tilde{n}} = \frac{d\theta_{\tilde{n}}}{du^*_{\tilde{n}}} + \sum_{i=\tilde{n}}^{B} \frac{d\psi_i}{dx^*_i} \leq 0$$

where C_B is the first critical job following $C_{\tilde{n}}$. Combining these two conditions implies

$$\frac{d\theta_{\tilde{n}}}{du^*_{\tilde{n}}} + \sum_{i=\tilde{n}}^{B} \frac{d\psi_i}{dx^*_i} \leq \frac{d\theta_{\tilde{n}}}{d\tilde{u}_{\tilde{n}}} + \frac{d\psi_{\tilde{n}}}{d\tilde{x}_{\tilde{n}}} = 0 \tag{59}$$

Since $x^*_{\tilde{n}} > \tilde{x}_{\tilde{n}}$ from (57), it follows from Assumption **A3** that $\frac{d\psi_{\tilde{n}}}{d\tilde{x}_{\tilde{n}}} < \frac{d\psi_{\tilde{n}}}{dx^*_{\tilde{n}}}$. Therefore, from (59) we get

$$\frac{d\theta_{\tilde{n}}}{du^*_{\tilde{n}}} + \sum_{i=\tilde{n}+1}^{B} \frac{d\psi_i}{dx^*_i} < \frac{d\theta_{\tilde{n}}}{d\tilde{u}_{\tilde{n}}} \tag{60}$$

In addition, since $\psi_i(\cdot)$ is monotonically increasing by Assumption **A3**, we have $\sum_{i=\tilde{n}+1}^{B} \frac{d\psi_i}{dx^*_i} \geq 0$ and (60) gives

$$\frac{d\theta_{\tilde{n}}}{du^*_{\tilde{n}}} < \frac{d\theta_{\tilde{n}}}{d\tilde{u}_{\tilde{n}}}$$

which implies that $u^*_{\tilde{n}} < \tilde{u}_{\tilde{n}}$ from Assumption **A1**. Then, recalling (57), we have $x^*_{\tilde{n}} = x^*_{\tilde{n}-1} + u^*_{\tilde{n}} > \tilde{x}_{\tilde{n}} = \tilde{x}_{\tilde{n}-1} + \tilde{u}_{\tilde{n}}$, therefore

$$x^*_{\tilde{n}-1} > \tilde{x}_{\tilde{n}-1} \tag{61}$$

Next, for $i = \tilde{n}-1, \tilde{n}-2, \ldots$ the optimality condition for $\tilde{u}_i$ requires that (see Theorem 3.1 and Lemma 5.1 in [10])

$$\tilde{\xi}^+_i = \frac{d\theta_i}{d\tilde{u}_i} + \sum_{j=i}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j} \geq 0$$

instead of (58) which applies to $i = \tilde{n}$. We can then replace (59) by

$$\frac{d\theta_i}{du^*_i} + \sum_{j=i}^{B} \frac{d\psi_j}{dx^*_j} \leq \frac{d\theta_i}{d\tilde{u}_i} + \sum_{j=i}^{\tilde{n}} \frac{d\psi_j}{d\tilde{x}_j}$$

and repeat the argument above recursively to obtain

$$x^*_i > \tilde{x}_i \text{ for all } i = B_1 + 1, \ldots, \tilde{n} \tag{62}$$

where B_1 is the first critical job preceding B in the optimal path BP, if such a job is present; otherwise, set $B_1 = t$, $k-1 \leq t < \tilde{n}$. Thus, there are two cases to consider.

Case 1: There is no critical job in the optimal path BP between C_{t+1} and $C_{\tilde{n}}$ (i.e., C_B is the next critical job after C_{t+1} in the optimal path). In this case, (62) becomes

$$x^*_i > \tilde{x}_i \text{ for all } i = t+1, ..., \tilde{n} \tag{63}$$

where $k-1 \leq t < \tilde{n}$. Next, consider two possible subcases as follows.

Case 1.1: If C_k starts a new BP on the RH path, then setting $t = k-1$ in (63) we get

$$x_i^* > \tilde{x}_i \text{ for all } i = k, ..., \tilde{n}$$

which contradicts the result of Lemma 4.1 stating that $\tilde{x}_i = x_i^*$ and completes the proof for this subcase.

Case 1.2: On the other hand, if C_k does not start a new BP on the RH path, then

$$x_{k-1}^* < a_k \leq \tilde{x}_{k-1}$$

since C_k starts a BP on the optimal path. Setting $t = k-1$ in (63) we obtain

$$x_i^* > \tilde{x}_i \text{ for all } i = k, ..., \tilde{n} \tag{64}$$

Combining these last two inequalities implies that $u_k^* > \tilde{u}_k$ and, using Assumption **A1**, we obtain

$$\frac{d\theta_k}{du_k^*} > \frac{d\theta_k}{d\tilde{u}_k} \tag{65}$$

Now once again recalling Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for u_k^* requires that

$$\xi_k^{*-} = \frac{d\theta_k}{du_k^*} + \sum_{i=k}^{B} \frac{d\psi_i}{dx_i^*} \leq 0$$

and the optimality condition for $\tilde{u}_k$ requires that

$$\tilde{\xi}_k^{+} = \frac{d\theta_k}{d\tilde{u}_k} + \sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} \geq 0$$

Therefore,

$$\frac{d\theta_k}{du_k^*} + \sum_{i=k}^{B} \frac{d\psi_i}{dx_i^*} \leq \frac{d\theta_k}{d\tilde{u}_k} + \sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i}$$

Using (65) we then get

$$\sum_{i=k}^{B} \frac{d\psi_i}{dx_i^*} < \sum_{i=k}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} \tag{66}$$

Further, using (64) and Assumption **A3**, and recalling that $B > \tilde{n}$ we get

$$\sum_{i=\tilde{n}+1}^{B} \frac{d\psi_i}{dx_i^*} < 0$$

which contradicts the monotonicity of $\psi_i(\cdot)$ in Assumption **A3**, thus completing the proof for this subcase as well.

Case 2: There is at least one critical job on the optimal path between C_{t+1} and $C_{\tilde{n}}$. Let us index this by B_1, which gives precisely (62). We shall show that this case is in fact infeasible.

First, we show that $x^*_{B_1} > \tilde{x}_{B_1}$ by establishing a contradiction. Assume that $x^*_{B_1} \leq \tilde{x}_{B_1}$. This inequality, together with $x^*_{B_1+1} > \tilde{x}_{B_1+1}$ from (62), implies that $u^*_{B_1+1} > \tilde{u}_{B_1+1}$. Therefore, from Assumption **A1** we obtain

$$\frac{d\theta_{B_1+1}}{du^*_{B_1+1}} > \frac{d\theta_{B_1+1}}{d\tilde{u}_{B_1+1}} \tag{67}$$

As done earlier, recalling Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for $u^*_{B_1+1}$ requires that

$$\xi^{*-}_{B_1+1} = \frac{d\theta_{B_1+1}}{du^*_{B_1+1}} + \sum_{i=B_1+1}^{B} \frac{d\psi_i}{dx^*_i} \leq 0$$

and the optimality condition for $\tilde{u}_{B_1+1}$ requires that

$$\tilde{\xi}^{+}_{B_1+1} = \frac{d\theta_{B_1+1}}{d\tilde{u}_{B_1+1}} + \sum_{i=B_1+1}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} \geq 0$$

Therefore,

$$\frac{d\theta_{B_1+1}}{du^*_{B_1+1}} + \sum_{i=B_1+1}^{B} \frac{d\psi_i}{dx^*_i} \leq \frac{d\theta_{B_1+1}}{d\tilde{u}_{B_1+1}} + \sum_{i=B_1+1}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i}$$

Using (67) we get

$$\sum_{i=B_1+1}^{B} \frac{d\psi_i}{dx^*_i} < \sum_{i=B_1+1}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i}$$

Further, using (62) and Assumption **A3** we have

$$\sum_{i=B_1+1}^{\tilde{n}} \frac{d\psi_i}{dx^*_i} > \sum_{i=B_1+1}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i}$$

Comparing the last two inequalities in view of the fact that $B > \tilde{n}$ gives

$$\sum_{i=\tilde{n}+1}^{B} \frac{d\psi_i}{dx^*_i} < 0$$

which contradicts the monotonicity of $\psi_i(\cdot)$ in Assumption **A3**. Thus, we have established that $x^*_{B_1} > \tilde{x}_{B_1}$. Since C_{B_1} is a critical job on the optimal path, we have $x^*_{B_1} = a_{B_1+1}$, therefore $\tilde{x}_{B_1} < a_{B_1+1}$. This asserts that the RH path BP ends with job C_{B_1} which contradicts the *Case* 2 assumption that C_{B_1} is a critical job between C_{t+1} and the end of this BP with $C_{\tilde{n}}$. Therefore, *Case* 2 is indeed infeasible. This completes the proof of the lemma. ■

Proof of Lemma 4.9. Let us assume there are r critical jobs ($r \geq 1$) in the optimal path BP (k, n) after C_{t+1} and without any loss of generality we index these critical jobs $t+1 \leq B_1 < \ldots < B_r < n$. Clearly, C_{B_r} is the last critical job of the optimal path BP (k, n). If $C_{\tilde{n}}$ is the last job of the RH BP that contains the job C_{t+1} then from Lemma 4.8, $\tilde{n} \geq n$. Further, from Theorem **??**, $\tilde{x}_i \geq x^*_i$ for all $i = B_r + 1, \ldots, n$ since $(r+1, n)$ is the last block of the optimal path BP. Therefore, clearly, none of the jobs $C_{B_r+1},\ldots,C_{n-1}$ can be critical in the RH path. Thus, we concentrate next on jobs prior to C_{B_r+1}. There are two cases to consider :

Case 1: None of the jobs $C_{B_1},\ldots,C_{B_r}$ is critical in the RH path. In this case, we have

$$\tilde{x}_{B_i} > x^*_{B_i} = a_{B_i+1} \text{ for all } i = 1,\ldots,r \tag{68}$$

Setting $i = r$ we have $\tilde{x}_{B_r} > x^*_{B_r} = a_{B_r+1}$. We will now show that $\tilde{x}_{B_r-1} \geq x^*_{B_r-1}$ by assuming that $\tilde{x}_{B_r-1} < x^*_{B_r-1}$ and establishing a contradiction. Since $\tilde{x}_{B_r-1} < x^*_{B_r-1}$ and $\tilde{x}_{B_r} > x^*_{B_r}$, it follows that

$$\tilde{u}_{B_r} > \tilde{u}^*_{B_r} \tag{69}$$

Recall that $\tilde{n} \geq n$ and that none of the jobs $C_{B_r},\ldots,C_{n-1}$ is critical in the RH path. Therefore, if $C_{\tilde{B}}$ is the next critical job after C_{B_r} in the RH path, then clearly $\tilde{B} \geq n$.

Recalling Theorem 3.1 and Lemma 5.1 in [10], the optimality condition for $\tilde{u}_{B_r}$ requires that

$$\tilde{\xi}^-_{B_r} = \frac{d\theta_{B_r}}{d\tilde{u}_{B_r}} + \sum_{j=B_r}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}^*_j} \leq 0$$

and the optimality condition for $u^*_{B_r}$ requires that

$$\xi^{*+}_{B_r} = \frac{d\theta_{B_r}}{du^*_{B_r}} + \sum_{j=B_r}^{n} \frac{d\psi_j}{dx^*_j} \geq 0$$

Therefore,

$$\frac{d\theta_{B_r}}{d\tilde{u}_{B_r}} + \sum_{j=B_r}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}^*_j} \leq \frac{d\theta_{B_r}}{du^*_{B_r}} + \sum_{j=B_r}^{n} \frac{d\psi_j}{dx^*_j}$$

Using (69) and Assumption **A1**, this reduces to

$$\sum_{j=B_r}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} \leq \sum_{j=B_r}^{n} \frac{d\psi_j}{dx^*_j} \tag{70}$$

where $\tilde{B} \geq n$. Since $\tilde{x}_{B_r} > x^*_{B_r}$ and $\tilde{x}_i \geq x^*_i$ for all $i = B_r+1,\ldots,n$, if $\tilde{B} = n$ then (70) contradicts Assumption **A3**. Therefore, let us consider $\tilde{B} > n$. By Assumption **A3**, we have

$$\sum_{j=B_r}^{n} \frac{d\psi_j}{d\tilde{x}_j} > \sum_{j=B_r}^{n} \frac{d\psi_j}{dx^*_j}$$

so that (70) yields

$$\sum_{j=n+1}^{\tilde{B}} \frac{d\psi_j}{d\tilde{x}_j} < 0$$

which contradicts the monotonicity of $\psi_i(\cdot)$ in Assumption **A3**. Thus, we have shown that $\tilde{x}_{B_r-1} \geq x^*_{B_r-1}$. Since C_{B_r-1} is not critical in the optimal path, i.e., $x^*_{B_{r-1}} > a_{B_r}$, it follows that it is also not critical in the RH path. Clearly this is a recursive relation, which can be repeated backwards to show that $\tilde{x}_i \geq x_i^*$ for all $i = B_r - 2, \ldots, B_{r-1} + 1$. Therefore, none of these jobs is critical in the RH path. Further, we also know that $\tilde{x}_{B_{r-1}} > x^*_{B_{r-1}} = a_{B_{r-1}+1}$. Thus, we repeat the exact same procedure for the previous block of the optimal path and obtain the same set of inequalities. Continuing over all previous blocks until we reach C_{t+1} we obtain

$$\tilde{x}_i \geq x_i^* \text{ for all } i = \{t+1, \ldots, B_1 - 1\}, \ldots, \{B_{r-1} + 1, \ldots, B_r - 1\}, \{B_r + 1, \ldots, n\} \tag{71}$$

Therefore, none of these jobs are critical in the RH path.

Case 2: At least one of the jobs $C_{B_1}, \ldots, C_{B_r}$ is critical in the RH path. Let us assume that C_{B_c} is such that it is critical in the RH path (as well as the optimal path) but none of C_{B_i}, $i = 1, \ldots, c-1$ are critical in the RH path, $i \leq c \leq r$. Immediately applying Lemma 4.1 we have

$$\tilde{x}_i = x_i^* \text{ for all } i = B_c, \ldots, n \tag{72}$$

i.e., the optimal and RH path coincide starting with C_{B_c} up to C_n. Thus, only C_{B_i} $i = c, \ldots, r$ are critical in the RH path but no other jobs can be.

It remains to show that none of the jobs $C_{t+1}, \ldots, C_{B_c - 1}$ is critical in the RH path. Let us assume that there exists at least one of $C_{B_{c-1}+1}, \ldots, C_{B_c - 1}$ which is critical in the RH path and let us index by $\tilde{B}$ the last of all such critical jobs which is critical in the RH path with $B_{c-1} + 1 \leq \tilde{B} \leq .B_c - 1$. Since $C_{\tilde{B}}$ is critical in the RH path but non-critical in the optimal path, we have $a_{\tilde{B}+1} = \tilde{x}_{\tilde{B}} < x^*_{\tilde{B}}$. On the other hand, setting $i = B_c$ in (72) we get $\tilde{x}_{B_c} = x^*_{B_c} = a_{B_c+1}$. Therefore, there must be some i, $\tilde{B} < i < B_c$, such that

$$\begin{aligned} \tilde{x}_j &< x_j^* \text{ for all } j = \tilde{B}, \ldots, i \\ \tilde{x}_j &\geq x_j^* \text{ for all } j = i+1, \ldots, B_c - 1 \end{aligned} \tag{73}$$

Now since $\tilde{x}_{B_c} = x^*_{B_c}$ but $\tilde{x}_{B_c - 1} \geq x^*_{B_c - 1}$, we get $\tilde{u}_{B_c} \leq u^*_{B_c}$. Applying Assumption **A1**, we have

$$\frac{d\theta_{B_c}}{d\tilde{u}_{B_c}} + \frac{d\psi_{B_c}}{d\tilde{x}_{B_c}} \leq \frac{d\theta_{B_c}}{du^*_{B_c}} + \frac{d\psi_{B_c}}{dx^*_{B_c}} \tag{74}$$

Recalling the definitions of ξ_i^- from [10], we have

$$\tilde{\xi}^-_{B_c} = \frac{d\theta_{B_c}}{d\tilde{u}_{B_c}} + \frac{d\psi_{B_c}}{d\tilde{x}^*_{B_c}} \text{ and } \xi^{*-}_{B_c} = \frac{d\theta_{B_c}}{du^*_{B_c}} + \frac{d\psi_{B_c}}{dx^*_{B_c}}$$

and it follows from (74) that

$$\tilde{\xi}^-_{B_c} \leq \xi^{*-}_{B_c} \tag{75}$$

Further since C_{i+1} and C_{B_C} are on the same block in the optimal and in the RH path as well, using Lemma 5.2 of [10] we can write

$$\tilde{\xi}^-_{B_c} = \tilde{\xi}^-_{i+1} \text{ and } \xi^{*-}_{B_c} = \xi^{*-}_{i+1}$$

It follows from (75) that

$$\tilde{\xi}^-_{i+1} \leq \xi^{*-}_{i+1} \tag{76}$$

Since C_{B_c} is the nearest critical job to C_{i+1} in both the optimal and the RH path, we have

$$\begin{aligned}\tilde{\xi}_{i+1}^{-} &= \frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{B_c} \frac{d\psi_j}{d\tilde{x}_j} \\ \xi_{i+1}^{*-} &= \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{B_c} \frac{d\psi_j}{dx_j^*}\end{aligned}$$

Then, from (76) we can write

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} + \sum_{j=i+1}^{B_c} \frac{d\psi_j}{d\tilde{x}_j} \leq \frac{d\theta_{i+1}}{du_{i+1}^*} + \sum_{j=i+1}^{B_c} \frac{d\psi_j}{dx_j^*} \tag{77}$$

Looking at (73), we have $\tilde{x}_j \geq x_j^*$ for all $j = i+1, \ldots, B_c - 1$. In addition, $\tilde{x}_{B_c} = x_{B_c}^*$. Therefore, using Assumption **A3**, we have

$$\sum_{j=i+1}^{B_c} \frac{d\psi_j}{d\tilde{x}_j} \geq \sum_{j=i+1}^{B_c} \frac{d\psi_j}{dx_j^*}$$

It then follows from (77) that

$$\frac{d\theta_{i+1}}{d\tilde{u}_{i+1}} < \frac{d\theta_{i+1}}{du_{i+1}^*}$$

Using Assumption **A1**, this implies that $\tilde{u}_{i+1} < u_{i+1}^*$. Moreover, setting $j = i+1$ in (73) we get $\tilde{x}_{i+1} \geq x_{i+1}^*$, i.e., $\tilde{x}_i + \tilde{u}_{i+1} \geq x_i^* + u_{i+1}^*$. In view of $\tilde{u}_{i+1} < u_{i+1}^*$, we obtain

$$\tilde{x}_i \geq x_i^*$$

which directly contradicts (73) for $j = i$. Therefore, $C_{\tilde{B}}$ cannot be critical in the RH path. Repeating the exact same process for any other presumed last critical job B', with $B_{c-1} + 1 \leq B' < \tilde{B}$, leads to a similar contradiction and proves that none of the jobs $C_{B_{c-1}+1}, \ldots, C_{B_c - 1}$ can be critical in the RH path.

Similarly, we can repeat the exact same process for all remaining blocks to show none of the jobs $\{C_{t+1}, \ldots, C_{B1}\}$, $\{C_{B_1+1}, \ldots, C_{B_2}), \ldots, \{C_{B_{c-1}+1}, \ldots, C_{B_c}\}$ can be critical and the proof is complete. ■

Proof of Lemma 4.10. We proceed by assuming that $u_j^* < \tilde{u}_j$ for some $j \in \{t+1, \ldots, n\}$ and establish a contradiction. From Theorems 4.1 and 4.2 we have $x_{j-1}^* \leq \tilde{x}_{j-1}$ which implies

$$x_j^* \leq \tilde{x}_j \tag{78}$$

We now consider the following two possible cases.

Case 1: The RH BP that contains C_j does not contain any critical job after job C_j. Suppose this RH BP ends with $C_{\tilde{n}}$. Then, from Lemma 4.5, we have $\tilde{n} \geq n$. Applying the optimality condition for controls $\tilde{u}_j$ and u_j^*(using Theorem 3.1 in [10]), we get

$$\frac{d\theta_j}{d\tilde{u}_j} + \sum_{i=j}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} = \frac{d\theta_j}{du_j^*} + \sum_{i=j}^{n} \frac{d\psi_i}{dx_i^*} = 0 \tag{79}$$

Since $u_j^* < \tilde{u}_j$, it follows from Assumption **A1** that $\frac{d\theta_j}{d\tilde{u}_j} > \frac{d\theta_j}{du_j^*}$, therefore (79) implies

$$\sum_{i=j}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} < \sum_{i=j}^{n} \frac{d\psi_i}{dx_i^*}$$

Moreover, from (78) and using Assumption **A2** we have $\frac{d\psi_j}{d\tilde{x}_j} > \frac{d\psi_j}{dx_j^*}$ and the inequality above yields

$$\sum_{i=j+1}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} < \sum_{i=j+1}^{n} \frac{d\psi_i}{dx_i^*} \tag{80}$$

However, in view of Theorems 4.1 and 4.2, we have $x_i^* \leq \tilde{x}_i$ for all $i = j+1, \ldots, n$, and using Assumption **A2** again we get

$$\sum_{i=j+1}^{n} \frac{d\psi_i}{d\tilde{x}_i} \geq \sum_{i=j+1}^{n} \frac{d\psi_i}{dx_i^*} \tag{81}$$

Comparing (80) and (81), we conclude that

$$\sum_{i=n}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} \leq 0 \tag{82}$$

Using the optimality condition for $\tilde{u}_n$ we have

$$\frac{d\theta_n}{d\tilde{u}_n} + \sum_{i=n}^{\tilde{n}} \frac{d\psi_i}{d\tilde{x}_i} = 0$$

which in the light of (82) gives

$$\frac{d\theta_n}{d\tilde{u}_n} \geq 0$$

This directly contradicts the property that $\theta_n(\cdot)$ is monotonically decreasing under Assumption **A1** and the proof for this case is complete.

Case 2: The RH BP that contains the job C_j does contain at least one critical job after C_j. Let the first job after C_j which is critical be C_B. It follows from Lemma 4.6, that $B \geq n$ and, in particular, $B = n$ or $B = n_b$ where C_{n_b} is the last job of the bth optimal path BP that follows (k, n), $b = 1, 2, \ldots$ Using the optimality condition for $\tilde{u}_j$ (see Theorem 3.1 and Lemma 5.1 in [10]), we have

$$\tilde{\xi}_j^- = \frac{d\theta_j}{d\tilde{u}_j} + \sum_{i=j}^{B} \frac{d\psi_i}{d\tilde{x}_i} < 0$$

whereas for u_j^*, which we know belongs to a BP with no critical jobs,

$$\frac{d\theta_j}{du_j^*} + \sum_{i=j}^{n} \frac{d\psi_i}{dx_i^*} = 0$$

Combining the above two relationships we get

$$\frac{d\theta_j}{d\tilde{u}_j} + \sum_{i=j}^{B} \frac{d\psi_i}{d\tilde{x}_i} < \frac{d\theta_j}{du_j^*} + \sum_{i=j}^{n} \frac{d\psi_i}{dx_i^*} \tag{83}$$

Since we have assumed $u_j^* < \tilde{u}_j$, it follows from Assumption **A1** that $\frac{d\theta_j}{d\tilde{u}_j} > \frac{d\theta_j}{du_j^*}$, so that (83) implies

$$\sum_{i=j}^{B} \frac{d\psi_i}{d\tilde{x}_i} < \sum_{i=j}^{n} \frac{d\psi_i}{dx_i^*} \tag{84}$$

Using Theorems 4.1 and 4.2 we have $\tilde{x}_i \geq x_i^*$ for all $i = j, \ldots, B$, therefore, using Assumption **A2**, we obtain

$$\sum_{i=j}^{B} \frac{d\psi_i}{d\tilde{x}_i} \geq \sum_{i=j}^{B} \frac{d\psi_i}{dx_i^*} \tag{85}$$

Now, as argued above, $B = n$ or $B = n_b$ where C_{n_b} is the last job of the bth optimal path BP that follows (k, n), $b = 1, 2, \ldots$ If $B = n$, then (84) and (85) immediately contradict each other. Otherwise, if $B = n_b > n$ for some b, comparing (84) and (85) we conclude that

$$\sum_{i=n+1}^{n_b} \frac{d\psi_i}{dx_i^*} < 0 \tag{86}$$

Let us assume there are b complete optimal path BPs between C_{n+1} and C_{n_b}: $(n+1, n_1), \ldots, (n_{b-1}+1, \ldots n_b)$. Considering the optimality condition of each $u_{n+1}^*, \ldots, u_{n_{b-1}+1}^*$ we have

$$\frac{d\theta_{n+1}}{du_{n+1}^*} + \sum_{i=n+1}^{n_1} \frac{d\psi_i}{dx_i^*} = \cdots = \frac{d\theta_{n_{b-1}+1}}{du_{n_{b-1}+1}^*} + \sum_{i=n_{b-1}+1}^{n_b} \frac{d\psi_i}{dx_i^*} = 0$$

Again, from Assumption **A1** we have $\frac{d\theta_{n+1}}{du_{n+1}^*} < 0, \ldots, \frac{d\theta_{n_{b-1}+1}}{du_{n_{b-1}+1}^*} < 0$, which in turn gives

$$\sum_{i=n+1}^{n_1} \frac{d\psi_i}{dx_i^*} > 0, \ldots, \sum_{i=n_{b-1}+1}^{n_b} \frac{d\psi_i}{dx_i^*} > 0$$

therefore, summing up yields $\sum_{i=n+1}^{n_b} \frac{d\psi_i}{dx_i^*} > 0$ which contradicts (86). This completes the proof under *Case 2* as well. ■

References

[1] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: Model and optimal control theory," *IEEE Trans. on Automatic Control*, vol. 43, no. 1, pp. 31–45, 1998.

[2] B. Piccoli, "Hybrid systems and optimal control," in *Proceedings of 37th IEEE Conf. On Decision and Control*, pp. 13–18, Dec. 1998.

[3] H. J. Sussmann, "A maximum principle for hybrid optimal control problems," in *Proceedings of 38th IEEE Conf. On Decision and Control*, pp. 425–430, Dec. 1999.

[4] S. Hedlund and A. Rantzer, "Optimal control of hybrid systems," *Proceedings of 38th IEEE Conf. On Decision and Control*, pp. 3972–3977, 1999.

[5] X. Xu and P. J. Antsaklis, "A dynamic programming approach for optimal control of switched systems," in *Proceedings of 39th IEEE Conf. On Decision and Control*, pp. 1822–1827, Dec. 2000.

[6] K. Gokbayrak and C. G. Cassandras, "Hybrid controllers for hierarchically decomposed systems," in *Proceedings of 3rd Intl. Workshop on Hybrid Systems: Computation and Control*, pp. 117–129, March 2000.

[7] X. Xu and P. J. Antsaklis, "Optimal control of switched systems: New results and open problems," *Proceedings of the ACC*, pp. 2683–2687, 2000.

[8] A. Giua, C. Seatzu, and C. V. D. Mee, "Optimal control of switched autonomous linear systems," in *Proceedings of 40th IEEE Conf. On Decision and Control*, pp. 2472–2477, Dec. 2001.

[9] D. L. Pepyne and C. G. Cassandras, "Modeling, analysis, and optimal control of a class of hybrid systems," *Journal of Discrete Event Dynamic Systems: Theory and Applications*, vol. 8, no. 2, pp. 175–201, 1998.

[10] C. G. Cassandras, D. L. Pepyne, and Y. Wardi, "Optimal control of a class of hybrid systems," *IEEE Trans. on Automatic Control*, vol. AC-46, no. 3, pp. 398–415, 2001.

[11] Y. C. Cho, C. G. Cassandras, and D. L. Pepyne, "Forward decomposition algorithms for optimal control of a class of hybrid systems," *Intl. Journal of Robust and Nonlinear Control*, vol. 11, no. 5, pp. 497–513, 2001.

[12] K. Gokbayrak and C. G. Cassandras, "Stochastic optimal control of a hybrid manufacturing system model," in *Proceedings of 38th IEEE Conf. On Decision and Control*, pp. 919–924, Dec. 1999.

[13] C. G. Cassandras and R. Mookherjee, "Receding horizon control for a class of hybrid systems with event uncertainties," in *Proc. of 2003 American Control Conf.*, 2003. To appear.

[14] D. Q. Mayne and L. Michalska, "Receding Horizon Control of Nonlinear Systems," *IEEE Trans. Automatic Control*, vol. AC-35, no. 7, pp. 814–824, 1990.

[15] C. G. Cassandras and R. Mookherjee, "Properties of receding horizon controllers for some hybrid systems with event uncertainties," *Submitted to IFAC Conf. On Analysis and Design of Hybrid Systems*, 2003.

[16] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems.* Kluwer Academic Publishers, 1999.

[17] P. Zhang and C. G. Cassandras, "An improved forward algorithm for optimal control of a class of hybrid systems," *IEEE Trans. Automatic Control*, vol. 47, no. 10, pp. 1735–1739, 2002.