A Unified Model for Credit Derivatives

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March 20, 2002

Abstract

A framework is provided for pricing derivatives on defaultable bonds and other credit-risky contingent claims. The framework includes structural models (those in which the time of default is determined by the value of the issuing firm), general reduced-form models (those in which default is exogenous), and reduced-form models in which default can occur only at specific times, such as coupon payment dates. Within the general framework, multiple recovery conventions for contingent claims are considered: recovery of a fraction of par, recovery of a fraction of a no-default version of the same claim, and recovery of a fraction of the pre-default value of the claim. These recovery conventions are matched to appropriate default protection contracts. A stochastic-integral representation for credit-risky contingent claims is provided, and the integrand for the credit exposure part of this representation is identified. In the case of intensity-based reduced-form models, credit spread and credit-risky term structure are studied.

JEL classification: G13
Mathematics Subject Classification (1991): 90A09, 60H30, 60G44

*Work supported by the National Science Foundation under grant DMS-98-02464 and DMS-01-01407.
†Work supported by the National Sciences and Engineering Council of Canada.

1 Introduction

1.1 Credit derivatives

Credit risk or *default risk* is the risk that an agent fails to fulfill contractual obligations. The classic example of an instrument bearing credit risk is a corporate bond. With the growth of over-the-counter derivatives markets, there has been a corresponding growth in risk of counter-party default.

Increased trading in instruments subject to credit risk has led to the creation of *credit derivatives*, instruments which partially or fully offset the credit risk of a deal. These provide an efficient means of hedging or acquiring credit risk, and permit investors to separate their views on credit risk from other market variables without jeopardizing relationships with borrowers. Furthermore, a deal which includes protection via credit derivatives may permit investors entry into markets from which they might be otherwise precluded. Hargreaves [18] estimates the size of the credit derivatives market to be $400 billion to $1 trillion notional outstanding.

We briefly describe some common credit derivatives. The reader will find a full treatment in Bielecki & Rutkowski [3]. In a *credit swap*, one party promises to make a payment in the event of a credit event (default or downgrade) for a reference security. The other party purchases this promise with periodic payments up to the time of the credit event. In a *spread swap*, two parties agree to swap the credit risk exposure of assets they own; if the asset held by the first party defaults, the second party compensates the first party, and vice versa. In a *total return swap*, for a series of periodic payments, one party sells the cash flow from an asset to a second party, and the second party is thus exposed to both market and credit risk associated with that asset. This is the synthetic equivalent of selling the asset, an action which may not be feasible for tax, regulatory or relationship management reasons. A *credit call (put)* gives its owner the right to buy (sell) a credit-risky asset at a predetermined price, regardless of credit events which may occur before expiration of the option. *Credit-linked notes* are instruments whose value is linked to performance of a credit-risky asset or portfolio of assets. *First-to-default options*, *last-to-default options* and *collateralized loan obligations* belong to this class.

1.2 Structural models

Structural models of credit risk, pioneered by Merton [35], view a firm's liabilities as contingent claims issued against its assets. Bankruptcy occurs if the asset value falls to a boundary determined by outstanding liabilities. Other early work on such models was done by Black & Cox [6] and Geske [16]. A host of subsequent papers have studied corporate capital structure by this approach. Bielecki & Rutkowski [2], [3] and Lando [31] review this literature.

Despite their conceptual appeal, structural models are difficult to use for pricing. Their underlying state is the value of a firm's assets, and the model assumes this can be observed and hence default is not a surprise. (One exception is the structural model of Zhou [40], in which the value of the firm's assets is allowed to jump.) In fact, the value of the firm's assets is not observable. Furthermore, in order to specify payoffs which would accrue to investors in the event of bankruptcy, one would need the full priority hierarchy of the firm's liabilities.

1.3 Reduced-form models

Reduced-form models for credit risk are less ambitious than structural models. They use risk-neutral pricing of contingent claims and take the time of default or other credit event as an exogenous random variable. *Recovery*, the amount which the owner of a defaulted claim receives upon default, is parametrized but does not take explicit account of hierarchy of liabilities. The model is calibrated to market data and then used to price credit derivatives.

Reduced-form models were developed by Artzner & Delbaen [1], Duffie, Schroder & Skiadas [13], Jarrow & Turnbull [25], and Madan & Unal [34]. Duffie & Lando [12] show how a reduced-form model can be obtained from a structural model with incomplete accounting information. Reduced-form models are of several types. In the simplest, there is an intensity for the arrival of default or credit migration, and recovery is an exogenous process. In Jarrow, Lando & Turnbull [24] the intensity for credit migration is constant; see also Litterman & Iben [33] for a Markov chain model of credit migration. In the papers Duffie et. al. [13], [14] and Lando [32], among others, the intensity of default is a random process. A common feature of these models is that default cannot be predicted and can occur at any time. Since their inception, reduced-form models have been used to price a wide variety of instruments; see, e.g., Das & Tufano [8], Duffie [10], Duffie & Singleton [14], Duffie & Huang [11]. Some recent papers on estimating the parameters of these models are Collin-Dufresne & Solnik [7] and Duffee [9]. Jarrow [23] sets up a reduced-from model in which estimation can be based on equity prices as well as bond prices. A systematic development of mathematical tools for reduced-form models is given by Elliott, Jeanblanc & Yor [15], Jeanblanc & Rutkowski [26], [28] and the recent book by Bielecki & Rutkowski [3].

1.4 Term-structure models

A third approach to credit risk modeling is to begin with the term-structure of risk-free bonds and the term-structure of defaultable bonds, building a model which includes both in an arbitrage-free framework. Some papers which discuss term structure of defaultable bonds are Duffie & Singleton [14], Pugachevsky [37], Bielecki & Rutkowski [2], Schönbucher [38] and Björk, Kabanov & Runggaldier [5]. A comprehensive treatment is provided by Bielecki & Rutkowski [3]. These models are not based on firm fundamentals, and a common theme in those papers is the relationship between reduced-form and term-structure models.

1.5 Preview of this paper

This paper provides a framework for pricing credit derivatives which includes structural and reduced-form models. The unified model is set out in Section 2. In Sections 1–5 and the first part of Section 6, the model also covers the situation in which default can occur only on certain dates, such as coupon payment dates. This is more general than the intensity-based reduced-form models cited above, except for those considered by [15], [26] and [28]. The last of these papers works in a framework which includes both structural and intensity-based reduced-form models, but not the model in which default can occur only on specified dates. Section 3 considers multiple recovery conventions: recovery of a fraction of par, recovery of a fraction of a no-default version of the same claim, and recovery of a fraction of the pre-default value of the claim. The last convention is developed in [13] under the assumption that the process $\widehat{V}$ of Theorem 2.8 does not jump at the default time. We do not impose that condition. In Section 4 the recovery conventions are matched to appropriate default protection contracts. Section 5 represents the discounted gain process associated with a credit-risky contingent claim as a stochastic integral, and the integrand for the credit exposure part of this integral is identified. In Section 6 we develop the default-risky forward rates for the intensity-based specialization of the model, and conclude with a necessary and sufficient condition for a term-structure model to be consistent with this intensity-based specialization. This condition shows that the nature of the recovery convention plays an important role, and in fact the recovery of market price may be the only convention which supports this necessary and sufficient condition within a manageable framework.

This paper is based on Wong [39]. A. Bélanger and D. Wong thank colleagues and management at Scotia Capital, where part of this research was done.

2 The model

2.1 Non-firm-specific information

We begin with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is given a standard, d-dimensional Brownian motion $W_t,\ 0 \leq t \leq \overline{T}$. The fixed positive number $\overline{T}$ is the final date in the model; all contracts expire at or before this date. We shall always take t to be in $[0, \overline{T}]$. We denote by $\{\mathcal{F}_t^W\}_{0 \leq t \leq \overline{T}}$ the filtration generated by W and by $\mathcal{F}_0$ the collection of null sets in $\mathcal{F}$. Then $\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_0$ is the usual augmented Brownian filtration. We interpret $\{\mathcal{F}_t\}_{0 \leq t \leq \overline{T}}$ as a model of the flow of public information which is not firm-specific. Because an augmented Brownian filtration is continuous, this information structure cannot generate a "surprise," and hence is inadequate to model default of a particular firm. It does suffice, however, to convey the information on which the interest rate is based. More specifically, we shall assume there is a nonnegative, $\{\mathcal{F}_t\}$-predictable *interest rate process* r_t, and we define the *discount processes*

$$\beta_t \triangleq e^{-\int_0^t r_u du}, \quad \beta(t,T) \triangleq e^{-\int_t^T r_u du}, \quad t, T \in [0, \overline{T}].$$

2.2 Firm-specific default

We introduce two additional objects in order to model default. The first is a nondecreasing, $\{\mathcal{F}_t\}$-predictable process Γ_t whose paths are right-continuous with left-limits (RCLL) and which satisfies $\Gamma_0 = 0$ almost surely. (Because $\{\mathcal{F}_t\}$ is generated by a Brownian motion, every RCLL adapted process is $\{\mathcal{F}_t\}$-predictable.) We also assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to support a strictly positive random variable Ξ independent of $\mathcal{F}_{\overline{T}}$. The *default time* τ is defined to be

$$\tau \triangleq \inf\{t \in [0, \overline{T}]; \Gamma_t \geq \Xi\},$$

where we follow the convention $\inf \emptyset = \infty$.

This model of default encompasses both the structural model and the reduced-form model and can also capture the situation in which default can occur only on specified dates. In particular, we may take Γ_t to jump on coupon payment dates and be constant between these dates, thereby creating a model in which default can occur only on coupon payment dates. The intensity-based reduced-form model is described by Definition 2.1 below. Finally, in one version of the structural model, there is some $\{\mathcal{F}_t\}$-predictable process X_t representing the value of the firm, and default occurs if X_t falls to some boundary $b < X_0$. We capture this situation by taking $\Gamma_t = \sup_{0 \leq s \leq t}(X_0 - X_s)$ and $\Xi = X_0 - b$, a positive constant.

Our set-up permits only one credit event, which we call default. Although we do not consider the more general case in this paper, one can create a similar model of credit migration in which there are multiple credit events as a firm undergoes changes in credit rating. Let $K = \{1, \ldots, n-1, n\}$ be the set of possible credit ratings, arranged in order of descending quality, with n representing default. We may choose a transition matrix and construct a Markov chain $\zeta_0, \zeta_1, \zeta_2, \ldots$ with state space K and n an absorbing state. This chain is independent of $\mathcal{F}_{\overline{T}}$. Let $\{\Xi_i; i = \mathbb{N}\}$ be a sequence of positive random variables, independent of $\mathcal{F}_{\overline{T}}$ and independent of one another, and let Γ_t again be an RCLL, nondecreasing, $\{\mathcal{F}_t\}$-adapted processes beginning at zero at time zero. We suppose the initial credit rating of the firm is ζ_0, and we take $\rho_0 = 0$. We construct $\{\rho_i; i \in \mathbb{N}\}$ recursively by the formula

$$\rho_{i+1} = \inf\{t > \rho_i; \Gamma_t - \Gamma_{\rho_i} \geq \Xi_{i+1}\},$$

and interpret ρ_i to be the time of the i-th credit event. At time ρ_i, the firm moves to credit class ζ_i.

2.3 Survival and hazard processes

We introduce the right-continuous cumulative distribution function for the random variable Ξ: $F(\xi) \triangleq \mathbb{P}\{\Xi \leq \xi\}$. Because $\{\tau \leq t\} = \{\Xi \leq \Gamma_t\}$ and Ξ is independent of $\mathcal{F}_{\overline{T}}$, we have

$$F(\Gamma_t) = \mathbb{P}\{\tau \leq t | \mathcal{F}_{\overline{T}}\} = \mathbb{P}\{\tau \leq t | \mathcal{F}_t\}. \tag{2.1}$$

We define the stopping time $\overline{\tau} \triangleq \inf\{t \geq 0; F(\Gamma_t) = 1\}$, which satisfies $\tau \leq \overline{\tau}$. We define the RCLL, non-increasing *survival processes*

$$S_t \triangleq 1 - F(\Gamma_t) = \mathbb{P}\{\tau > t | \mathcal{F}_{\overline{T}}\} = \mathbb{P}\{\tau > t | \mathcal{F}_t\}, \; 0 \leq t \leq \overline{T}. \tag{2.2}$$

for which $S_0 = 1$ and $S_t > 0$ for $0 \leq t < \overline{\tau}$, and we set

$$S(t,T) \triangleq \mathbb{I}_{\{S_t > 0\}} \frac{S_T}{S_t}, \quad 0 \leq t \leq \overline{T}.$$

We do not rule out the possibility that $S_t = 0$ and even $\mathbb{P}\{\tau > t\} = 0$ for some values of t, and we adopt throughout the convention that $\mathbb{I}_{\{S_t>0\}} \frac{1}{S_t} = 0$ if $S_t = 0$. We further define the RCLL, nondecreasing *hazard process*

$$\Lambda_t \triangleq - \int_{]0,t]} \mathbb{I}_{\{S_{u-}>0\}} \frac{dS_u}{S_{u-}}, \; 0 \leq t \leq \overline{T}. \tag{2.3}$$

where $S_{u-} \triangleq \lim_{v \uparrow u} S_v$ and $S_{0-} \triangleq 1$. We may write (2.3) in the differential form $dS_t = -S_{t-} d\Lambda_t$, which leads to the integrated form (e.g., [36], page 77 and (4.52) of [26])

$$S_t = \exp\big(-\Lambda_t^c\big) \prod_{0<u\leq t} \big(1 - \Delta\Lambda_u\big), \; 0 \leq t \leq \overline{T}, \tag{2.4}$$

where $\Delta\Lambda_u \triangleq \Lambda_u - \Lambda_{u-}$ and $\Lambda_t^c \triangleq \Lambda_t - \sum_{0<u\leq t} \Delta\Lambda_u$ is the continuous part of Λ. If $\overline{\tau} \leq \overline{T}$ and S jumps to zero at $\overline{\tau}$, then $\Lambda_t = \Lambda_{\overline{\tau}-} + 1$ for $\overline{\tau} \leq t \leq \overline{T}$, whereas if $S_{\overline{\tau}-} = 0$, then (2.4) implies that $\Lambda_t = \Lambda_{\overline{\tau}-} = \infty$ for $\overline{\tau} \leq t \leq \overline{T}$. We caution the reader than when there are jumps in S_t, the hazard process defined here differs from the one defined elsewhere, e.g., [26]; compare our equation (2.4) with Proposition 4.10(ii) of [26].

For the structural model, Ξ is nonrandom, $S_t = \mathbb{I}_{\{\tau>t\}}$ and $\tau = \overline{\tau}$.

With the exception of [15] and [26], previous reduced-form models have been "intensity-based" in the sense of the following definition.

Definition 2.1 *The* intensity-based reduced-form model *is the special case that* $\Gamma_t = \int_0^t \lambda_u \, du$ *for some nonnegative,* $\{\mathcal{F}_t\}$*-predictable* intensity process λ, *and* $F(\xi) = 1 - e^{-\xi}$, *so that* $S_t = e^{-\Gamma_t}$ *and* $\Lambda_t = \Gamma_t$.

2.4 Filtrations

We introduce a process to indicate that the firm is in default,

$$H_t \triangleq \mathbb{I}_{\{\tau \leq t\}} = \mathbb{I}_{\{\Xi \leq \Gamma_t\}},$$

which permits us to rewrite (2.1), (2.2) as

$$\mathbb{E}[1 - H_t | \mathcal{F}_{\overline{T}}] = \mathbb{E}[1 - H_t | \mathcal{F}_t] = S_t. \tag{2.5}$$

Because $\tau \leq \overline{\tau}$, $S_t = 0$ implies $H_t = 1$, or equivalently,

$$\mathbb{I}_{\{S_t>0\}}(1 - H_t) = 1 - H_t. \tag{2.6}$$

The information available at time t is captured by $\mathcal{F}_t^H \triangleq \sigma\{H_s; 0 \leq s \leq t\}$ and $\mathcal{F}_t$. We define the corresponding filtration

$$\mathcal{H}_t \triangleq \mathcal{F}_t \vee \mathcal{F}_t^H, \quad 0 \leq t \leq \overline{T}. \tag{2.7}$$

We shall also need the filtration

$$\mathcal{G}_t \triangleq \mathcal{F}_{\overline{T}} \vee \mathcal{F}_t^H, \quad 0 \leq t \leq \overline{T}. \tag{2.8}$$

Both these filtrations are right-continuous and contain all null sets of $\mathcal{F}$ (see Appendix A). Furthermore, if X is $\mathcal{F}_{\overline{T}}$-measurable, then $E[X|\mathcal{H}_t] = E[X|\mathcal{F}_t]$. It follows that every $\{\mathcal{F}_t\}$-martingale is an $\{\mathcal{H}_t\}$-martingale. This property is called "Condition (H)" by [15], [26], who give several conditions equivalent to it.

For the structural model, $S_t = 1 - H_t$ and because of the convention we adopted, $\mathbb{I}_{\{S_t>0\}}\frac{1}{S_t} = S_t = 1 - H_t$. Moreover, the filtrations $\{\mathcal{F}_t\}$ and $\{\mathcal{H}_t\}$ coincide.

2.5 $\{\mathcal{H}_t\}$-semimartingales

Itô's formula (see, e.g., [36], p. 74) permits calculations with $\{\mathcal{H}_t\}$-semimartingales, which by definition are right-continuous with left limits. We shall need the following special case of this formula.

Lemma 2.2 (Ito's product rule) *Let X and Y be $\{\mathcal{H}_t\}$-semimartingales, and assume X is a finite-variation process. Then*

$$X_t Y_t = X_0 Y_0 + \int_{]0,t]} X_{u-} dY_u + \int_{]0,t]} Y_u dX_u. \tag{2.9}$$

One may write (2.9) in differential form as $d(X_t Y_t) = X_{t-} dY_t + Y_t dX_t$.

We define

$$J_t \triangleq \mathbb{I}_{\{S_t>0\}} \frac{1 - H_t}{S_t}. \tag{2.10}$$

Because $\mathcal{G}_t$ is the join of the independent filtrations $\mathcal{F}_{\overline{T}}$ and $\sigma(\Xi)$, we have for $0 \leq t_1 \leq t_2 \leq \overline{T}$

$$\mathbb{E}[1 - H_{t_2}|\mathcal{G}_{t_1}] = \mathbb{I}_{\{H_{t_1}=0\}} \frac{\mathbb{P}\{H_{t_2} = 0|\mathcal{F}_{\overline{T}}\}}{\mathbb{P}\{H_{t_1} = 0|\mathcal{F}_{\overline{T}}\}} = J_{t_1} S_{t_2}. \tag{2.11}$$

Moreover, for $0 \leq t_1 \leq t_2 \leq \overline{T}$, we have

$$E\left[\int_{]t_1,t_2]} J_{u-} dS_u \middle| \mathcal{G}_{t_1}\right] = \int_{]t_1,t_2]} \mathbb{E}[1 - H_{u-}|\mathcal{G}_{t_1}] \mathbb{I}_{\{S_{u-}>0\}} \frac{dS_u}{S_{u-}} = J_{t_1}(S_{t_2} - S_{t_1}).$$

It follows now by direct computation that with $A_t = \Lambda_{t\wedge\tau}$, the process

$$M_t \triangleq H_t - A_t \tag{2.12}$$

is an $\{\mathcal{H}_t\}$-adapted $\{\mathcal{G}_t\}$-martingale, hence also an $\{\mathcal{H}_t\}$-martingale.

Remark 2.3 Because M is a finite-variation process and the jumps in A are positive and bounded by 1, we have

$$[M,M]_t = \sum_{0<s\le t} (\Delta M_s)^2 \le \sum_{0<s\le t} \big((\Delta H_s)^2 + (\Delta A_s)^2\big) \le 1 + \sum_{0<s\le t} \Delta A_s \le 1 + A_t.$$

If ψ is a bounded, $\{\mathcal{H}_t\}$-predictable process, then

$$\mathbb{E}\int_0^{\overline{T}} \psi_t^2 d[M,M]_t \le C(1+\mathbb{E}A_t) = C(1+\mathbb{E}H_t) \le 2C < \infty.$$

It follows that $\int_0^t \psi_u dM_u$ is an $\{\mathcal{H}_t\}$-martingale.

The process J is nonnegative and may have a discontinuity at $t = \overline{\tau}$, the first time S falls to zero. Indeed, if $S_{\overline{\tau}-} > 0$ then it is also possible that $H_{\overline{\tau}-} = 0$, although we are guaranteed by (2.6) that $H_{\overline{\tau}} = 1$. Consequently, it can occur that J has a jump of size $-\frac{1}{S_{\overline{\tau}-}}$ at time $\overline{\tau}$.

It is also possible that $\overline{\tau} \le \overline{T}$ and $S_{\overline{\tau}-} = 0$, but despite this, we cannot have $J_{\overline{\tau}-} = \infty$, as we now show. Let $A = \{\overline{\tau} \le \overline{T}; S_{\overline{\tau}-} = 0\}$. On the set A, we have

$$\Gamma_{\overline{\tau}-} = \xi^* \triangleq \min\{\xi \ge 0; F(\xi) = 1\}.$$

Moreover, F must be continuous at ξ^*. By the definition of $\overline{\tau}$, we have $\Gamma_t < \xi^*$ for $t < \overline{\tau}$. It follows that $A \cap \{\tau < \overline{\tau}\} = A \cap \{\Xi < \xi^*\}$, but since $\mathbb{P}\{\Xi < \xi^*\} = 1$, we must have $\mathbb{P}\big(A \cap \{\tau < \overline{\tau}\}\big) = \mathbb{P}(A)$. Hence, on the set A, the numerator in $J_t = \mathbb{I}_{\{S_t>0\}} \frac{1-H_t}{S_t}$ falls to zero strictly prior to time $\overline{\tau}$.

We have from Itô's product rule (Lemma 2.2) that

$$(1-H_t) = S_t J_t = \int_{]0,t]} J_{u-} dS_u + \int_{]0,t]} S_u dJ_u = \int_{]0,t]} S_u dJ_u - A_t,$$

and hence $\int_{]0,t]} S_u dJ_u = 1 - M_t$. It follows that

$$\int_{]0,t]} \mathbb{I}_{\{S_u>0\}} dJ_u = 1 - \int_{]0,t]} \mathbb{I}_{\{S_u>0\}} \frac{1}{S_u} dM_u. \tag{2.13}$$

2.6 No-arbitrage pricing

In order to rule out arbitrage, we assume the probability measure $\mathbb{P}$ is a martingale measure in the sense of Harrison & Kreps [19]. That is, the discounted value of every asset at time t plus the discounted value of all cash flows generated by the asset up to time t is an $\{\mathcal{H}_t\}_{0\le t\le \overline{T}}$-martingale under $\mathbb{P}$. We do not assume $\mathbb{P}$ is the only martingale measure for this model, i.e., we do not assume the model is complete. The prices we define, e.g., in Definition 2.6 below, may not be the only arbitrage-free prices consistent with the model. In this regard we are following the industry practice of setting up one arbitrage-free system of pricing and depending on model calibration to make this system useful.

More specifically, fix $T \leq \overline{T}$ and let C_t, $0 \leq t \leq T$, be a nonnegative, nondecreasing, RCLL, $\{\mathcal{F}_t\}$-adapted and hence $\{\mathcal{F}_t\}$-predictable process representing the *promised cumulative cash flow* paid by an asset up to time t. We assume $C_0 = 0$, i.e., there is no payment at date zero. This cash flow will be paid on the time interval $]0, T]$ if there is no default, but only on the time interval $]0, \tau[$ if $\tau \leq T$. In particular, the promised payment is not made at the default time. However, we assume that upon default the asset holder is entitled to an immediate recovery. The recovered amount is Z_τ, where the *recovery process* Z_t, $0 \leq t \leq T$, is a nonnegative $\{\mathcal{F}_t\}$-predictable process.

The pair (C, Z) specifies a contingent claim with expiration date T. We shall always make the following assumptions.

Assumption 2.4 *The cumulative cash flow C satisfies* $\mathbb{E}\left[\int_{]0,T]} \beta_t dC_t\right] < \infty$.

Assumption 2.5 *The recovery process Z satisfies* $\mathbb{E}\left[\sup_{t\in]0,T]} \beta_t Z_t\right] < \infty$.

The above assumptions guarantee integrability of the random variables V_t and G_t in the following definition.

Definition 2.6 *The* price process V *for the contingent claim* (C, Z) *is*

$$V_t \triangleq \mathbb{E}\left[\int_{]t,T]} (1 - H_u)\beta(t, u)dC_u + \int_{]t,T]} \beta(t, u)Z_u dH_u \,\middle|\, \mathcal{H}_t\right], \; 0 \leq t \leq T. \tag{2.14}$$

The gain process G *for* (C, Z) *is*

$$G_t \triangleq V_t + \left[\int_{]0,t]} (1 - H_u)\beta(t, u)dC_u + \int_{]0,t]} \beta(t, u)Z_u dH_u\right], \; 0 \leq t \leq T. \tag{2.15}$$

The right-hand side of (2.14) does not include the cash flow at time t. In particular, $V_T = 0$. The cash flow at time t is included in the right-hand side of (2.15). For $t = 0$, there is no possibility of default and recovery (by construction, $\mathbb{P}\{\Xi \leq \Gamma_0\} = 0$). In particular, $G_0 = V_0$.

Remark 2.7 Suppose an agent begins with ownership of the defaultable contingent claim (C, Z), and invests the cash flow generated by this claim in the money market. Let X_t denote the value of his portfolio at time t. Then $X_0 = V_0$ and

$$dX_t = dV_t + (1 - H_t)dC_t + Z_t dH_t + r_t(X_t - V_t)dt.$$

The change in portfolio value is due to four effects: the change in the value of the underlying defaultable contingent claim, the cash flow paid by the contingent claim, the recovery paid in the event of default, and the interest earnings on the part of the portfolio invested in the money market. Consequently, $d(\beta_t X_t) = d(\beta_t V_t) + (1-H_t)\beta_t dC_t + \beta_t Z_t dH_t$, and integration of this equation yields $\beta_t X_t = \beta_t G_t$, or equivalently, $X_t = G_t$. We can thus interpret G_t as the value of the portfolio described above. It can be regarded as a non-dividend-paying asset, and as expected, the discounted gain process

$$\beta_t G_t = \mathbb{E}\left[\int_{]0,T]} (1-H_u)\beta_u dC_u + \int_{]0,T]} \beta_u Z_u dH_u \middle| \mathcal{H}_t \right] \tag{2.16}$$

is an $\{\mathcal{H}_t\}$-martingale under $\mathbb{P}$. ♢

We next define

$$\widehat{V}_t \triangleq \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \int_{]t,T]} \beta_u Z_u dS_u \middle| \mathcal{F}_t \right], \tag{2.17}$$

which we interpret as the date-t price of the contingent claim (C, Z) conditioned on no default prior to date t. The following theorem generalizes a theorem of Duffie, Schroder and Skiadas [13], who consider a contingent claim making a single payment in a reduced-form intensity-based model. It is also Proposition 3.1 of [26], except that the model assumptions in [26] are slightly different. We provide the main steps in the proof for the sake of completeness.

Theorem 2.8 (Price Representation) *The price process V_t for the contingent claim (C, Z) satisfies $V_t = (1-H_t)\widehat{V}_t$, $0 \le t \le T$.*

Proof: For $0 \le t \le T$, we compute

$$\begin{aligned}
\mathbb{E}\left[\int_{]t,T]} (1-H_u)\beta_u dC_u \middle| \mathcal{H}_t \right] &= \mathbb{E}\left[\mathbb{E}\left[\int_{]t,T]} (1-H_u)\beta_u dC_u \middle| \mathcal{G}_t \right] \middle| \mathcal{H}_t \right] \\
&= \mathbb{E}\left[\int_{]t,T]} \mathbb{E}[1-H_u|\mathcal{G}_t]\beta_u dC_u \middle| \mathcal{H}_t \right] \qquad (2.18) \\
&= \frac{1-H_t}{S_t}\mathbb{E}\left[\int_{]t,T]} S_u\beta_u dC_u \middle| \mathcal{H}_t \right] \\
&= \frac{1-H_t}{S_t}\mathbb{E}\left[\int_{]t,T]} S_u\beta_u dC_u \middle| \mathcal{F}_t \right].
\end{aligned}$$

According to Remark 2.3,

$$\mathbb{E}\left[\int_{]t,T]} n \wedge (\beta_u Z_u) dH_u \middle| \mathcal{H}_t \right] = \mathbb{E}\left[\int_{]t,T]} n \wedge (\beta_u Z_u) dA_u \middle| \mathcal{H}_t \right].$$

Letting $n \uparrow \infty$ and using (2.11), we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dH_u \middle| \mathcal{H}_t\right] &= \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dA_u \middle| \mathcal{H}_t\right] \\
&= -\mathbb{E}\left[\int_{]t,T]} \beta_u Z_u (1 - H_{u-}) \frac{dS_u}{S_{u-}} \middle| \mathcal{H}_t\right] \qquad (2.19)\\
&= -\mathbb{E}\left[\mathbb{E}\left[\int_{]t,T]} \beta_u Z_u (1 - H_{u-}) \frac{dS_u}{S_{u-}} \middle| \mathcal{G}_t\right] \middle| \mathcal{H}_t\right] \\
&= -\mathbb{E}\left[\int_{]t,T]} \beta_u Z_u \mathbb{E}[1 - H_{u-} | \mathcal{G}_t] \frac{dS_u}{S_{u-}} \middle| \mathcal{H}_t\right] \\
&= -\frac{1 - H_t}{S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dS_u \middle| \mathcal{H}_t\right] \\
&= -\frac{1 - H_t}{S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dS_u \middle| \mathcal{F}_t\right].
\end{aligned}$$

Dividing (2.18) and (2.19) by β_t and summing, we obtain

$$V_t = \frac{1 - H_t}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \int_{]t,T]} \beta_u Z_u dS_u \middle| \mathcal{F}_t\right] = (1 - H_t)\widehat{V}_t,$$

with the last equality a consequence of (2.6). $\diamondsuit$

For the structural model, V_t and $\widehat{V}_t$ agree since $S_u = 1 - H_u$ and the term $\frac{\mathbb{I}_{\{S_t > 0\}}}{S_t}$ on the right-hand side of (2.17) is $1 - H_t$ by convention.

3 Models of recovery

3.1 Derivatives of contingent claims

We shall obtain prices for promised cumulative cash flows under a variety of assumptions about the nature of recovery from default. To facilitate this, we introduce some processes derived from a contingent claim C. We note that the process $\widehat{V}$ of (2.17) can be written in terms of the notation and (3.3) and (3.4) below as

$$\widehat{V}_t = CFP_t[C] + DRP_t[Z]. \qquad (3.1)$$

Definition 3.1 *Let (C, Z) be a contingent claim satisfying Assumptions 2.4 and 2.5. We define the following processes.*

NDP *The* **N***o-***D***efault-***P***rice of C is*

$$NDP_t[C] \triangleq \mathbb{E}\left[\int_{]t,T]} \beta(t,u) dC_u \middle| \mathcal{F}_t\right]. \qquad (3.2)$$

This is the price of the promised cash flow C if there is no risk of default (set $H \equiv 0$ in (2.14)).

CFP *The* **C***ash-***F***low-***P***remium of C is*

$$CFP_t[C] \triangleq \mathbb{I}_{\{S_t>0\}} \mathbb{E}\left[\int_{]t,T]} \beta(t,u) S(t,u) dC_u \middle| \mathcal{F}_t \right]. \tag{3.3}$$

Conditioned on the claim not being in default at date t, this is the price of the claim if there were zero recovery from any later default (set $Z \equiv 0$ in (2.17)).

DRP *The* **D***efault-***R***ecovery-***P***remium of Z is*

$$DRP_t[Z] \triangleq \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[-\int_{]t,T]} \beta_u Z_u dS_u \middle| \mathcal{F}_t \right]. \tag{3.4}$$

Conditioned on the claim not being in default at date t, this is the value of the recovery from any later default (set $C \equiv 0$ in (2.17)).

If we let C be the cumulative dividends paid by a firm up to time of bankruptcy, then $CFP_t[C]$ is the present value of these dividends, i.e., the value of the stock. This observation permits the inclusion of stock prices in the estimation of default probabilities; see Jarrow [23].

Let $\delta \in [0,1]$ be given. We shall say a cumulative cash flow C is subject to **R***ecovery of* **P***ar* **V***alue* if, in the event of default, the owner of the defaulted cash flow receives an immediate bullet payment δ, a fraction of the so-called *par value*, which we take to be 1. In other words, the contingent claim under the recovery of par value convention is (C,δ). According to Theorem 2.8, conditioned on not being in default, the price of this claim is

$$RPV_t[C,\delta] \triangleq \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \delta \int_{]t,T]} \beta_u dS_u \middle| \mathcal{F}_t \right]. \tag{3.5}$$

A simplification of this formula when S is continuous is provided in Subsection 3.2.

For $\delta \in [0,1]$, we say a cumulative cash flow C is subject to **R**ecovery of **T**reasury **V**alue if, in the event of default, the owner of the defaulted cash flow receives δ times the value of the no-default-risk version of the same cash flow, including the promised payment at the time of default. This is not a standard market convention, but it is a useful mathematical model for recovery (Duffie & Singleton [14], Jarrow & Turnbull [25], Lando [32]). We shall see in Proposition 3.3 below that, conditioned on not being in default, the price of this claim is

$$\begin{aligned} RTV_t[C,\delta] &\triangleq CFP_t[C] + DRP\big[NDP^-[\delta C]\big] \\ &= (1-\delta) CFP_t[C] + \delta \mathbb{I}_{\{S_t>0\}} NDP_t[C], \end{aligned} \tag{3.6}$$

where

$$NDP_t^-[\delta C] \triangleq \delta \mathbb{E}\left[\int_{[t,T]} \beta(t,u) dC_u \,\middle|\, \mathcal{F}_t \right] \tag{3.7}$$

$$= \lim_{s \uparrow t} \frac{\delta}{\beta_s} \left\{ \mathbb{E}\left[\int_{[t,T]} \beta_u dC_u \,\middle|\, \mathcal{F}_s \right] + \mathbb{E}\left[\int_{]s,t[} \beta_u C_u \,\middle|\, \mathcal{F}_s \right] \right\}$$

$$= \lim_{s \uparrow t} NDP_s[\delta C] \;=\; NDP_{t-}[\delta C].$$

Finally, for $\delta \in [0,1]$, we wish to take the recovery process to be $\delta \widehat{V}_{t-}$, i.e., **R***ecovery* is a fraction of the pre-default **M***arket* **V***alue* of the claim. We shall see in Subsection 3.4 that, conditioned on not being in default, the price of the claim under the recovery of market value convention is

$$RMV_t[C,\delta] \triangleq \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t \kappa_t} \mathbb{E}\left[\int_{]t,T]} \beta_v \kappa_v dC_v \,\middle|\, \mathcal{F}_t \right], \tag{3.8}$$

where

$$Y_t \triangleq \textstyle\int_{]0,t]} \delta \mathbb{I}_{\{S_{u-}>0\}} \frac{dS_u}{S_{u-}} = -\delta \Lambda_t, \tag{3.9}$$

$$X_t \triangleq e^{Y_t^c} \textstyle\prod_{0<u\leq t}(1+\Delta Y_u), \quad \kappa_t \triangleq \mathbb{I}_{\{S_t>0\}} \dfrac{S_t}{X_t}. \tag{3.10}$$

Equation (3.8) is of the form of the no-default-price equation (3.2), but with the discount β replaced by $\beta\kappa$. We may interpret κ as the extra discount due to credit risk.

For the intensity-based, reduced-form model, $\kappa(t) = e^{-\int_0^t (1-\delta)\lambda_u du}$. Under the recovery of market value convention, (3.8) and the Price Representation Theorem 2.8 imply that the price of a "zero-coupon" contingent claim $C_t \triangleq X\mathbb{I}_{\{T\}}(t)$ is

$$V_t = (1-H_t)\mathbb{E}\left[e^{-\int_t^T \left(r_u + (1-\delta)\lambda_u\right) du} X \,\middle|\, \mathcal{F}_t \right].$$

This formula is due to [14].

For the structural model, $\kappa_t = 1 - H_t$ and (3.8) becomes

$$RMV_t[C,\delta] = \frac{1}{\beta_t} \mathbb{E}\left[\int_{]t,T]} \beta_u (1-H_u)\, dC_u \,\middle|\, \mathcal{F}(t) \right] = CFP_t[C].$$

The price of the contingent claim is the same as its price with zero recovery; the recovery fraction δ of pre-default market value does not enter this formula because pre-default market value is zero. Indeed,

$$RMV_{\tau-}[C,\delta] = \frac{1}{\beta_\tau} \mathbb{E}\left[\int_{[\tau,T]} \beta_u (1-H_u)\, dC_u \,\middle|\, \mathcal{F}_\tau \right] = 0.$$

3.2 Recovery of par value

Let δ be a constant in $[0,1]$ and assume the recovery process is $Z \equiv \delta$. If, in addition, S is continuous, then the price of the cash flow C with recovery of a fraction δ of par value agrees with the price of the higher cash flow $C+\delta\Lambda$ with no recovery in the event of default. We state and prove this fact.

Proposition 3.2 *If S is continuous, then for all $\delta \in [0,1]$, we have*

$$RPV_t[C,\delta] = CFP_t[C+\delta\Lambda], \; 0 \le t \le T. \tag{3.11}$$

Proof: The continuity of S implies $dS_t = -S_{t-}d\Lambda_t = -S_t d\Lambda_t$. Therefore,

$$\begin{aligned} RPV_t[C,\delta] &\triangleq \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \delta \int_{]t,T]} \beta_u dS_u \middle| \mathcal{F}_t\right] \\ &= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{t,T]} \beta_u S_u dC_u + \delta \int_{]t,T]} \beta_u S_u d\Lambda_u \middle| \mathcal{F}_t\right] \\ &= CFP_t[C+\delta\Lambda]. \end{aligned}$$

$\diamondsuit$

If there is a positive probability that S is not continuous, as would be the case if there is a positive probability of default on a particular date, (3.11) does not hold. The contingent claim (C,δ) pays out recovery δ at the default time, and Λ has a corresponding jump at this time, but the contingent claim $(C+\delta\Lambda, 0)$ defaults before making the payment associated with that jump. Because of this, $RPV_0[C,\delta]$ is strictly larger than $CFP_0[C+\delta\Lambda]$.

3.3 Recovery of treasury value

Let δ be a constant in $[0,1]$ and assume that upon default, the holder of the contingent claim receives δ times a no-default-risk version of the same claim, this no-default-risk version including the promised payment at the time of default. More precisely, the recovery process is $Z_t = NDP^-[\delta C]$ of (3.7).

Proposition 3.3 *The contingent claim $(C, NDP^-[\delta C])$ has price process*

$$\begin{aligned} V_t &= (1-H_t)\big\{CFP_t[C] + DRP\big[NDP^-[\delta C]\big]\big\} \\ &= (1-H_t)\big\{(1-\delta)CFP_t[C] + \delta\mathbb{I}_{\{S_t>0\}}NDP_t[C]\big\}. \end{aligned}$$

The proof of Proposition 3.3 uses the following lemma, which asserts that the price of a no-default-risk version of the contingent claim C is the sum of the price of a defaultable version plus the value of a recovery convention which pays the remainder of C upon default, including the promised payment at the time of default.

Lemma 3.4 *We have $\mathbb{I}_{\{S_t>0\}}NDP_t[C] = CFP_t[C] + DRP_t[NDP^-[C]]$.*

Proof: We compute

$$
\begin{aligned}
DRP_t\big[NDP^-[C]\big] &= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[-\int_{]t,T]}\beta_u\,NDP_{u-}[C]dS_u\middle|\,\mathcal{F}_t\right]\\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[-\int_{]t,T]}\mathbb{E}\left[\int_{[u,T]}\beta_v dC_v\middle|\,\mathcal{F}_u\right]dS_u\middle|\,\mathcal{F}_t\right]\\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[-\int_{]t,T]}\int_{[u,T]}\beta_v dC_v dS_u\middle|\,\mathcal{F}_t\right]\\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[-\int_{]t,T]}\int_{]t,v]}\beta_v dS_u dC_v\middle|\,\mathcal{F}_t\right] \qquad (3.12)\\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]}\beta_v(S_t-S_v)dC_v\middle|\,\mathcal{F}_t\right]\\
&= \mathbb{I}_{\{S_t>0\}}NDP_t[C]-CFP_t[C].
\end{aligned}
$$

Proof of Proposition 3.3: According to Price Representation Theorem 2.8 and (3.1), the price process for the contingent claim $(C, NDP^-[\delta C])$ is

$$
\begin{aligned}
V_t &= (1-H_t)\Big(CFP_t[C]+DRP_t\big[NDP^-[\delta C]\big]\Big)\\
&= (1-H_t)\delta\Big(CFP_t[C]+DRP_t\big[NDP^-[C]\big]\Big)+(1-H_t)(1-\delta)CFP_t[C]\\
&= (1-H_t)\big(\delta\mathbb{I}_{\{S_t>0\}}NDP_t[C]+(1-\delta)CFP_t[C]\big)\\
&= (1-H_t)RTV_t[C,\delta].
\end{aligned}
$$

Remark 3.5 In the Jarrow & Turnbull [25] model, Ξ is exponentially distributed and $\Gamma=\lambda t$ for some $\lambda>0$. The security considered is a zero-coupon bond with maturity date T, i.e., $C(t)\triangleq\mathbb{I}_{\{T\}}(t)$. The recovery process is a bullet payment of δ at maturity if there is default at or prior to maturity. In other words, for this security $\delta NDP^-[C]$ is the recovery process of [25]. Prior to default, the price of the bond in [25] is $RTV_t[C,\delta]$. After default, Jarrow & Turnbull price the bond based on the value of the anticipated recovery, whereas we set the price equal to zero. In other words, the Jarrow & Turnbull price is

$$
P(t)\triangleq(1-H_t)RTV_t[C,\delta]+H_t\delta\,NDP_t^-[C],\ 0\le t\le T, \qquad (3.13)
$$

whereas we price the bond as $(1-H_t)RTV_t[C,\delta]$. Our gain process (2.15) is

$$
G(t)=(1-H_t)RTV_t[C,\delta]+H_t e^{\int_\tau^t r_u du}\delta NDP_\tau^-[C]. \qquad (3.14)
$$

The difference between the right-hand sides of (3.13) and (3.14) is due to the fact that the Jarrow & Turnbull recovery is paid at maturity, whereas our recovery is immediate upon default, and is then invested in the money market.

3.4 Recovery of market value

For a contingent claim (C, Z), we have the representation $V_t = (1 - H_t)\widehat{V}_t$ of the price process, where $\widehat{V}_t$ of (2.17) is the price of the contingent claim conditioned on no default at or prior to date t. We would like to take the recovery process to be

$$Z_t = \delta \widehat{V}_{t-}, \ 0 < t \leq T, \tag{3.15}$$

where $\delta \in [0, 1]$ is constant. In case of default, recovery is a δ-portion of the pre-default value of the claim. This is a common informal viewpoint among bond traders. (We do not need to specify a value for Z_0 because the probability of default at date 0 is zero.)

Substituting (3.15) into (2.17), we obtain the integral relation for $\widehat{V}$:

$$\widehat{V}_t = \frac{\mathbb{I}_{\{S_t > 0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \int_{]t,T]} \delta \beta_u \widehat{V}_{u-} dS_u \middle| \mathcal{F}_t \right], \ 0 \leq t \leq T. \tag{3.16}$$

The following proposition justifies (3.8). It shows that (3.16) may be written as

$$RMV_t[C, \delta] = CFP_t[C] + \delta DRP_t\big[RMV^-[C, \delta]\big], \tag{3.17}$$

where, as usual, $RMV_t^-[D, \delta] \triangleq RMV_{t-}[C, \delta]$.

Proposition 3.6 *Equation (3.16) for $\widehat{V}$ is satisfied by the process*

$$RMV_t[C, \delta] \triangleq \frac{\mathbb{I}_{\{S_t > 0\}}}{\beta_t \kappa_t} \mathbb{E}\left[\int_{]t,T]} \beta_v \kappa_v dC_v \middle| \mathcal{F}_t \right], \tag{3.18}$$

where κ_t is given by (3.9), (3.10). Furthermore,

$$CFP_t[C] \leq RMV_t[C, \delta] \leq NDP_t[C]. \tag{3.19}$$

If $\delta \in [0, 1[$ and C_T is bounded, uniformly in ω, then $\beta RMV[C, \delta]$ is bounded, uniformly in t and ω, and is the only such process satisfying (3.16).

PROOF: If $\delta = 0$, the claimed results are immediate. Hence, we assume $\delta \neq 0$.

We first show that κ is non-increasing. The jumps in Y occurring on the time interval $[0, T] \cap [0, \overline{\tau}[$ are strictly greater than -1, and hence X is positive on this interval. According to the Doleans-Dade exponential formula (e.g., [36], page 77),

$$X_t = 1 + \int_{]0,t]} X_{u-} dY_u = 1 + \delta \int_{]0,t]} \mathbb{I}_{\{S_{u-} > 0\}} \frac{1}{\kappa_{u-}} dS_u, \ t \in [0, T] \cap [0, \overline{\tau}[. \tag{3.20}$$

According to Itô's formula,

$$\begin{aligned}
\frac{S_t}{X_t} &= S_0 + \int_{]0,t]} \frac{1}{X_{u-}} dS_u^c - \int_{]0,t]} \frac{S_{u-}}{X_{u-}^2} dX_u^c + \sum_{0<u\le t} \left(\frac{S_u}{X_u} - \frac{S_{u-}}{X_{u-}} \right) \\
&= 1 + (1-\delta) \int_{]0,t]} \frac{1}{X_{u-}} dS_u^c + \sum_{0<u\le t} \left(\frac{S_u}{X_u} - \frac{S_{u-}}{X_{u-}} \right), \ t \in [0,T] \cap [0,\overline{\tau}[.
\end{aligned}$$

Because S is non-increasing, so is $(1-\delta)\int_{]0,t]} \frac{1}{X_{u-}} dS_{u-}^c$. It remains to show that $\frac{S_u}{X_u} \le \frac{S_{u-}}{X_{u-}}$, or equivalently,

$$\Delta S_u X_{u-} \le S_{u-} \Delta X_u. \tag{3.21}$$

But $\Delta X_u = \delta \mathbb{I}_{\{S_{u-}>0\}} \frac{X_{u-}}{S_{u-}} \Delta S_u$, and $X_{u-} \ge 0$, $\Delta S_u \le 0$, so relation (3.21) holds.

As observed in Subsection 2.3, if $\overline{\tau} \le \overline{T}$ and $S_{\overline{\tau}-} = 0$, then $\Lambda_{\overline{\tau}-} = \infty$. In this case, $Y_{\overline{\tau}-} = -\infty$. Consequently, $X_{\overline{\tau}-}$ might be 0. However, X is positive on $[0,\overline{\tau}[$ and $\frac{S}{X}$ is non-increasing there. With $\kappa_t \triangleq 0$ for $\overline{\tau} \le t \le T$, we see that κ is non-increasing on all of $[0,T]$. Because S and κ are non-increasing, (3.18) implies (3.19).

By an argument analogous to (3.7), we see that $RMV[C,\delta]$ given by (3.18) satisfies

$$RMV_{u-}[C,\delta] = \mathbb{I}_{\{S_{u-}>0\}} \frac{1}{\beta_u \kappa_{u-}} \mathbb{E}\left[\int_{[u,T]} \beta_v \kappa_v dC_v \middle| \mathcal{F}_u \right]. \tag{3.22}$$

We compute the right-hand side of (3.16) using (3.22) and (3.20):

$$\begin{aligned}
&\frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u - \delta \int_{]t,T]} \beta_u RMV_{u-}[C,\delta]\, dS_u \middle| \mathcal{F}_t \right] \\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u \middle| \mathcal{F}_t \right] - \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\Bigg[\delta \int_{]t,T]} \mathbb{I}_{\{S_{u-}>0\}} \frac{1}{\kappa_{u-}} \\
&\qquad \times \mathbb{E}\left[\int_{[u,T]} \beta_v \kappa_v dC_v \middle| \mathcal{F}_u \right] dS_u \Bigg| \mathcal{F}_t \Bigg] \\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\delta \int_{]t,T]} \int_{[u,T]} \mathbb{I}_{\{S_{u-}>0\}} \frac{1}{\kappa_{u-}} \beta_v \kappa_v dC_v dS_u \middle| \mathcal{F}_t \right] \\
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t} \mathbb{E}\left[\delta \int_{]t,T]} \int_{]t,v]} \mathbb{I}_{\{S_{u-}>0\}} \frac{1}{\kappa_{u-}} \beta_v \kappa_v dS_u dC_v \middle| \mathcal{F}_t \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u \middle| \mathcal{F}_t\right] \\
&\quad -\frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]} \mathbb{I}_{\{S_v>0\}}\frac{\beta_v S_v}{X_v}(X_v - X_t)dC_v \middle| \mathcal{F}_t\right] \\
&= \mathbb{I}_{\{S_t>0\}}\frac{X_t}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]} \mathbb{I}_{\{S_v>0\}}\frac{\beta_v S_v}{X_v}dC_v \middle| \mathcal{F}_t\right] \;=\; RMV_t[C,\delta].
\end{aligned}
$$

Hence, $RMV[C,\delta]$ defined by (3.18) satisfies (3.16).

It remains to prove uniqueness under the assumption that $\delta \in]0,1[$ and $C_T \leq K < \infty$ almost surely for some constant K. The boundedness of C_T and (3.19) imply the boundedness of $\beta RMV[C,\delta]$. Suppose $\widehat{V}$ is another process for which $\beta\widehat{V}$ is bounded and which satisfies (3.16). Then

$$
\beta_t(RMV[C,\delta]_t - \widehat{V}_t) = \delta\frac{\mathbb{I}_{\{S_t>0\}}}{S_t}\mathbb{E}\left[\int_{]t,T]} \beta_u \left(RMV[C,\delta]_{u-} - \widehat{V}_{u-}\right)(-dS_u) \middle| \mathcal{F}_t\right].
$$

With $K_t \triangleq \operatorname{ess\,sup}_{\omega\in\Omega}\sup_{t\leq u\leq T}\beta_u\big|RMV[C,\delta]_u - \widehat{V}_u\big|$, this implies

$$
\beta_t\big|RMV[C,\delta]_t - \widehat{V}_t\big| \leq \delta K_t\frac{\mathbb{I}_{\{S_t>0\}}}{S_t}\mathbb{E}\left[\int_{]t,T]}(-dS_u)\middle| \mathcal{F}_t\right] \leq \delta K_t. \tag{3.23}
$$

But then we have $\beta_u\big|RMV[C,\delta]_u - \widehat{V}_u\big| \leq \delta K_u \leq \delta K_t,\ u \in [t,T]$, which yields $K_t \leq \delta K_t$. Because $\delta < 1$, this inequality implies $K_t = 0$.

Remark 3.7 A simple example of non-uniqueness in (3.16) arises if we take $\delta = 1$ and assume $S_T = 0$ almost surely. If $\widehat{V}$ solves (3.16), then $\widehat{V} + \frac{c}{\beta_t}\mathbb{I}_{\{S_t>0\}}$ does as well for any constant c

However, if $\delta = 1$, (3.9), (3.10) and (2.4) imply $S_t = X_t$ for all t. Therefore $\kappa_t = 1$, $0 \leq t < \overline{\tau}$ and $RMV[C,\delta]_t$ given by (3.18) is $\mathbb{I}_{\{S_t>0\}}NDP_t[C]$. This is the "right" price, the one corresponding to our expectation that if full market value is recovered upon default, then conditioned on no default prior to date t, the date-t price of the contingent claim is the no-default price. ♢

Remark 3.8 Proposition 3.6 generalizes a theorem of [13], who considered only the intensity-based reduced-form model but allowed δ to be an $\{\mathcal{F}_t\}$-predictable process. A straight-forward modification of the proof of Proposition 3.6 shows that it is still valid in the case of an $\{\mathcal{F}_t\}$-predictable process δ.

Example 3.9 (Continuous survival function) If S is continuous, then $\kappa_t = S_t^{1-\delta}$. To see this, we first use Itô's rule to derive the formula

$$
dS_t^\delta = 1 + \delta\int_0^t \mathbb{I}_{\{S_{u-}>0\}}S_{u-}^\delta\frac{dS_u}{S_{u-}} = 1 + \int_0^t S_{u-}^\delta dY_u.
$$

In other words, S_t^δ satisfies the integral equation (3.20). The solution to this equation is unique (e.g., [36], page 77); hence $X_t = S_t^\delta$ and $\kappa_t = S_t^{1-\delta}$. Equation (3.8) becomes

$$RMV_t[C,\delta] = \mathbb{I}_{\{S_t>0\}}\mathbb{E}\left[\int_{]t,T]}\beta(t,u)S^{1-\delta}(t,u)dC_u\middle|\mathcal{F}_t\right]. \tag{3.24}$$

3.5 Comparison of recovery conventions

We assume in this section that S is continuous. We recall that

$$NDP_t[C] = \mathbb{E}\left[\int_{]t,T]}\beta(t,u)dC_u\middle|\mathcal{F}_t\right], \tag{3.25}$$

$$CFP_t[C] = \mathbb{I}_{\{S_t>0\}}\mathbb{E}\left[\int_{]t,T]}\beta(t,u)S(t,u)dC_u\middle|\mathcal{F}_t\right], \tag{3.26}$$

$$RPV_t[C,\delta] = CFP[C+\delta\Lambda], \tag{3.27}$$

$$RTV_t[C,\delta] = (1-\delta)CFP_t[C] + \delta\mathbb{I}_{\{S_t>0\}}NDP_t[C], \tag{3.28}$$

$$RMV_t[C,\delta] = \mathbb{I}_{\{S_t>0\}}\mathbb{E}\left[\int_{]t,T]}\beta(t,u)S^{1-\delta}(t,u)dC_u\middle|\mathcal{F}_t\right]. \tag{3.29}$$

These formulas lead immediately to the following proposition.

Proposition 3.10 (Linearity) *Suppose $C^{(1)}$ and $C^{(2)}$ are contingent claims and γ_1, γ_2 are nonnegative numbers. Then*

$$\begin{aligned}
NDP[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}] &= \gamma_1 NDP[C^{(1)}] + \gamma_2 NDP[C^{(2)}],\\
CFP[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}] &= \gamma_1 CFP[C^{(1)}] + \gamma_2 CFP[C^{(2)}],\\
RPV[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}, (\gamma_1+\gamma_2)\delta] &= \gamma_1 RPV[C^{(1)},\delta] + \gamma_2 RPV[C^{(2)},\delta],\\
RTV[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}, \delta] &= \gamma_1 RTV[C^{(1)},\delta] + \gamma_2 RTV[C^{(2)},\delta],\\
RMV[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}, \delta] &= \gamma_1 RMV[C^{(1)},\delta] + \gamma_2 RMV[C^{(2)},\delta].
\end{aligned}$$

Note that in contrast to the last two equations,

$$\begin{aligned}
RPV[\gamma_1 C^{(1)} + \gamma_2 C^{(2)}, \delta] &= \gamma_1 RPV[C^{(1)},\delta] + \gamma_2 RPV[C^{(2)},\delta]\\
&\quad +(1-\gamma_1-\gamma_2)\delta CFP(\Lambda).
\end{aligned}$$

In particular,

$$RPV[C^{(1)} + C^{(2)}, \delta] \le RPV[C^{(1)},\delta] + RPV[C^{(2)},\delta];$$

when recovering a bullet payment δ, it is better to divide $C^{(1)} + C^{(2)}$ into two claims, each of which will recover δ. We also have

$$RMV[C,\delta] \le RTV[C,\delta], \tag{3.30}$$

with equality if $\delta = 1$ or if $S_T = 1$. To see this, we use the inequality $S_u^{1-\delta} \le \delta + (1-\delta)S_u$ in (3.28) and (3.29), noting that the above inequality is an equality if either $\delta = 1$ or $S_T = 1$ (in which case $S_u = 1$ for $0 \le u \le T$).

Proposition 3.11 *For any contingent claim C, we have*

$$CFP[C] \le RMV[C,\delta] \le RTV[C,\delta] \le NDP[C,\delta].$$

Proof: The first inequality follows from the fact that $S(u,t) \le 1$ for $t \le u \le T$, and hence $S(u,t) \le S^{1-\delta}(u,t)$. The second inequality is (3.30). The third inequality is a consequence of $CFP[C] \le NDP[C]$ and (3.28). ♢

Proposition 3.12 *For any contingent claim C, the following are equivalent:*

$$\begin{aligned} RTV_0[C,\delta] &\le RPV_0[C,\delta]\ \forall\delta \in [0,1], & (3.31)\\ NDP_0[C] &\le CFP_0[C+\Lambda], & (3.32)\\ RTV_0[C,1] &\le RPV_0[C,1]. & (3.33) \end{aligned}$$

Proof: Since

$$\begin{aligned} RPV_0[C,\delta] - RTV_0[C,\delta] &= CFP_0[C+\delta\Lambda] - (1-\delta)CFP_0[C] - \delta NDP_0[C] \\ &= \delta\big(CFP_0[C+\Lambda] - NDP_0[C]\big), \end{aligned}$$

relations (3.31) and (3.32) are equivalent. Furthermore, $RTV_0[C,1] = NDP_0[C]$ and $RPV_0[C,1] = CFP_0[C+\Lambda]$, so (3.32) and (3.33) are also equivalent. ♢

Remark 3.13 Condition (3.32) tells us that whenever Λ is high enough so that the contingent claim $C+\Lambda$ with default risk and zero recovery is more valuable than the no-default price of C, then the price of C with recovery of par is greater than the price with recovery of treasury. Moreover, condition (3.33) says that this situation prevails if and only if it prevails for $\delta = 1$. Finally, because for a zero-coupon bond with face value 1, recovery of par with $\delta = 1$ results in a recovery payment at least as large as recovery of treasury, the inequalities in Proposition 3.12 hold for such bonds.

4 Default protection

An investor who holds a defaultable contingent claim (C,Z) may wish to buy a contract which will make a payment to him upon default of the claim. To purchase this, he may make either a single payment, in which case the contract is called a *default option*, or he may make a series of scheduled payments up to the time of default, in which case the contract is called a *default swap*. In the market, the payoff of a default swap is typically the difference between par value (if the protected claim has a "par" value) and recovery.

We model the default protection by a nonnegative, $\{\mathcal{F}_t\}$-predictable process $Y(t)$, $0 \le t \le T$. It is often the case that the protection expires before date T. In this case, formulas like those in Proposition 4.1 below can still be worked out, but are more complicated.

If the default time τ is less than or equal to T, the protection makes the one-time payment Y_τ at time τ. This is like the payment of the recovery Z_τ at time τ, and the value at time t of the payment Y_τ can be determined via the Price Representation Theorem 2.8, i.e., it is $(1-H_t)DRP_t[Y]$.

We can choose the default protection process Y to "match" the recovery convention of the contingent claim (C,Z). More specifically, let $\delta \in]0,1]$ be a constant, and define

$$Y^{RPV} \triangleq 1-\delta, \quad Y^{RTV} \triangleq (1-\delta)NDP_t^-[C], \quad Y^{RMV} \triangleq (1-\delta)RMV_t^-[C,\delta].$$

These are matched to the respective conventions recovery of par, recovery of treasury and recovery of market value in the sense of the following proposition.

Proposition 4.1 *We have*

$$DRP_t[Y^{RPV}] = \frac{(1-\delta)\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[-\int_{]t,T]}\beta_u dS_u \middle| \mathcal{F}_t\right], \tag{4.1}$$

$$DRP_t[Y^{RTV}] = \mathbb{I}_{\{S_t>0\}}NDP_t[C] - RTV_t[C], \tag{4.2}$$

$$DRP_t[Y^{RMV}] \triangleq \frac{1-\delta}{\delta}\big(RMV_t[C,\delta] - CFP_t[C]\big). \tag{4.3}$$

and

$$RPV_t[C,\delta] + DRP_t[Y^{RPV}] = RPV_t[C,1], \tag{4.4}$$

$$RTV_t[C,\delta] + DRP_t[Y^{RTV}] = \mathbb{I}_{\{S_t>0\}}NDP_t[C], \tag{4.5}$$

$$RMV[C,\delta] + DRP_t[Y^{RMV}] = CFP_t[C] + DRP_t\big[RMV^-[C,\delta]\big]. \tag{4.6}$$

Equation (4.4) says that the protection Y^{RPV} added to a contingent claim with recovery of a fraction δ of par value yields the equivalent of a contingent claim with recovery of full par value 1. Equation (4.5) states that the protection Y^{RTV} added to a contingent claim with recovery of treasury value yields the equivalent of a no-default version of C. Equation (4.6) asserts that the protection Y^{RMV} added to a contingent claim $RMV[C,\delta]$ with recovery of a fraction δ of market value yields the equivalent of a defaultable, zero-recovery version of the contingent claim C plus a recovery of full market value of the claim $RMV[C,\delta]$ immediately prior to default.

Proof of Proposition 4.1: Equation (4.1) follows immediately from (3.4). Equation (3.5) then gives

$$\begin{aligned} &RPV_t[C,\delta] + DRP_t[Y^{RPV}] \\ &= \frac{\mathbb{I}_{\{S_t>0\}}}{\beta_t S_t}\mathbb{E}\left[\int_{]t,T]}\beta_u S_u dC_u - \int_{]t,T]}\beta_u S_u \middle| \mathcal{F}_t\right] = RPV[C,1], \end{aligned}$$

which is (4.4). To obtain (4.2), which is the same as (4.5), we use Lemma 3.4 and equation (3.6) to write

$$\begin{aligned} DRP_t[Y^{RTV}] &= (1-\delta)DRP_t\big[NDP^-[C]\big] \\ &= (1-\delta)\mathbb{I}_{\{S_t>0\}}NDP_t[C] - (1-\delta)CFP_t[C] \\ &= \mathbb{I}_{\{S_t>0\}}NDP_t[C] - RTV_t[C,\delta]. \end{aligned}$$

Equations (4.3) and (4.6) follow immediately from (3.17). ◊

If default protection, modeled by a process Y, is to be purchased by a single payment at time zero, this payment should be $DRP_0[Y]$. If it is to be purchased by a series of scheduled payments, made up to the time of default, the determination of the size of those payments is more complicated. Specifically, suppose a purchaser of default protections agrees to pay a fixed *default swap rate* s on each of the dates $0 < t_1 < \cdots < t_n \leq T$, except that no payment is made at or after the time of default. In the event of no default, the cumulative payment made by time t is $X(t) = s\sum_{i=1}^{n} \mathbb{I}_{[t_i,T]}(t), \quad 0 \leq t \leq T$. Because these payments will terminate upon default, the value of this series of payments is

$$
\begin{aligned}
CFP_0[X] &= \mathbb{E}\left[\int_{]0,T]} \beta_u S_u dX_u\right] = s\sum_{i=1}^{n} \mathbb{E}[\beta_{t_i} S_{t_i}] \\
&= s\sum_{i=1}^{n} \big(\mathbb{E}[S_{t_i}]\mathbb{E}[\beta_{t_i}] + Cov(S_{t_i}, \beta_{t_i})\big) \qquad (4.7) \\
&= s\sum_{i=1}^{n} \big(\mathbb{P}\{\tau > t_i\}P(0,t_i) + Cov(S_{t_i}, \beta_{t_i})\big),
\end{aligned}
$$

where $P(0,t_i) \triangleq NDP[\mathbb{I}_{[t_i,T]}]$ is the price of the default-free zero-coupon bond paying 1 at time t_i. We determine the default swap rate by setting the expression in (4.7) equal to $DRP_0[Y]$ and solving for s.

Now suppose that S is possibly random but is constant on each of the intervals $[t_{i-1}, t_i[$, $i = 1, \ldots, n$, where $t_0 = 0$, so there is no credit exposure between the payment dates of X. If the default protection is $Y^{RPV} = 1 - \delta$, then according to (4.1),

$$
\begin{aligned}
DRP_0[Y^{RPV}] &= (1-\delta)\sum_{i=1}^{n} \mathbb{E}[\beta_{t_i}(S_{t_{i-1}} - S_{t_i})] \qquad (4.8) \\
&= (1-\delta)\sum_{i=1}^{n} \big(\mathbb{E}\beta_{t_i}(\mathbb{E}S_{t_{i-1}} - \mathbb{E}S_{t_i}) + Cov(\beta_{t_i}, S_{t_{i-1}}) - Cov(\beta_{t_i}, S_{t_i})\big) \\
&= (1-\delta)\sum_{i=1}^{n} \big(\mathbb{P}\{\tau = t_i\}P(0,t_i) + Cov(\beta_{t_i}, S_{t_{i-1}}) - Cov(\beta_{t_i}, S_{t_i})\big).
\end{aligned}
$$

If, in addition, the conditional survival process S and the interest rate process r are independent under $\mathbb{P}$, then the covariance terms in (4.7) and (4.8) vanish, and equating these expressions, we obtain the *default swap rate formula*

$$
s = (1-\delta)\frac{\sum_{i=1}^{n} \mathbb{P}\{\tau = t_i\}P(0,t_i)}{\sum_{i=1}^{n-1} \mathbb{P}\{\tau \geq t_{i+1}\}P(0,t_i)}. \qquad (4.9)
$$

This can be simplified further if the conditional default probability is a constant $p \in]0,1[$, i.e., $\mathbb{P}\{\tau = t_i | \tau \geq t_i\} = p$, or equivalently, $\mathbb{P}\{\tau = t_i\} = p\mathbb{P}\{\tau \geq t_i\}$,

$i = 1, \ldots, n$. In this case, (4.9) becomes

$$s = (1-\delta)p \frac{\sum_{i=1}^{n} \mathbb{P}\{\tau \geq t_i\} P(0,t_i)}{\sum_{i=1}^{n-1} \mathbb{P}\{\tau \geq t_{i+1}\} P(0,t_i)},$$

For high-grade instruments, $\sum_{i=1}^{n} \mathbb{P}\{\tau \geq t_i\} P(0,t_i) / \sum_{i=1}^{n-1} \mathbb{P}\{\tau \geq t_{i+1}\} P(0,t_i)$ is approximately equal to 1, and we obtain the *default swap rate approximation* $s \approx (1-\delta)p$. For a coupon-paying bond, the default swap rate is approximately the fraction of par value which will not be recovered times the probability of default at the next coupon date.

5 Martingale representation

According to Remark 2.7, the gain process G given by (2.15) for the defaultable contingent claim (C, Z) may be regarded as an asset, and the discounted gain process is an $\{\mathcal{H}_t\}$-martingale. Using general martingale representation results due to Jacod [22] (see Kusuoka [30], Theorem 3.2 for the application here; related representations appear in Hugonnier [21], Jeanblanc & Rutkowski [26], [27]), one can show that this martingale has a stochastic integral representation. Rather than directly appeal to these results, we work out the details of the construction implicit in them in order to identify the integrand for the martingale M.

This martingale representation is a first step toward developing hedges for defaultable securities. Greenfield [17] carries this process to its conclusion, constructing hedges for defaultable derivative securities in a model in which market risk and credit risk are independent under the risk-neutral measure. The primary instruments in these hedges are default-free zero-coupon bonds, defaultable zero-coupon bonds, and "tail options," which are options to receive a no-default version of the defaultable derivative security being hedged.

Theorem 5.1 (Martingale representation) *The discounted gain process βG may be decomposed as*

$$\beta_{t\wedge\tau} G_{t\wedge\tau} = G_0 + \int_{]0,t\wedge\tau]} \beta_u \varphi_u dM_u + N_{t\wedge\tau}, \quad 0 \leq t \leq T, \tag{5.1}$$

where

$$\varphi_u = Z_u - CFP_{u-}[C] - DRP_u[Z] \tag{5.2}$$

is an $\{\mathcal{F}_t\}$-predictable process, N is an $\{\mathcal{F}_t\}$-local-martingale, and M is the $\{\mathcal{H}_t\}$-martingale of (2.12).

The integrand φ in Theorem 5.1 can be understood as follows. In the event of default at time u, the contingent claim (C, Z) receives recovery Z_u and loses the promise of future cash flow, a promise whose value is $CFP_{u-}[C]$ because it includes the cash flow at time u, and also loses the prospect of future recovery, a prospect whose value is $DRP_u[Z]$.

If we have a second contingent claim (D, Y) with expiration date $T' \leq T$, the same default time τ, and with gain process denoted $G^{D,Y}$, then Theorem 5.1 implies that with $\tilde{\varphi}_u = Y_u - CFP_{u-}[D] - DRP_u[Y]$, there exists an $\{\mathcal{F}_t\}$-local-martingale $\widetilde{N}$ such that

$$\beta_t G_t^{D,Y} = G_0^{D,Y} + \int_{]0,t]} \beta_u \tilde{\varphi}_u dM_u + \widetilde{N}_t, \quad 0 \leq t \leq T' \wedge \tau.$$

If φ_u is nonzero whenever $\tilde{\varphi}_u$ is nonzero, we can solve (5.1) for $\beta_u \tilde{\varphi}_u dM_u$ in terms of $d(\beta G)$ and dN and, under mild integrability conditions on $\mathbb{I}_{\{\varphi_t \neq 0\}} \frac{\tilde{\varphi}_t}{\varphi_t}$, we can represent $\beta G^{D,Y}$ in terms of βG, the $\{\mathcal{F}_t\}$-local-martingales N and $\widetilde{N}$, and the $\{\mathcal{F}_t\}$-predictable processes φ and $\tilde{\varphi}$:

$$\beta_{t\wedge\tau} G_{t\wedge\tau}^{D,Y} = G_0^{D,Y} + \int_{]0,t\wedge\tau]} \mathbb{I}_{\{\varphi_u \neq 0\}} \frac{\tilde{\varphi}_u}{\varphi_u} d(\beta_u G_u) - \int_{]0,t\wedge\tau]} \mathbb{I}_{\{\varphi_u \neq 0\}} \frac{\tilde{\varphi}_u}{\varphi_u} dN_u + \widetilde{N}_{t\wedge\tau}.$$

Moreover, we have a formula for the hedge ratio $\mathbb{I}_{\{\varphi_u \neq 0\}} \frac{\tilde{\varphi}_u}{\varphi_u}$. If there are sufficient default-free assets in the market to permit hedging of the d sources of uncertainty inherent in the d-dimensional driving Brownian W, then the defaultable contingent claim (D, Y) can be replicated by dynamically trading these assets, the money market, and the defaultable contingent claim (C, Z).

The proof of Theorem 5.1 requires the following lemma, which is almost Proposition 3.4 of [15]. Unlike [15], we permit S_t to be zero and we do not assume X is an $\{\mathcal{F}_t\}$-predictable process.

Lemma 5.2 *Let $X(\omega, t)$ be a nonnegative, $\mathcal{F}_T \times \mathcal{B}[0,T]$-measurable function, where $\mathcal{B}[0,T]$ denotes the Borel σ-algebra in $[0,T]$. We use the notation $X_v = X(\omega, v)$, $X_\tau = X\big(\omega, \tau(\omega)\big)$. For $s, t \in [0,T]$, define $Y_{s,t} \triangleq \mathbb{E}[X_s | \mathcal{F}_t]$. Then*

$$\mathbb{E}\left[X_\tau \mathbb{I}_{\{\tau \leq T\}} \middle| \mathcal{H}_t \right] = -J_t \mathbb{E}\left[\int_{]t,T]} X_v dS_v \middle| \mathcal{F}_t \right] + H_t Y_{\tau,t}, \quad 0 \leq t \leq T, \qquad (5.3)$$

where $J_t \triangleq \mathbb{I}_{\{S_t > 0\}} \frac{1 - H_t}{S_t}$ is defined in (2.10).

PROOF: Because nonnegative random variables can be approximated by simple random variables, it suffices to show that (5.3) holds whenever $X = \mathbb{I}_A$ is the indicator of an $\mathcal{F}_T \times \mathcal{B}[0,T]$-measurable set A. For this we use Dynkin's π-λ theorem (e.g., [4], p. 37).

Let $\mathcal{L}$ be the collection of all sets $A \in \mathcal{F}_T \times \mathcal{B}[0,T]$ for which (5.3) is satisfied by $X = \mathbb{I}_A$. If $A = \Omega$ so that $X \equiv 1$, then (5.3) reduces to $\mathbb{E}[H_T | \mathcal{H}_t] = -J_t \mathbb{E}[S_T | \mathcal{F}_t] + 1$, and this equation can be verified by taking the $\mathcal{H}_t$ conditional expectation on both sides of (2.11). It follows that $\Omega \in \mathcal{L}$. This fact and the linearity of (5.3) implies that $\mathcal{L}$ is closed under complementation. Finally, if $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{L}$, then $\cup_{n=1}^{\infty} A_n$ is also in $\mathcal{L}$. Hence, $\mathcal{L}$ is a λ-system.

We next define $\mathcal{P}$ to be the collection of all sets of the form $A \times]t_1, t_2]$, where $A \in \mathcal{F}_T$ and $0 \leq t_1 \leq t_2 \leq T$. This collection is closed under pairwise intersection (a π-system), and it generates the σ-algebra $\mathcal{F}_T \times \mathcal{B}[0,T]$. We show that $\mathcal{P} \subset \mathcal{L}$, and then appeal to the π-λ theorem to assert that every set in $\mathcal{F}_T \times \mathcal{B}[0,T]$ is in $\mathcal{L}$. To show that $\mathcal{P} \subset \mathcal{L}$, we fix t in (5.3) and consider $X = \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}$. We need only consider the cases $0 \le t_1 \le t_2 \le t$ and $t \le t_1 \le t_2 \le T$; the case $X = \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}$ for $0 \le t_1 \le t \le t_2 \le T$ can then be gotten from the decomposition $X = \mathbb{I}_A \mathbb{I}_{]t_1,t]} + \mathbb{I}_A \mathbb{I}_{]t,t_2]}$. For $0 \le t_1 \le t_2 \le t$, we have $Y_{s,t} = \mathbb{I}_{]t_1,t_2]}(s)\mathbb{E}[\mathbb{I}_A|\mathcal{F}_t]$, and the right-hand side of (5.3) becomes

$$-J_t \mathbb{E}\left[\mathbb{I}_A \int_{]t,T]} \mathbb{I}_{]t_1,t_2]}(u)dS_u \middle| \mathcal{F}_t\right] + H_t \mathbb{I}_{]t_1,t_2]}(\tau)\mathbb{E}[\mathbb{I}_A|\mathcal{F}_t] = \mathbb{I}_{]t_1,t_2]}(\tau)\mathbb{E}[\mathbb{I}_A|\mathcal{F}_t],$$

which agrees with the left-hand side.

It remains to prove (5.3) when $X = \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}$ with $t \le t_1 \le t_2 \le T$ and $A \in \mathcal{F}_T$. For such an X, we have $Y_{s,t}{=}0$ when $0 \le s \le t$, and hence $H_t Y_{\tau,t} = 0$. To prove (5.3), it suffices then to show

$$\mathbb{E}\left[\mathbb{I}_B \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)\mathbb{I}_{\{\tau \le T\}}\right] = -\mathbb{E}\left[\mathbb{I}_B J_t \mathbb{E}\left[\mathbb{I}_A \int_{]t,T]} \mathbb{I}_{]t_1,t_2]}(v)dS_v \middle| \mathcal{F}_t\right]\right] \tag{5.4}$$

for every set $B \in \mathcal{H}_t$. We first do this for sets $B \in \mathcal{F}_t$. For such a set, starting with the right-hand side of (5.4) and using (2.2), we have

$$\begin{aligned}
-E\left[\mathbb{I}_B J_t \mathbb{E}\left[\mathbb{I}_A \int_{]t,T]} \mathbb{I}_{]t_1,t_2]}(v)dS_v \middle| \mathcal{F}_t\right]\right] &= \mathbb{E}\big[\mathbb{I}_B J_t \mathbb{E}[\mathbb{I}_A (S_{t_1} - S_{t_2})|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[\mathbb{I}_B J_t \mathbb{E}[\mathbb{I}_A \mathbb{E}(S_{t_1} - S_{t_2}|\mathcal{F}_T)|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[\mathbb{I}_B J_t \mathbb{E}[\mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)|\mathcal{F}_t]\big] \\
&= \mathbb{E}\big[\mathbb{I}_B \mathbb{E}[J_t|\mathcal{F}_t]\mathbb{E}[\mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)|\mathcal{F}_t]\big].
\end{aligned}$$

According to (2.5), $\mathbb{E}[J_t|\mathcal{F}_t] = 1$ on the set $\{S_t > 0\}$. On the set $\{S_t = 0\}$, we still have $\mathbb{E}[J_t|\mathcal{F}_t]\mathbb{E}[\mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)|\mathcal{F}_t] = \mathbb{E}[\mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)|\mathcal{F}_t]$ because both sides are zero. Therefore,

$$\begin{aligned}
-\mathbb{E}\left[\mathbb{I}_B J_t \mathbb{E}\left[\mathbb{I}_A \int_{]t,T]} \mathbb{I}_{]t_1,t_2]}(v)dS_v \middle| \mathcal{F}_t\right]\right] &= \mathbb{E}\big[\mathbb{I}_B \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)\big] \\
&= \mathbb{E}\left[\mathbb{I}_B \mathbb{I}_A \mathbb{I}_{]t_1,t_2]}(\tau)\mathbb{I}_{\{\tau \le T\}}\right],
\end{aligned}$$

and we have obtained the left-hand side of (5.4).

The collection of sets $B \in \mathcal{H}_t$ for which (5.4) holds is a λ-system. We have just shown that Ω belongs to this collection, and because of the linearity of (5.4), this collection is closed under complementation and disjoint unions. Furthermore, the collection of sets $B = C \cap \{\tau \le s\}$, where $C \in \mathcal{F}_t$ and $0 \le s \le t$, together with the collection of sets $B \in \mathcal{F}_t$, forms a π-system which generates $\mathcal{H}_t$. Equation (5.4) holds for every set in this π-system. We have already established this for sets $B \in \mathcal{F}_t$. For sets $B = C \cap \{\tau \le s\}$ with

$0 \le s \le t$, both sides of (5.4) are zero. The Dynkin π-λ system theorem implies that (5.4) holds for every $B \in \mathcal{H}_t$. $\diamondsuit$

PROOF OF THEOREM 5.1: We observe from (2.16) that if $\tau \le T$, then

$$\beta_T G_T = \beta_\tau G_\tau = \int_{]0,\tau[} \beta_u dC_u + \beta_\tau Z_\tau,$$

whereas if $\tau > T$, then $\beta_T G_T = \int_{]0,T]} \beta_u dC_u$. Combining these two observations, we obtain the formula

$$\begin{aligned} \beta_T G_T \;=\; \beta_{T\wedge\tau} G_{T\wedge\tau} \;&=\; \int_{]0,T]} \beta_u dC_u + \mathbb{I}_{\{\tau \le T\}} \left(- \int_{[\tau,T]} \beta_u dC_u + \beta_\tau Z_\tau \right) \\ &=\; \overline{X} + \mathbb{I}_{\{\tau \le T\}} X_\tau, \end{aligned}$$

where $\overline{X} \triangleq \int_{]0,T]} \beta_u dC_u$, $X_t \triangleq -\int_{[t,T]} \beta_u dC_u + \beta_t Z_t$.

We define the $\{\mathcal{F}_t\}$-martingales

$$\begin{aligned} R_t \;&\triangleq\; \mathbb{E}[\overline{X}|\mathcal{H}_t] \;=\; \mathbb{E}[\overline{X}|\mathcal{F}_t], \\ Q_t \;&\triangleq\; \mathbb{E}\left[\int_{]0,T]} \beta_u S_u dC_u - \int_{]0,T]} \beta_u Z_u dS_u \,\middle|\, \mathcal{F}_t \right]. \end{aligned}$$

Because βG is an $\{\mathcal{H}_t\}$-martingale (see (2.16)), Lemma 5.2 implies

$$\beta_t G_t = \mathbb{E}[\overline{X}|\mathcal{H}_t] + \mathbb{E}[\mathbb{I}_{\{\tau \le T\}} X_\tau|\mathcal{H}_t] = R_t - J_t \mathbb{E}\left[\int_{]t,T]} X_u dS_u \,\middle|\, \mathcal{F}_t \right] + H_t Y_{\tau,t}, \tag{5.5}$$

where

$$Y_{s,t} \triangleq \mathbb{E}[X_s|\mathcal{F}_t] = -R_t + \int_{]0,s[} \beta_u dC_u + \beta_s Z_s, \;\; 0 < s \le t \le T. \tag{5.6}$$

Using (3.12) to justify the second equality below, we compute

$$\begin{aligned} &\mathbb{E}\left[\int_{]t,T]} X_u dS_u \,\middle|\, \mathcal{F}_t \right] \\ &\quad= \mathbb{E}\left[- \int_{]t,T]} \int_{[u,T]} \beta_v dC_v dS_u \,\middle|\, \mathcal{F}_t \right] + \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dS_u \,\middle|\, \mathcal{F}_t \right] \\ &\quad= \beta_t S_t \, NDP_t[C] - \beta_t S_t \, CFP_t[C] + \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dS_u \,\middle|\, \mathcal{F}_t \right] \qquad (5.7) \\ &\quad= S_t \mathbb{E}\left[\int_{]t,T]} \beta_u dC_u \,\middle|\, \mathcal{F}_t \right] - \mathbb{E}\left[\int_{]t,T]} \beta_u S_u dC_u \,\middle|\, \mathcal{F}_t \right] \\ &\qquad\quad + \mathbb{E}\left[\int_{]t,T]} \beta_u Z_u dS_u \,\middle|\, \mathcal{F}_t \right] \end{aligned}$$

$$= \quad S_t\left(R_t - \int_{]0,t]} \beta_u dC_u\right) - Q_t + \int_{]0,t]} \beta_u S_u dC_u - \int_{]0,t]} \beta_u Z_u dS_u.$$

Substituting (5.6) and (5.7) into (5.5), we obtain

$$\begin{aligned}
\beta_t G_t \quad &= \quad -J_t\left(-Q_t + \int_{]0,t]} \beta_u S_u dC_u - \int_{]0,t]} \beta_u Z_u dS_u\right) \\
&\qquad +(1-H_t)\int_{]0,t]} \beta_u dC_u + H_t\left(\int_{]0,\tau[} \beta_u dC_u + \beta_\tau Z_\tau\right).
\end{aligned}$$

We may rewrite this using Itô's product rule (Lemma 2.2) as

$$\begin{aligned}
\beta_t G_t \quad &= \quad G_0 + \int_{]0,t]} J_{u-} dQ_u + \int_{]0,t]} Q_u dJ_u \\
&\qquad - \int_{]0,t]} J_u \beta_u S_u dC_u - \int_{]0,t]}\int_{]0,u[} \beta_v S_v dC_v dJ_u \\
&\qquad + \int_{]0,t]} J_{u-}\beta_u Z_u dS_u + \int_{]0,t]}\int_{]0,u]} \beta_v Z_v dS_v dJ_u \qquad (5.8) \\
&\qquad + \int_{]0,t]} (1-H_u)\beta_u dC_u - \int_{]0,t]}\int_{]0,u[} \beta_v dC_v dH_u \\
&\qquad + \int_{]0,t]}\int_{]0,u[} \beta_v dC_v dH_u + \int_{]0,t]} \beta_u Z_u dH_u \\
&= \quad G_0 + \int_{]0,t]} J_{u-} dQ_u + \int_{]0,t]} \beta_u Z_u dM_u + \int_{]0,t]} \psi_u dJ_u,
\end{aligned}$$

where

$$\psi_u \triangleq \left(Q_u - \int_{]0,u[} \beta_v S_v dC_v + \int_{]0,u]} \beta_v Z_v dS_v\right).$$

We note that

$$\begin{aligned}
\mathbb{I}_{\{S_u=0\}}\psi_u \quad &= \quad \mathbb{I}_{\{\overline{\tau}\le u\}}\mathbb{E}\left[\int_{[u,T]} \beta_v S_v dC_v - \int_{]u,T]} \beta_v Z_v dS_v \,\middle|\, \mathcal{F}_u\right] \\
&= \quad \mathbb{E}\left[\mathbb{I}_{\{\overline{\tau}\le u\}}\left(\int_{[u,T]} \beta_v S_v dC_v - \int_{]u,T]} \beta_v Z_v dS_v\right)\middle|\, \mathcal{H}_u\right] \; = \; 0
\end{aligned}$$

because $S_v = 0$ for $v \in [\overline{\tau}, T]$. Recalling (2.13), we conclude that

$$\int_{]0,t]} \psi_u dJ_u = \int_{]0,t]} \mathbb{I}_{\{S_u>0\}}\psi_u dJ_u + \int_{]0,t]} \mathbb{I}_{\{S_u=0\}}\psi_u dJ_u = -\int_{]0,t]} \mathbb{I}_{\{S_u>0\}}\frac{\psi_u}{S_u} dM_u.$$

Substitution of this into (5.8) gives (5.1) with

$$N_{t\wedge\tau} \triangleq \int_{]0,t\wedge\tau]} J_{u-}dQ_u = \int_{]0,t\wedge\tau]} \mathbb{I}_{\{S_{u-}>0\}}\frac{1}{S_{u-}}dQ_u \tag{5.9}$$

$$\varphi_u \triangleq Z_u + \frac{1}{\beta_u}\mathbb{E}\left[\mathbb{I}_{\{S_u>0\}}\frac{1}{S_u}\left(-\int_{[u,T]}\beta_v S_v dC_v + \int_{]u,T]}\beta_v Z_v dS_v\right)\middle|\mathcal{F}_u\right] \tag{5.10}$$

From (5.10), (3.3) and (3.4), we obtain (5.2). ♢

6 Default-prone term structure

The price of a default-risk-free zero-coupon bond paying \$1 at maturity T is

$$P(t,T) \triangleq \mathbb{E}[\beta(t,T)|\mathcal{F}_t] = NDP_t[\mathbb{I}_{[T,\overline{T}]}],\ 0 \le t < T \le \overline{T}.$$

In this section, we consider the term structure associated with a set of defaultable zero-coupon bonds with different maturities but simultaneous default. We assume $\mathbb{P}\{S_{\overline{T}} > 0\} = 1$ throughout.

6.1 Defaultable zero-coupon bonds

Consider a family of defaultable zero-coupon bonds, indexed by maturity $T \in]0,\overline{T}]$. For each maturity T, $\{Z(t,T); 0 \le t \le T\}$ is a recovery process satisfying Assumption 2.5. For $0 \le t \le T \le \overline{T}$, we define

$$P^Z(t,T) \triangleq \mathbb{E}\left[\beta(t,T)\mathbb{I}_{\{\tau>T\}} + \int_{]t,T]}\beta(t,u)Z(u,T)dH_u\middle|\mathcal{H}_t\right].$$

For $0 \le t < T$, this is the price of a defaultable bond promising \$1 at maturity T and paying recovery $Z(\tau,T)$ in the event of default at time $\tau \le T$ (see (2.14)). According to the Price Representation Theorem 2.8, $P^Z(t,T) = (1-H_t)\widehat{P}^Z(t,T)$ for $0 \le t < T$, where

$$\widehat{P}^Z(t,T) \triangleq \frac{1}{\beta_t S_t}\mathbb{E}\left[\beta_T S_T - \int_{]t,T]}\beta_u Z(u,T)dS_u\middle|\mathcal{F}_t\right],\ 0 \le t \le T. \tag{6.1}$$

Because $\mathbb{P}\{S_{\overline{T}} > 0\} = 1$, we have $\widehat{P}^Z(t,T) > 0$ and can define the *recovery ratio*

$$\rho^Z(t,T) \triangleq \frac{Z(t,T)}{\widehat{P}^Z(t,T)}. \tag{6.2}$$

We assume throughout that $\rho^Z(t,T)$ takes values in $[0,1]$, i.e., $0 \le Z(t,T) \le \widehat{P}^Z(t,T)$ for $0 \le t \le T \le \overline{T}$.

Theorem 6.1 *For each T, there is an $\{\mathcal{F}_t\}$-predictable, d-dimensional* volatility vector $\sigma^Z(t,T)$ *such that*

$$\frac{d\widehat{P}^Z(t,T)}{\widehat{P}^Z(t,T)} = r_t dt + \big(1-\rho^Z(t,T)\big)d\Lambda_t + \sigma^Z(t,T)\cdot dW_t, \quad 0 \le t \le T. \qquad (6.3)$$

PROOF: We define the $\{\mathcal{H}_t\}$- and $\{\mathcal{F}_t\}$-martingale

$$Q(t,T) \quad \triangleq \quad \mathbb{E}\left[\beta_T S_T - \int_{]0,T]} \beta_u Z(u,T) dS_u \,\middle|\, \mathcal{F}_t\right].$$

According to (6.1), $Q(t,T)+\int_{]0,t]}\beta_u Z(u,T)dS_u = \beta_t S_t \widehat{P}^Z(t,T)$, and Itô's product rule (Lemma 2.2) implies

$$\begin{aligned} & dQ(t,T) + \beta_t Z(t,T) dS_t \\ & \quad = \quad S_{t-} d\big(\beta_t \widehat{P}^Z(t,T)\big) + \beta_t \widehat{P}^Z(t,T) dS_t \\ & \quad = \quad -r_t \beta_t S_{t-} \widehat{P}^Z(t,T)dt + \beta_t S_{t-} d\widehat{P}^Z(t,T) + \beta_t \widehat{P}^Z(t,T) dS_t, \end{aligned}$$

which, divided by $\beta_t S_{t-}$, becomes

$$d\widehat{P}^Z(t,T) = \widehat{P}^Z(t,T)\big(r_t dt + (1-\rho^Z(t,T)d\Lambda_t)\big) + \frac{1}{\beta_t S_{t-}} dQ(t,T). \qquad (6.4)$$

Being an $\{\mathcal{F}_t\}$-martingale, $Q(t,T)$ has an Itô integral representation

$$Q(t,T) = Q(0,T) + \int_0^t \gamma^Z(u,T)\cdot dW_u \qquad (6.5)$$

for some d-dimensional $\{\mathcal{F}_t\}$-predictable process $\gamma^Z(t,T)$. We set $\sigma^Z(t,T) = \frac{\gamma^Z(t,T)}{\beta_t S_{t-}\widehat{P}^Z(t,T)}$, and then substitution of (6.5) into (6.4) yields (6.3). $\diamondsuit$

For integrable random variables X and Y, the conditional covariance is

$$\begin{aligned} \mathrm{Cov}[X,Y|\mathcal{F}_t] \quad & \triangleq \quad \mathbb{E}\big[\big(X - E[X|\mathcal{F}_t]\big)\big(Y - E[Y|\mathcal{F}_t]\big)\big|\mathcal{F}_t\big] \\ & = \quad \mathbb{E}[XY|\mathcal{F}_t] - E[X|\mathcal{F}_t]\cdot\mathbb{E}[Y|\mathcal{F}_t]. \end{aligned}$$

We denote by $\widehat{P}^{CFP}(t,T)$ the price of a zero-coupon bond promising \$1 at time T under the cash flow premium (CFP) convention (i.e., zero recovery), and use analogous notation for the other three conventions, recovery of par value (RPV), recovery of treasury value (RTV) and recovery of market value (RMV).

Proposition 6.2 *For $\delta \in [0,1]$ constant, we have*

$$\begin{aligned} \widehat{P}^{CFP}(t,T) \quad &= \quad \mathbb{E}[S(t,T)|\mathcal{F}_t]\cdot\mathbb{E}[\beta(t,T)|\mathcal{F}_t] + Cov[S(t,T),\beta(t,T)|\mathcal{F}_t], \quad (6.6) \\ \widehat{P}^{RTV}(t,T) \quad &= \quad \delta P(t,T) + (1-\delta)\widehat{P}^{CFP}(t,T). \quad (6.7) \end{aligned}$$

If S is nonrandom (but possibly discontinuous), then

$$\widehat{P}^{RPV}(t,T) = P(t,T)S(t,T) - \frac{\delta}{S_t}\int_t^T P(t,u)dS_u. \tag{6.8}$$

If S is continuous (but possibly random), then

$$\begin{aligned}\widehat{P}^{RMV}(t,T) &= \mathbb{E}\left[\beta(t,T)S^{1-\delta}(t,T)\middle|\mathcal{F}_t\right] \\ &\leq \delta P(t,T) + (1-\delta)\widehat{P}^{CFP}(t,T).\end{aligned} \tag{6.9}$$

Proof: Equations (6.6), (6.7) and (6.8) follow from (6.1), (3.6) and (3.5), respectively. To obtain (6.9), we first use (3.24) and then the inequality

$$s^{1-\delta} \leq \delta + (1-\delta)s, \quad 0 \leq s \leq 1. \tag{6.10}$$

Remark 6.3 Under the recovery of treasury convention, the recovery ratio is

$$\rho^{RTV}(t,T) = \frac{\delta P(t,T)}{\widehat{P}^{RTV}(t,T)} = \frac{\delta P(t,T)}{\delta P(t,T) + (1-\delta)\widehat{P}^{CFP}(t,T)}. \tag{6.11}$$

If S is nonrandom, then the recovery ratio under the recovery of par value convention is $\rho^{RPV}(t,T) = \frac{\delta}{\widehat{P}^{RPV}(t,T)}$. One way to ensure this is in $[0,1]$ is to assume that $P(0,T)S(T) \geq \delta$, since $\widehat{P}^{RPV}(t,T) \geq P(0,T)S(T)$. If S is continuous, then the recovery ratio under the recovery of market value convention is $\rho^{RMV}(t,T) = \delta$. Only in the case of recovery of market value is the recovery ratio independent of the maturity variable T. ♢

Proposition 6.2 allows us to estimate the probability of default under the martingale measure $\mathbb{P}$ from bond prices under the different recovery conventions.

Corollary 6.4 *Assume S and β are independent. Then*

$$\mathbb{P}\{\tau > T\} = \frac{\widehat{P}^{CFP}(0,T)}{P(0,T)} = \frac{1}{1-\delta}\left[\frac{\widehat{P}^{RTV}(0,T)}{P(0,T)} - \delta\right]. \tag{6.12}$$

It S is nonrandom (but possibly discontinuous), then

$$\begin{aligned}\mathbb{P}\{\tau > T\} &= \exp\left\{-\frac{1}{1-\delta}\int_0^T \frac{dP(0,u)}{P(0.u)}\right\} \\ &\quad + \frac{1}{1-\delta}\int_0^T \exp\left\{-\int_t^T \frac{dP(0,u)}{P(0,u)}\right\}\cdot\frac{d\widehat{P}^{RPV}(0,t)}{P(0,t)}.\end{aligned} \tag{6.13}$$

If S is continuous (but possibly random), then

$$\mathbb{P}\{\tau > T\} \geq \frac{1}{1-\delta}\left[\frac{\widehat{P}^{RMV}(0,T)}{P(0,T)} - \delta\right]. \tag{6.14}$$

PROOF: Recalling that $\mathbb{P}\{\tau > T\} = \mathbb{E}S_t$, we see that (6.12) and (6.14) follow immediately from (6.6), (6.7) and (6.9), respectively. It remains to prove (6.13). Because r_u is nonnegative for all u, $P(0,t) = \mathbb{E}e^{-\int_0^t r_u du}$ is nondecreasing in t. Defining

$$I_t \triangleq \exp\left\{\frac{1}{1-\delta}\int_0^t \frac{dP(0,u)}{P(0,u)}\right\},$$

we have $dI_t = I_t dP(0,t)/(1-\delta)P(0,t)$. From (6.8) we see that (Lemma 2.2) $d\widehat{P}^{RPV}(0,t) = (1-\delta)P(0,t)dS_t + S_{t-}dP(0,t)$, which implies $d(I_tS_t) = I_t dS_t + S_{t-}dI_t = \frac{I_t d\widehat{P}^{RPV}(0,t)}{(1-\delta)P(0,t)}$. Integration yields $I_TS_T = 1 + \frac{1}{1-\delta}\int_0^T I_t \frac{d\widehat{P}^{RPV}(0,t)}{P(0,t)}$, which is (6.13). ♢

Remark 6.5 Inequality (6.10) is sharp for s near 1, and hence (6.9) is sharp for $S(t,T)$ near 1. This is typically the case, since $S(t,T)$ is the probability of survival between t and T, given that there has been no default by time t.

6.2 Credit spreads

In this subsection, we make the intensity-based reduced-form model assumptions of Definition 2.1. We define the *effective credit spot rate*

$$r^Z(t,T) \triangleq r_t + \big(1 - \rho^Z(t,T)\big)\lambda_t, \tag{6.15}$$

so that (6.3) becomes

$$\frac{d\widehat{P}^Z(t,T)}{\widehat{P}^Z(t,T)} = r^Z(t,T)dt + \sigma^Z(t,T)\cdot dW_t, \quad 0 \le t \le T. \tag{6.16}$$

Theorem 6.6 *For the intensity-based reduced-form model of Definition 2.1, we have*

$$\widehat{P}^Z(t,T) = \mathbb{E}\left[e^{-\int_t^T r^Z(u,T)du} \middle| \mathcal{F}_t \right]. \tag{6.17}$$

PROOF: Equation (6.16) and Itô's product rule imply

$$d\left(e^{-\int_0^t r^Z(u,T)du}\widehat{P}^Z(t,T)\right) = e^{-\int_0^t r^Z(u,T)du}\widehat{P}^Z(t,T)\sigma^Z(t,T)\cdot dW_t,$$

and hence $e^{-\int_0^t r^Z(u,T)du}P^Z(t,T)$ is an $\{\mathcal{F}_t\}$-local-martingale. From (6.1),

$$\begin{aligned}
e^{-\int_0^t r^Z(u,T)du}\widehat{P}^Z(t,T) &\le e^{-\int_0^t r_u du}\widehat{P}^Z(t,T) \\
&= \frac{1}{S_t}\mathbb{E}\left[\beta_T S_T - \int_{]t,T]} \beta_u Z(u,T)dS_u \middle| \mathcal{F}_t\right] \\
&\le \mathbb{E}\left[1 + (\sup_{0\le u\le T}\beta_u Z(u,T))\left(1 - \frac{S_T}{S_t}\right)\middle| \mathcal{F}_t\right] \\
&\le 1 + \mathbb{E}[\sup_{0\le u\le T}\beta_u Z(u,T)|\mathcal{F}_t].
\end{aligned}$$

In light of Assumption 2.5, this last expression is an $\{\mathcal{F}_t\}$-martingale with the last element $1 + \sup_{0 \le u \le T} \beta_u Z(u,T)$. Being bounded above by such a martingale, the nonnegative local martingale $e^{-\int_0^t r^Z(u,T)du}\widehat{P}^Z(t,T)$ is itself a martingale. In particular, $e^{-\int_0^t r^Z(u,T)du}\widehat{P}^Z(t,T) = \mathbb{E}\left[e^{-\int_0^T r^Z(u,T)du}\widehat{P}^Z(T,T)\Big|\, \mathcal{F}_t\right]$, and (6.17) follows. ◊

We determine the credit spread under each of the four recovery conventions. We use the notation $r^{CFP}(t,T)$ and $\rho^{CFP}(t,T)$ when the recovery process Z corresponds to the cash flow premium convention of zero recovery, and we use analogous notation for the other three conventions.

Proposition 6.7 *We have the following spread formulas:*

$$r^{CFP}(t,T) - r_t = \lambda_t, \tag{6.18}$$

$$r^{RPV}(t,T) - r_t = \left(1 - \frac{\delta}{\widehat{P}^{RPV}(t,T)}\right)\lambda_t, \tag{6.19}$$

$$r^{RTV}(t,T) - r_t = (1-\delta)\frac{\widehat{P}^{CFP}(t,T)}{\widehat{P}^{RTV}(t,T)}\lambda_t, \tag{6.20}$$

$$r^{RMV}(t,T) - r_t = (1-\delta)\lambda_t. \tag{6.21}$$

PROOF: Equation (6.18) is a trivial consequence of (6.15) because $\rho^{CFP}(t,T) = 0$. Equation (6.19) is also trivial, the result of setting $Z \equiv \delta$ in (6.2). For (6.20), we first recall (3.6) in the form $\widehat{P}^{RTV}(t,T) = \delta\widehat{P}^{NDP}(t,T) + (1-\delta)\widehat{P}^{CFP}(t,T)$. The recovery process is $Z_t = \delta\widehat{P}^{NDP}(t,T)$. This process is continuous in the intensity-based reduced-form model, so there is no need to take the left limit. Substitution of these formulas into (6.2), (6.15) gives the stated result. Finally, (6.21) comes from the formula $Z(t,T) = \delta\widehat{P}^{RMV}(t,T)$, where again there is no need to take the left limit. ◊

We note that in the four cases of Proposition 6.7, the spreads are typically random. In the cases of cash flow premium and recovery of market value, they are independent of maturity T.

6.3 Forward curves

For this section and the next, we shall impose the following condition on the intensity-based reduced form model of Definition 2.1:

Assumption 6.8 *There exist finite constants $p > 1$, $L > 0$ and $\gamma > 1$ such that the nonnegative processes r_t and λ_t satisfy the continuity condition*

$$\mathbb{E}|r_{t_1} - r_{t_2}|^{2p} + \mathbb{E}|\lambda_{t_1} - \lambda_{t_2}|^{2p} \le L|t_1 - t_2|^{2\gamma}. \tag{6.22}$$

The processes $Z(t,T)$ are bounded, uniformly in (t,T) and ω. For all $0 \le t \le T \le T^*$, $\frac{\partial}{\partial T}Z(t,T)$ exists, and for all $T_1, T_2 \in [0,\overline{T}]$, the processes $Z(t,T)$ and

$\frac{\partial}{\partial T}Z(t,T)$ *satisfy the continuity and integrability conditions*

$$E\Big|Z(T_1,T_1)-Z(T_2,T_2)\Big|^{2p} \leq L|T_1-T_2|^{2\gamma}, \quad (6.23)$$

$$\sup_{t\in[0,T_1\wedge T_2]} \mathbb{E}\left|\frac{\partial}{\partial T}Z(t,T_1)-\frac{\partial}{\partial T}Z(t,T_2)\right|^{2p} \leq L|T_1-T_2|^{2\gamma}, \quad (6.24)$$

$$\sup_{t\in[0,\overline{T}]} E\left|\frac{\partial}{\partial T}Z(t,\overline{T})\right|^{2p} < \infty. \quad (6.25)$$

Remark 6.9 There is a constant C such that for all $a\geq 0$, $b\geq 0$,

$$a^{2p}+b^{2p}\leq (a+b)^{2p}\leq C(a^{2p}+b^{2p}),\ \ a^p+b^p\leq (a+b)^p\leq C(a^p+b^p). \quad (6.26)$$

Because r_0 and λ_0 are constant, (6.22) and (6.26) imply the uniform integrability condition

$$\sup_{t\in[0,\overline{T}]}\big(\mathbb{E}|r_t|^{2p}+\mathbb{E}|\lambda_t|^{2p}\big)\leq \sup_{t\in[0,\overline{T}]}\mathbb{E}(r_t+\lambda_t)^{2p}<\infty. \quad (6.27)$$

Similarly, (6.24), (6.25) and (6.26) imply

$$\sup_{t,T\in[0,\overline{T}],t\leq T}\mathbb{E}\left|\frac{\partial}{\partial T}Z(t,T)\right|^{2p}<\infty. \quad (6.28)$$

Theorem 6.10 *Under Assumption 6.8, we have*

$$\frac{\partial}{\partial T}\widehat{P}^Z(t,T)=\mathbb{E}\Bigg[-(r_T+\lambda_T)\beta(t,T)S(t,T)+\lambda_T\beta(t,T)S(t,T)Z(T,T) \quad (6.29)$$
$$+\int_t^T\lambda_u\beta(t,u)S(t,u)\frac{\partial}{\partial T}Z(u,T)du\Bigg|\mathcal{F}_t\Bigg],\ 0\leq t\leq T<\overline{T},$$

and there is a version of this process which is jointly continuous in (t,T).

Proof: Define

$$X_T\triangleq -(r_T+\lambda_T)\beta_TS_T+\lambda_T\beta_TS_TZ(T,T)+\int_0^T\lambda_v\beta_vS_v\frac{\partial}{\partial T}Z(v,T)dv,\ 0\leq T\leq\overline{T}.$$

We derive a continuity property of X. Recall that β and S are both bounded between 0 and 1. For $0\leq T_1\leq T_2\leq\overline{T}$, we may write

$$\begin{aligned}|X_{T_2}-X_{T_1}| \leq\ & \beta_{T_2}S_{T_2}\big|-r_{T_2}-\lambda_{T_2}+r_{T_1}+\lambda_{T_1}\big|\\ &+(r_{T_1}+\lambda_{T_1})\beta_{T_1}S_{T_1}\big(1-\beta(T_1,T_2)S(T_1,T_2)\big)\\ &+\lambda_{T_2}\beta_{T_2}S_{T_2}\big|Z(T_2,T_2)-Z(T_1,T_1)\big|\\ &+\beta_{T_2}S_{T_2}\big|Z(T_1,T_1)(\lambda_{T_2}-\lambda_{T_1})\big|\\ &+\lambda_{T_1}\big|Z(T_1,T_1)\beta_{T_1}S_{T_1}\big(\beta(T_1,T_2)S(T_1,T_2)-1\big)\big|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{T_1} \lambda_v \beta_v S_v \left| \frac{\partial}{\partial T} Z(v,T_2) - \frac{\partial}{\partial T} Z(v,T_1) \right| dv \\
&+\int_{T_1}^{T_2} \lambda_v \beta_v S_v \left| \frac{\partial}{\partial T} Z(v,T_2) \right| dv \ \leq \ \sum_{i=1}^{7} A_i,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &\triangleq |r_{T_2} - r_{T_1}| + |\lambda_{T_2} - \lambda_{T_1}|, \qquad A_2 \triangleq (r_{T_1} + \lambda_{T_1}) \int_{T_1}^{T_2} (r_v + \lambda_v) dv, \\
A_3 &\triangleq \lambda_{T_2} \big| Z(T_2, T_2) - Z(T_1, T_1) \big|, \qquad A_4 \triangleq \big| Z(T_1, T_1) \big| \cdot |\lambda_{T_2} - \lambda_{T_1}|, \\
A_5 &\triangleq \lambda_{T_1} |Z(T_1, T_1)| \int_{T_1}^{T_2} (r_v + \lambda_v) dv, \\
A_6 &\triangleq \int_0^{T_1} \lambda_v \left| \frac{\partial}{\partial T} Z(v,T_2) - Z(v,T_1) \right| dv, \qquad A_7 \triangleq \int_{T_1}^{T_2} \lambda_v \left| \frac{\partial}{\partial T} Z(v,T_2) \right| dv.
\end{aligned}
$$

According to (6.26), there is a constant C such that $\mathbb{E}|X_{T_2} - X_{T_1}|^p \leq C \sum_{i=1}^{7} \mathbb{E} A_i^p$. From (6.22) and Hölder's inequality, $\mathbb{E} A_1^p \leq \left(\mathbb{E} A_1^{2p} \right)^{1/2} \leq \sqrt{L}\, |T_2 - T_1|^\gamma$. From Jensen's inequality, Hölder's inequality, and (6.27), we have

$$
\begin{aligned}
\mathbb{E} A_2^p &\leq \mathbb{E} \left[(r_{T_1} + \lambda_{T_1})^p (T_2 - T_1)^p \left(\int_{T_1}^{T_2} (r_v + \lambda_v) \frac{dv}{T_2 - T_1} \right)^p \right] \\
&\leq \mathbb{E} \left[(r_{T_1} + \lambda_{T_1})^p (T_2 - T_1)^p \int_{T_1}^{T_2} (r_v + \lambda_v)^p \frac{dv}{T_2 - T_1} \right] \\
&\leq (T_2 - T_1)^{p-1} \mathbb{E} \int_{T_1}^{T_2} (r_{T_1} + \lambda_{T_1})^p (r_v + \lambda_v)^p dv \\
&\leq (T_2 - T_1)^{p-1} \left(\int_{T_1}^{T_2} \mathbb{E} (r_{T_1} + \lambda_{T_1})^{2p} dv \right)^{1/2} \left(\int_{T_1}^{T_2} \mathbb{E} (r_v + \lambda_v)^{2p} dv \right)^{1/2} \\
&\leq C (T_2 - T_1)^p
\end{aligned}
$$

for some finite constant C. From (6.27), (6.23) and Hölder's inequality, we have $\mathbb{E} A_3^p \leq C |T_2 - T_1|^\gamma$ for some finite constant C. Because Z is uniformly bounded,

$$
\mathbb{E} A_4^p \leq C \mathbb{E} |\lambda_{T_2} - \lambda_{T_1}|^p \leq C \left(\mathbb{E} |\lambda_{T_2} - \lambda_{T_1}|^{2p} \right)^{1/2} \leq C L |T_1 - T_2|^\gamma
$$

for some constant C. Because Z is bounded, $\mathbb{E} A_5^p$ is bounded by a constant times $\mathbb{E} A_2^p$. Using Jensen's inequality, Hölder's inequality, (6.27) and (6.24) on A_6, we obtain

$$
\begin{aligned}
\mathbb{E} A_6^p &\leq T_1^p \mathbb{E} \left(\int_0^{T_1} \lambda_v \left| \frac{\partial}{\partial T} Z(v,T_2) - \frac{\partial}{\partial T} Z(v,T_1) \right| \frac{dv}{T_1} \right)^p \\
&\leq T_1^{p-1} \mathbb{E} \int_0^{T_1} \lambda_v^p \left| \frac{\partial}{\partial T} Z(v,T_2) - \frac{\partial}{\partial T} Z(v,T_1) \right|^p dv
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; T_1^{p-1}\left(\int_0^{T_1}\mathbb{E}\lambda_{T_1}^{2p}dv\right)^{1/2}\left(\int_0^{T_1}\mathbb{E}\left|\frac{\partial}{\partial T}Z(v,T_2)-\frac{\partial}{\partial T}Z(v,T_1)\right|^{2p}dv\right)^{1/2}\\
&\leq \; C|T_1-T_2|^{\gamma}
\end{aligned}
$$

for some finite constant C. The argument used to bound $\mathbb{E}A_2^p$ shows that $\mathbb{E}A_7^p \leq C(T_2-T_1)^p$ for some finite constant C. We conclude that for all $T_1, T_2 \in [0,\overline{T}]$, $\mathbb{E}|X_{T_2}-X_{T_1}|^p \leq C|T_2-T_1|^{p\wedge\gamma}$ for some constant C

We next define $Y(t,T) \triangleq \mathbb{E}[X_T|\mathcal{F}_t]$ for $0 \leq t \leq \overline{T}$, $0 \leq T \leq \overline{T}$. For each fixed T, $Y(\cdot,T)$ is a martingale with respect to a Brownian filtration, and hence has a version with paths continuous in t. We choose this version. For fixed T, we regard $Y(\cdot,T)$ as a random variable taking values in the space $C[0,\overline{T}]$ of continuous functions on $[0,\overline{T}]$, and we denote by $\|\cdot\|_\infty$ the supremum norm on this space. We may further regard $\{Y(\cdot,T); 0 \leq T \leq \overline{T}\}$ as a $C[0,\overline{T}]$-valued random process. According to Doob's maximal martingale inequality,

$$
\mathbb{E}\|Y(\cdot,T_1)-Y(\cdot,T_2)\|_\infty^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|X_{T_1}-X_{T_2}|^p \leq C|T_1-T_2|^{p\wedge\gamma}.
$$

The Kolmogorov-Čentsov theorem (e.g., [29], p. 53) extends to $C[0,\overline{T}]$-valued processes, and implies the existence of a continuous version of $\{Y(\cdot,T); 0 \leq T \leq \overline{T}\}$. In other words, except for a null set of $\omega \in \Omega$, $Y(t,T)$ is jointly continuous in (t,T), and the continuity in T is uniform over t. We compute

$$
\begin{aligned}
\int_t^T Y(t,u)du &= \mathbb{E}\left[\int_t^T X_u du\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\beta_T S_t - \beta_t S_t - \int_t^T \beta_u Z(u,T)dS_u\middle|\mathcal{F}_t\right]\\
&= \beta_t S_t\left(\widehat{P}^Z(t,T)-1\right),
\end{aligned}
$$

or equivalently, $\widehat{P}^Z(t,T) = 1 + \frac{1}{\beta_t S_t}\int_t^T Y(t,u)du$. Differentiation with respect to T yields (6.29). ♢

Corollary 6.11 *Assume there exist finite constants $p > 1$, $L > 0$ and $\gamma > 1$ such that the nonnegative processes r_t and λ_t satisfy the continuity condition*

$$
\mathbb{E}|r_{t_1}-r_{t_2}|^{4p} + \mathbb{E}|\lambda_{t_1}-\lambda_{t_2}|^{4p} \leq L|t_1-t_2|^{4\gamma}. \tag{6.30}
$$

Let $\delta \in [0,1]$ be constant. If $Z \equiv \delta$ (recovery of par), or $Z(t,T) = \delta\mathbb{E}[\beta(t,T)|\mathcal{F}_t] = \delta P(t,T)$ (recovery of treasury), or $Z(t,T) = \delta\mathbb{E}[\beta(t,T)S^{1-\delta}(t,T)|\mathcal{F}_t]$ (recovery of market value; see Example 3.9), then the conclusion of Theorem 6.10 holds.

Proof: We need to verify, under each of the three recovery conventions, that Assumption 6.8 is satisfied. Condition (6.30) implies (6.22). If $Z \equiv \delta$, then (6.23)–(6.25) are also clearly satisfied.

The conditions imposed on r by (6.30) ensure that

$$\frac{\partial}{\partial T} P(t,T) = -\mathbb{E}[r_T \beta(t,T)|\mathcal{F}_t],$$

and there is a version of this process jointly continuous in (t,T); this is a special case of Theorem 6.10 with $\lambda \equiv 0$, $Z \equiv 0$. Furthermore, for $0 \le T_1 \le T_2 \le \overline{T}$,

$$\begin{aligned}
&\mathbb{E}\left|\frac{\partial}{\partial T} P(t,T_2) - \frac{\partial}{\partial T} P(t,T_1)\right|^{2p} \\
\le\ & C\mathbb{E}\left|\mathbb{E}\big[r_{T_2}\beta(t,T_1)\big(1-\beta(T_1,T_2)\big)\big|\mathcal{F}_t\big]\right|^{2p} + C\mathbb{E}\left|\mathbb{E}\big[\beta(t,T_1)\big|r_{T_1}-r_{T_2}\big|\big|\mathcal{F}_t\big]\right|^{2p} \\
\le\ & C\mathbb{E}\left[r_{T_2}\big(1-\beta(T_1,T_2)\big)\right]^{2p} + C\mathbb{E}|r_{T_1}-r_{T_2}|^{2p} \\
\le\ & C(T_2-T_1)^{2p}\mathbb{E}\left[\int_{T_1}^{T_2} r_{T_2} r_v \frac{dv}{T_2-T_1}\right]^{2p} + C\left(E|r_{T_1}-r_{T_2}|^{4p}\right)^{1/2} \qquad (6.31)\\
\le\ & C(T_2-T_1)^{2p-1}\mathbb{E}\left[\int_{T_1}^{T_2} r_{T_2}^{2p} r_v^{2p} dv\right] + C\left(E|r_{T_1}-r_{T_2}|^{4p}\right)^{1/2} \\
\le\ & C(T_2-T_1)^{2p-1}\left(\int_{T_1}^{T_2}\mathbb{E}r_{T_2}^{4p}dv\right)^{1/2}\left(\int_{T_1}^{T_2}\mathbb{E}r_v^{4p}dv\right)^{1/2} + C\sqrt{L}\,|T_1-T_2|^{2\gamma} \\
\le\ & C'|T_1-T_2|^{2p} + C\sqrt{L}\,|T_1-T_2|^{2\gamma}\ \le\ C''|T_1-T_2|^{2(p\wedge\gamma)}.
\end{aligned}$$

This establishes (6.24) with γ replaced by $p \wedge \gamma$ when $Z(t,T) = \delta P(t,T)$. Conditions (6.23) and (6.25) are similarly proved under this recovery convention.

If $Z(t,T) = \delta\mathbb{E}[\beta(t,T)S^{1-\delta}(t,T)|\mathcal{F}_t]$, then replacement of β by $\beta S^{1-\delta}$ in the argument just given for recovery of treasury shows that again, Z satisfies conditions (6.23)–(6.25). ♢

Under Assumption 6.8, we define the forward rate for the defaultable bond in the intensity-based reduced-form model to be

$$g^Z(t,T) \triangleq -\frac{\partial}{\partial T}\log \widehat{P}^Z(t,T) = -\frac{\frac{\partial}{\partial T}\widehat{P}^Z(t,T)}{\widehat{P}^Z(t,T)},\ 0 \le t \le T \le \overline{T}, \qquad (6.32)$$

where we rely on Theorem 6.10 to choose a version which is jointly continuous in (t,T). Integration yields

$$\widehat{P}^Z(t,T) = e^{-\int_t^T g^Z(t,u)du},\ 0 \le t \le T \le \overline{T}. \qquad (6.33)$$

From (6.29) we have

$$g^Z(t,t) = r_t + \big(1 - Z(t,t)\big)\lambda_t = r_t + \big(1-\rho^Z(t,t)\big)\lambda_t. \qquad (6.34)$$

In particular, $g^Z(t,t) = r^Z(t,T)$ (see (6.15)) if and only if $r^Z(t,T)$, or equivalently, $\rho^Z(t,T)$ is independent of T. This is the case under recovery of market value (see (6.21)). Under recovery of par, recovery of treasury and recovery of market value, we have $Z(t,t) = \delta$ and $g^Z(t,t) = r_t + (1-\delta)\lambda_t$.

6.4 Compatibility with Heath-Jarrow-Morton model

Heath et. al. [20] construct arbitrage-free zero-coupon bond prices from a model for forward rates. In this section, we begin with a model for forward rates for defaultable bonds and determine necessary and sufficient conditions for this to be consistent with the intensity-based reduced-form model of Definition 2.1.

More specifically, let $g(t,T)$ be a family of forward rates, indexed by $T \in [0,\overline{T}]$. For each T, $\{g(t,T); 0 \le t \le T\}$ is an $\{\mathcal{F}_t\}$-predictable process within the setting of Subsection 2.1, and as a function of its three variables (t,T,ω), g is measurable. We model the evolution of forward rates by the Heath-Jarrow-Morton (HJM) equation

$$g(t,T) = g(0,T) + \int_0^t \alpha(v,T)dv + \int_0^t \nu(v,T)\cdot dW_v,\ 0 \le t \le T, \tag{6.35}$$

and we impose the HJM conditions

(i) $g(0,T)$ is a nonrandom, measurable function of T,

(ii) $\alpha(t,T,\omega)$ is jointly measurable, the process $\{\alpha(t,T); 0 \le t \le T\}$ is $\{\mathcal{F}_t\}$-predictable for every T, and $\int_0^T |\alpha(t,T)|dt < \infty$ a. s. for every T;

(iii) $\nu(t,T,\omega)$, a d-dimensional vector, is jointly measurable, $\{\nu(t,T); 0 \le t \le T\}$ is $\{\mathcal{F}_t\}$-predictable, and $\int_0^T \|\nu(t,T)\|^2 dt < \infty$ a. s. for every T;

C.2 $\int_0^{\overline{T}} |g(0,T)|dt < \infty$, $\int_0^{\overline{T}}\int_0^u |\alpha(t,u)|dtdu < \infty$ a. s.

C.3 The following conditions hold almost surely:

$$\int_0^t \left[\int_v^t \nu_i(v,u)du\right]^2 dv < \infty,\ i=1,\dots,d,\ t\in[0,\overline{T}],$$

$$\int_0^t \left[\int_t^T \nu_i(v,u)dv\right]^2 du < \infty,\ i=1,\dots,d,\ t,T\in[0,\overline{T}],$$

$$\int_t^T \left[\int_0^t \nu_i(v,u)dW_i(v)\right] du \text{ is continuous in } t \text{ for } i=1,\dots,d,\ T\in[0,\overline{T}].$$

Following HJM, we define $\alpha^*(t,T) \triangleq \int_t^T \alpha(t,u)du$, $\nu^*(t,T) \triangleq \int_t^T \nu(t,u)du$. We assume that the nonnegative local martingale

$$Q_t \triangleq \exp\left\{\int_0^t \nu^*(v,T)\cdot dW_v - \frac{1}{2}\int_0^t \|\nu^*(v,T)\|^2 dv\right\} \tag{6.36}$$

is a martingale.

Heath, Jarrow and Morton show that if bond prices are given by the formula $\overline{P}(t,T) \triangleq e^{-\int_t^T g(t,u)du}$, then

$$\frac{d\overline{P}(t,T)}{\overline{P}(t,T)} = \left(g(t,t) - \alpha^*(t,T) + \frac{1}{2}\|\nu^*(t,T)\|^2\right)dt - \nu^*(t,T)\cdot dW_t. \tag{6.37}$$

We seek an intensity-based reduced-form model, including a family of recovery processes $Z(t,T)$, such that $\overline{P}(t,T)$ agrees with $\widehat{P}^Z(t,T)$ given by (6.1). In this case, $g(t,T)$ will agree with $g^Z(t,T)$ of (6.32).

Theorem 6.12 *Let r_t be an $\{\mathcal{F}_t\}$-predictable process and let $\rho^Z(t,T)$, defined for $0 \le t \le T \le \overline{T}$, be jointly measurable and such that $\rho^Z(t,T)$ and $\rho^Z(t,t)$ are $\{\mathcal{F}_t\}$-predictable processes for each T. Assume $0 \le r_t \le g(t,t)$ and $0 \le \rho^Z(t,T) < 1$ for $0 \le t \le T \le \overline{T}$ and with*

$$\lambda_t \triangleq \frac{g(t,t) - r_t}{1 - \rho^Z(t,t)}, \tag{6.38}$$

assume

$$\left(\rho^Z(t,T) - \rho^Z(t,t)\right)\lambda_t = \alpha^*(t,T) - \frac{1}{2}\|\nu^*(t,T)\|^2,\ 0 \le t \le T \le \overline{T}. \tag{6.39}$$

Assume further that

$$\mathbb{E}\left[\sup_{t\in]0,T]} e^{-\int_0^t r_u du}\rho^Z(t,T)\overline{P}(t,T)\right] < \infty\ \forall T \in]0,\overline{T}]. \tag{6.40}$$

Then the intensity-based reduced-form model with r, λ as above and $Z(t,T) \triangleq \rho^Z(t,T)\overline{P}(t,T)$ satisfies $\widehat{P}^Z(t,T) = \overline{P}(t,T)$ for $0 \le t \le T \le \overline{T}$.

Conditions (6.38) and (6.39) are necessary for the intensity-based reduced form model to be consistent with the HJM model. Indeed, in order for $\overline{P}(t,T)$ to agree with $\widehat{P}^Z(t,T)$, $g^Z(t,t)$ of (6.34) must agree with $g(t,t)$, which is (6.38). Furthermore, the drift terms in (6.16) and (6.37) must agree, and in the presence of (6.38), this reduces to (6.39).

If the case that $\rho(t,T)$ is independent of T, condition (6.39) reduces to the usual HJM condition $\alpha(t,T) = \nu(t,T)\int_t^T \nu(t,u)du$, $0 \le t \le T \le \overline{T}$. The necessity of this result was obtained by [14].

Proof of Theorem 6.12: From (6.37)–(6.39) and with $r^Z(t,T)$ defined by (6.15),

$$\frac{d\overline{P}(t,T)}{\overline{P}(t,T)} = r^Z(t,T)dt - \nu^*(t,T)\cdot dW_t.$$

It follows that $e^{-\int_0^t r^Z(u,T)du}\overline{P}(t,T) = \overline{P}(0,T)Q_t$, which is an $\{\mathcal{F}_t\}$-martingale. In particular,

$$e^{-\int_0^t r^Z(u,T)du}\overline{P}(t,T) = \mathbb{E}\left[e^{-\int_0^T r^Z(u,T)du}\middle|\mathcal{F}_t\right].$$

In light of (6.40), Assumption 2.5 is satisfied, and Theorem 6.6 shows that $P^Z(t,T) = \overline{P}(t,T)$. ◇

Remark 6.13 From Theorem 6.12 we see that there are two ways to build an intensity-based reduced-form model. The first is to construct an intensity process λ_t as in Definition 2.1, and adopt a recovery convention. Calibration is then used to determine λ_t. The second method is to build an HJM model for default-free forward rates, build a second HJM model (6.35) for defaultable forward rates, and adopt a recovery convention. These two HJM models are then calibrated to market data. The models must be constructed so that $g(t,t)-r_t \geq 0$, i.e., the spread for credit risk must be nonnegative, and so that $\rho^Z(t,t) < 1$, i.e., recovery is less than 100%. Then the intensity of default λ_t is no longer exogenous; it is given by (6.38). The defaultable bond volatility $\sigma^Z(t,T)$ in (6.16) is $-\nu^*(t,T)$ of (6.37). Finally, condition (6.39) must be satisfied.

Condition (6.39) is substantial. Since it holds with both sides equal to zero when $T = t$, it is equivalent to the equation obtained by differentiating both sides with respect to T:

$$\alpha(t,T) - \nu(t,T) \cdot \int_t^T \nu(t,u)\,du = \frac{\partial}{\partial T}\rho^Z(t,T). \tag{6.41}$$

Under the recovery of market value convention, $\rho(t,T) = \delta$ and (6.41) gives the risk-neutral defaultable forward rate drift formula $\alpha(t,T) = \nu(t,T)\cdot\int_0^t \nu(t,u)du$, the same formula obtained by Heath, Jarrow and Morton for default-free forward rates. In this case, one builds the defaultable HJM model by choosing only $\nu(t,T)$ and the recovery fraction δ.

Under the recovery of treasury value convention used in Chapter 13 of [3], (6.41) becomes

$$\begin{aligned} &\alpha(t,T) - \nu(t,T) \cdot \int_t^T \nu(t,u)\,du \\ &= \delta \frac{\partial}{\partial T}\Big(\frac{P(t,T)}{\overline{P}^{RTV}(t,T)}\Big) \\ &= \delta\big(g(t,T) - f(t,T)\big)e^{\int_t^T (g(t,u)-f(t,u))du},\ 0 \leq t \leq T \leq \overline{T}, \end{aligned} \tag{6.42}$$

where $f(t,T)$ is the default-free forward rate given by an equation of the form

$$f(t,T) = f(0,T) + \int_0^t \beta(v,T)\,dv + \int_0^t \gamma(v,T)\cdot dW_v,\ 0 \leq t \leq T. \tag{6.43}$$

In order to avoid arbitrage among the default-free bonds, we must have

$$\beta(t,T) = \gamma(t,T)\cdot\int_0^T \gamma(t,u)du,\ 0 \leq t \leq T \leq \overline{T}. \tag{6.44}$$

It is not apparent how to choose $\alpha(t,T)$, $\nu(t,T)$ and $\gamma(t,T)$ to satisfy (6.42), (6.43) and (6.44). The situation with recovery of par value is likewise unpleasant.

Finally, regardless of the recovery convention, we see that the only way to force $\lambda_t = g(t,t) - r_t$ (Assumption HJM.6 in Section 13.1 of [3]) is to choose $\rho^Z(t,t) = 0$. In other words, recovery at the maturity date must be zero. Absent this situation, the intensity of default λ_t must strictly exceed the spread $g(t,t) - r_t$. ◊

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A Right-continuity of filtrations

In this appendix we show that the filtrations

$$\mathcal{H}_t \triangleq \mathcal{F}_t \vee \mathcal{F}_t^H \quad \text{and} \quad \mathcal{G}_t \triangleq \mathcal{F}_T \vee \mathcal{F}_t^H, \ 0 \le t \le T,$$

of (2.7), (2.8) are right-continuous.

Lemma A.1 *Let $0 \le s < t \le T$ be given. If $A \in \mathcal{F}_t^W \vee \mathcal{F}_t^H$, then there is a set $B \in \mathcal{F}_t^W \vee \mathcal{F}_s^H$ such that $A \Delta B \subset \{s < \tau \le t\}$.*

PROOF: Being in $\mathcal{F}_t^W \vee \mathcal{F}_t^H$, the set A must be of the form

$$\left\{\left(W_{s_1}, W_{s_2}, \ldots; H_{s_1}, H_{s_2}, \ldots; W_{t_1}, W_{t_2}, \ldots; H_{t_1}, H_{t_2}, \ldots\right) \in C\right\}$$

for some sequences $\{s_i\} \subset [0,s]$, $\{t_i\} \subset (s,t]$ and some Borel set $C \subset \left(\mathbb{R}^d\right)^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times \left(\mathbb{R}^d\right)^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. We define

$$B = \left\{\left(W_{s_1}, W_{s_2}, \ldots; H_{s_1}, H_{s_2}, \ldots; W_{t_1}, W_{t_2}, \ldots; H_s, H_s, \ldots\right) \in C\right\},$$

a set in $\mathcal{F}_t^W \vee \mathcal{F}_s^H$. Then $A \Delta B \subset \{s < \tau \le t\}$. ♢

A modification of the proof of Lemma A.1 results in the following.

Lemma A.2 *Let $0 \le s < t \le T$ be given. If $A \in \mathcal{F}_T^W \vee \mathcal{F}_t^H$, then there is a set $B \in \mathcal{F}_T^W \vee \mathcal{F}_s^H$ such that $A \Delta B \subset \{s < \tau \le t\}$.*

Lemma A.3 *Let $A \in \mathcal{H}_{s+}$ be given. Then $A \in \bigcap_{s<t\le T}(\mathcal{F}_t \vee \mathcal{H}_s)$.*

PROOF: Let $A \in \mathcal{H}_{s+}$ be given. Let $\{t_n\}$ be a sequence in $(s,T]$ converging down to s. For every n, we have $A \in \mathcal{H}_{t_n}$, and hence there is an equivalent set $\widetilde{A}_n \in \mathcal{F}_{t_n}^W \vee \mathcal{F}_{t_n}^H$. According to Lemma A.1, we may then choose $B_n \in \mathcal{F}_{t_n}^W \vee \mathcal{F}_s^H$ such that

$$\limsup_{n\to\infty} \mathbb{P}\{A \Delta B_n\} \le \lim_{n\to\infty} \mathbb{P}\{s < \tau \le t_n\} = \lim_{n\to\infty} \mathbb{E}\left[S_s - S_{t_n}\right] = 0.$$

We may then choose a subsequence $\{B_{n_k}\}_{k=1}^\infty$ satisfying $\sum_{k=1}^\infty \mathbb{P}\{A \Delta B_{n_k}\} < \infty$. We define three decreasing sequences of sets,

$$C_m \triangleq \bigcup_{k=m}^\infty (B_{n_k} \setminus A), \quad D_m \triangleq \bigcup_{k=m}^\infty B_{n_k}, \quad E_m \triangleq \bigcup_{k=m}^\infty (A \setminus B_{n_k}),$$

and their limits $C_\infty = \cap_{m=1}^\infty C_m$, $D_\infty = \cap_{m=1}^\infty D_m$, $E_\infty = \cap_{m=1}^\infty E_m$. It is easily verified that $A \cup C_\infty = D_\infty \cup E_\infty$, or equivalently,

$$A = (D_\infty \cup E_\infty) \setminus (C_\infty \setminus A).$$

Both $C_\infty \setminus A$ and E_∞ are null sets, i.e., in $\mathcal{F}_0$, and D_∞ is in $\cap_{s<t\le T}(\mathcal{F}_t \vee \mathcal{H}_s)$. ♢

Proposition A.4 *The filtration $\{\mathcal{G}_t\}_{0\le t\le T}$ is right-continuous.*

Proof: If $A \in \mathcal{G}_{s+}$, then we may follow the proof of Lemma A.3, choosing sets $\widetilde{A}_n \in \mathcal{F}_T^W \vee \mathcal{F}_{t_n}^H$ equivalent to A, where $t_n \downarrow s$, and then appeal to Lemma A.2 to choose sets $B_n \in \mathcal{F}_T^W \vee \mathcal{F}_s^H$ with $\lim_{n\to\infty} \mathbb{P}\{A\Delta B_n\} = 0$. We conclude from the argument in the second paragraph of the proof of Lemma A.3 that $A \in \mathcal{F}_0 \vee \mathcal{F}_T^W \vee \mathcal{F}_s^H = \mathcal{G}_s$. $\diamondsuit$

The proof of right-continuity of $\{\mathcal{H}_t\}_{0\le t\le T}$ is more difficult. To conclude that proof, we introduce the σ-algebras, defined for $0 \le s < t \le T$,

$$\mathcal{F}_{s,t}^W \triangleq \sigma\{W_u - W_s; s < u \le t\}, \tag{A.1}$$

$$\mathcal{K}_{s,t} \triangleq \mathcal{F}_t \vee \mathcal{H}_s = \mathcal{F}_0 \vee \mathcal{F}_{s,t}^W \vee \mathcal{H}_s, \tag{A.2}$$

$$\mathcal{K}_{s+} = \bigcap_{s<t\le T} \mathcal{K}_{s,t}. \tag{A.3}$$

Lemma A.5 *Let $A \in \mathcal{K}_{s+}$ be given. There exists an $\mathcal{H}_s$-measurable random variable X, and for each $t \in (s,T]$, there exist an $\mathcal{F}_{s,t}^W$-measurable random variable Y_t and a Borel function $f_t\colon \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{I}_A = f_t(X, Y_t)$ almost surely.*

Proof: Being in $\mathcal{K}_{s+}$, the set A must also be in $\mathcal{K}_{s,T}$. Therefore, there is an $\mathcal{H}_s$-measurable random variable X, an $\mathcal{F}_{s,T}^W$-measurable random variable Y_T and a Borel function f_T such that $\mathbb{I}_A = f(X, Y_T)$ almost surely. On the other hand, for each $t \in (s,T]$, Y_T is $\mathcal{F}_{s,t}^W \vee \mathcal{F}_{t,T}^W$-measurable and can thus be written as $Y_T = g_t(Y_t, Z_t)$ for some $\mathcal{F}_{s,t}^W$-measurable Y_t, $\mathcal{F}_{t,T}^W$-measurable Z_t and Borel function g_t. We have then $\mathbb{I}_A = f_T\big(X, g_t(Y_t, Z_t)\big)$ almost surely. Taking conditional expectations on both sides and using the $\mathcal{K}_{s,t}$-measurability of (X, Y_t) and the independence of Z_t from $\mathcal{K}_{s,t}$, we obtain

$$\mathbb{I}_A = \mathbb{E}\left[f_T\big(X, g_t(Y_t, Z_t)\big) \big| \mathcal{K}_{s,t}\right] = f_t(X, Y_t) \text{ almost surely,}$$

where $f_t(x,y) = \mathbb{E} f_T\big(x, g_t(y, Z_t)\big)$. $\diamondsuit$

Proposition A.6 *The filtration $\{\mathcal{H}_t\}_{0\le t\le T}$ is right-continuous.*

Proof: According to Lemma A.3, the σ-algebra $\mathcal{H}_{s+}$ is contained in $\mathcal{K}_{s+}$. Let $A \in \mathcal{K}_{s+}$ be given. It suffices to show that $A \in \mathcal{H}_s$.

Let $\{t_n\}$ be a sequence in $(s,T]$ converging down to s. According to Lemma A.5, there exists an $\mathcal{H}_s$-measurable random variable X and for each $n \in \mathbb{N}$ there exist an $\mathcal{F}_{s,t_n}^W$-measurable random variable Y_n and a Borel function f_n such that $\mathbb{I}_A = f_n(X, Y_n)$ almost surely.

For each $B \in \mathcal{F}_{s,T}^W$, we define $g_n^B(x) \triangleq \mathbb{E}\left[\mathbb{I}_B f_n(x, Y_n)\right]$. Let $\mathcal{C}$ be a countable collection of subsets of $\mathcal{F}_{s,T}^W$ which is closed under intersection and generates $\mathcal{F}_{s,T}^W$. Because X is $\mathcal{H}_s$-measurable and Y_n and $C \in \mathcal{C}$ are independent of $\mathcal{H}_s$, we have for all $n \in \mathbb{N}$ and $C \in \mathcal{C}$ that

$$g_n^C(X) = \mathbb{E}[\mathbb{I}_C f_n(X, Y_n)|\mathcal{H}_s] = \mathbb{E}[\mathbb{I}_C \mathbb{I}_A|\mathcal{H}_s] = \mathbb{E}[\mathbb{I}_C f_1(X, Y_1)|\mathcal{H}_s] = g_1^C(X)$$

almost surely. In other words, $g_n^C(x) = g_1^C(x)$ for μ_X-almost every x, where μ_X is the measure induced in $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ by X. In particular, there is a Borel subset D of $\mathbb{R}$ such that $\mu_X(D) = 1$ and

$$g_n^C(x) = g_1^C(x) \ \forall x \in D,\ n \in \mathbb{N},\ C \in \mathcal{C}.$$

The collection of sets $B \in \mathcal{F}_{s,T}^W$ which satisfy

$$g_n^B(x) = g_1^B(x) \ \forall x \in D,\ n \in \mathbb{N} \tag{A.4}$$

forms a Dynkin system, and by the Dynkin system theorem, equation (A.4) holds for every $B \in \mathcal{F}_{s,T}^W$. In other words, $\mathbb{E}\left[\mathbb{I}_B f_n(x, Y_n)\right] = \mathbb{E}\left[\mathbb{I}_B f_1(x, Y_1)\right]$ for all $B \in \mathcal{F}_{s,T}^W$, and it follows that

$$f_n(x, Y_n) = f_1(x, Y_1) \text{ almost surely } \forall x \in D,\ n \in \mathbb{N}. \tag{A.5}$$

For $x \in D$,

$$f(x) \triangleq \liminf_{n\to\infty} f_n(x, Y_n) = f_1(x, Y_1) \text{ almost surely} \tag{A.6}$$

is both $\mathcal{F}_{s+} = \mathcal{F}_s$ and $\mathcal{F}_{s,T}^W$-measurable, and since these two σ-algebras are independent, $f(x)$ is almost surely constant (independent of ω, but dependent on x).

The null set of ω for which (A.6) fails may depend on x. However, if we let μ_{Y_1} be the measure induced on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by Y_1, (A.6) may be restated as

$$f(x) = f_1(x, y) \text{ for } \mu_X \times \mu_{Y_1}\text{-almost every } x, y. \tag{A.7}$$

But X and Y_1 are independent, and from (A.7) we conclude $f(X) = f_1(X, Y_1) = \mathbb{I}_A$ almost surely. This implies $A \in \mathcal{H}_s$. $\diamondsuit$