Spaces to Play: Topo-games

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Logical games may provide a useful paradigm to analyze—and also enjoy—topology. We have been investigating Ehrenfeucht-Fraïssé style model comparison games for topological models (topological spaces equipped with a valuation function) of modal languages.

1. Topo-Games: the rules

Spoiler and Duplicator play over two topological models $\langle X, O, \nu \rangle$, $\langle X', O', \nu' \rangle$ starting from two given points $x \in X$, $x' \in X'$, which we call *current* points, for a given number of rounds n. We refer to such game as $TG(X, X', n, x, x')$. Intuitively, Spoiler is trying to prove that the two points are 'topologically' different, while Duplicator is doing the opposite. Spoiler starts by choosing a model containing the current point in that model. Duplicator replies by an open set in the other space also containing the current point. The round is not over yet, as Spoiler has now to pick a point within Duplicator's open. The new current point of that model. Duplicator replies by picking a corresponding point in Spoiler's open. The new current point of that model. The first round has thus ended. By these sequences of rounds, the two players are constructing sequences of related points. If these points always agree pairwise in all atomic propositions, Duplicator has won, otherwise Spoiler has. Winning strategies (*w.s.*) and infinite games are defined as usual.

2. The languages

The underlying language we use is the modal logic S4 with its original topological interpretation, [Tar38]. In this setting, every formula represents a region of a topological space and the box is interpreted as the interior operator:

$$M, x \models \Box\varphi \text{ iff } \exists o \in O : \; x \in o \wedge \forall y \in o : \; M, y \models \varphi$$

E.g., $\Diamond\Box p$ denotes the closure of the interior of the set p. Of interest is the second order truth definition for the modal operator. We find two nested quantifications: one over sets and one over points. The first one is reflected in the first half of a round of a TG game: choosing an open set. While the second one is for the second nested quantifier: choosing a point in the open set. As for Ehrenfeucht-Fraïssé games, a notion of adequacy is available.

Theorem 1 (Adequacy) Duplicator has a winning strategy in $TG(X, X', n, x, x')$ iff x and x' satisfy the same formulas of modal rank up to n.

From the proof of this theorem—[AvB99]—one extracts an effective method for building winning strategies for both players.

3. An example

In the figure on the next page, we have three example games: the two same spoons are played upon. We view the spoons as subsets p of different copies

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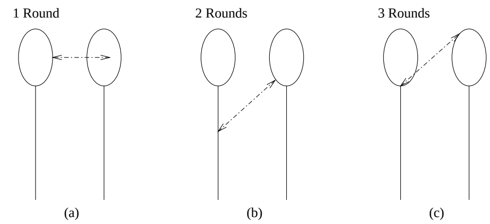

of the space $\mathbb{R}^2$. The starting points are different though. (a) The leftmost game starts by comparing a point on the frontier of the spoon with an interior point of the other spoon. Spoiler can win this game in one round, since $\Box p$ is true of the starting point of the right spoon, and its negation $\Diamond\neg p$ on the left. (b) In the central game, a point on the handle is compared with a point on the boundary of the container of the spoon. Again, Spoiler has a *w.s.* in the two round game: $\Diamond\Box\neg p$ holds on the starting point on the left spoon, but not of the starting point of the right one. (c) Finally, in the game on the left, the junction point between handle and container is related with a boundary point of the container. Spoiler has a *w.s*, since $\Diamond\Box p \wedge \Diamond(p \wedge \neg\Diamond\Box p)$ is true of the starting point on the left, but not on the right.

4. Game extensions

We defined, together with a family of languages of increasing expressive power, a family of TG games, [AvB99]. One extension is in terms of infinite games, for which we have that if Duplicator has a winning strategy in the infinite round game, then the two points are bisimilar. Another expressive power extensions goes towards globality (there are no starting points). Finally, we have defined several extensions towards geometry (e.g., in addition to opens we consider segments in the rounds of a game).

5. Fields of application

The definition of TG games is not only interesting from a merely game theoretic point of view. Its definition has brought new insights in logic, topology, and computer vision. Logic: more on interpretations of modal logics different from Kripke semantics. Topology: *w.s.* (and the related concept of bisimulation) have a strong connection with homeomorphism in topology and the correspondence can be refined to give a modal analysis of continuous mappings. Furthermore, bisimulation provides means to transfer information across spaces (e.g., connectedness is a bisimulation preserved property). Computer vision: an abstract take on languages to describe spatial patterns.

References

[AvB99] M. Aiello and J. van Benthem. Logical Patterns in Space. Technical report, ILLC, University of Amsterdam, 1999.

[Tar38] A. Tarski. Der Aussagenkalkül und die Topologie. *Fund. Math.*, 31:103–134, 1938.