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DIMENSIONAL REASONING WITH QUALITATIVE AND QUANTITATIVE DISTANCES

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April 1997

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Abstract

We outline a model of dimensional reasoning on time and space scales which integrates quantitative and qualitative knowledge about distances. At the core of this model lie constraints on interval boundaries, partial ordering and subsumption relations on interval relations and interval boundary constraints, as well as the transformation of interval relations to interval boundary constraints and vice versa.

Dimensional Reasoning with Qualitative and Quantitative Distances

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Abstract

We outline a model of dimensional reasoning on time and space scales which integrates quantitative and qualitative knowledge about distances. At the core of this model lie constraints on interval boundaries, partial ordering and subsumption relations on interval relations and interval boundary constraints, as well as the transformation of interval relations to interval boundary constraints and vice versa.

Keywords: Qualitative and Quantitative Distances, Distance Constraints, Subsumption, Reasoning on Scales

Introduction

When humans reason about temporal or spatial relations or even degree expressions in evaluative discourse (*"tall"*, *"fast"*, etc.), they are highly proficient at seamlessly integrating quantitative data and qualitative distances in these *dimensional reasoning* processes. Unfortunately, formal models for dimensional reasoning have so far been restricted to relations with either quantitative or qualitative distances, while any attempt at dealing with both types of knowledge in an integrated framework is lacking so far.[1] In this paper, we develop a formal framework which tries to fill this gap.

In order to illustrate the need for such an integrated account, consider example (1). If you planned to meet John and Heinrich at their arrivals, you would have to integrate quantitative (1a,b) and qualitative information (1c) in order to draw a conclusion such as (1d).

(1) a. Heinrich's flight from Frankfurt to New York-JFK takes eight hours.

 b. John's plane from Chicago to New York-La Guardia will start one hour after Heinrich's.

 c. John's flight will be rather short.

 (One plausible) Conclusion:

 d. John will reach New York before Heinrich.

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[1]As Cohn (1996), p. 138, remarks: "... qualitative and quantitative reasoning are complementary techniques and research is needed to ensure they can be integrated ..."

This paper describes how previous research on quantitative temporal distances by Badaloni & Berati (1996) and work on qualitative spatial distances by Hernández et al. (1995) can be restated in order to allow for a tight coupling between quantitative and qualitative knowledge.

Moreover, these proposals only support reasoning mechanisms on fairly low-level quantitative scales and rather "atomic" qualitative relations. What is lacking then are adequate means to adjust the level of inferences being carried out according to the needs of various levels of abstraction (this distinction between fine-grained and coarser types of knowledge is often discussed in terms of different *granularities* of knowledge (Hobbs, 1985)).

We, first, introduce the interval boundary constraint representation for Allen's well-known interval relation system (Allen, 1983) and then extend the approach to interval relations and boundary constraints with distances. A crucial point of our approach is the easy conversion between boundary constraints and higher-level interval relations. Reasoning by composition on distance constraints is then described in the section on different types of distances. Whereas, at first, we are able to abstract from the kinds of distances involved, these will become very important for the definition of the composition rules.

We only mention that as with temporal approaches like those of Badaloni & Berati (1996) or spatial approaches like those of Hernández et al. (1995) which both can be generalized to dimensional reasoning, our proposal can be applied to inferences with space, time or degree expressions as well. However, for the ease of presentation we will concentrate on examples with temporal information.

Representation by Interval Boundary Constraints

In order to represent and reason with qualitative knowledge (Allen, 1983), quantitative distances (Badaloni & Berati, 1996), and qualitative distances (Hernández et al., 1995), we aim to combine two sorts of requirements. On the one hand, we use relations on intervals that allow for flexibility and a high degree of abstraction in order to express the relevant level of temporal and spatial dimensions. On the other hand, it is often more efficient and — as it will become plausible in the course of this paper — much simpler to in-

tegrate knowledge about distances into constraints between the boundaries of intervals.

Constraints without Distances

Abstracting, for the time being, from distances, knowledge about interval relations can obviously be encoded in terms of constraints on interval boundaries. Figure 1, which was adapted from Freksa (1992), shows clearly how Allen's primitive interval relations and the corresponding interval boundary constraints are interrelated.

Figure 1: Changing between Representations

Allen's primitive relations between two events can be characterized by the conjunction of at most three constraints between their beginnings (X_b, Y_b) and endings (X_e, Y_e).

A very simple way to introduce a notation for interval relations in terms of constraints on their boundaries is by way of qualitative constraint arrays.

Definition 1 (Qualitative Constraints) *">",">"$\geq$"   and "$\top$" are qualitative constraints.*

The first two describe the common relations, the third one denotes a non-restricting constraint.

Definition 2 (Qualitative Constraint Array) *A qualitative constraint array for two intervals, X and Y, is an array $[c_1, \ldots, c_8]$ of eight qualitative constraints. These constraints describe all possible restrictions between the beginnings (X_b, Y_b) and endings (X_e, Y_e) of these intervals in the following ordering:[2] $X_b c_1 Y_b \wedge Y_b c_2 X_b \wedge X_b c_3 Y_e \wedge Y_e c_4 X_b \wedge X_e c_5 Y_b \wedge Y_b c_6 X_e \wedge X_e c_7 Y_e \wedge Y_e c_8 X_e.$*

Each of Allen's primitive relations can be defined by a single constraint array. Moreover, many non-primitive relations can be expressed that relate intervals by coarser knowledge, namely by disjunctions of neighboring primitive relations (like {m,o}; cf. Fig. 1). For instance, considering the example (1b), the relation between the time intervals that are needed for the flights (we here do not consider

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[2] We always assume that $X_e > X_b \wedge Y_e > Y_b$.

the information conveyed by *"1 hour"*) can be denoted by the disjunction $\{di, fi, o, m, <\}$ in Allen's notation and by the corresponding array set $\{[\top, >, \top, \top, \top, \top, \top, \top]\}$ with the single constraint $Y_b > X_b$ in ours (cf. also Fig. 1).

The largest benefit that can be attributed to our proposal derives from the additional representation and reasoning power we get from the smooth extension of qualitative constraint arrays, an issue we will elaborate on in depth. There are, fortunately, no penalties brought about with this extension, mainly due to three reasons. First, the data structures we use for constraint arrays, at first sight, may look more complicated than those provided for Allen's interval relations. But empty constraints, the source of seemingly increased complexity in our model, are only made explicit here for the sake of structural arguments, and, hence, need not be accounted for in an implementation tuned for efficiency. Second, a conjunction of constraints (i.e., a constraint array with more than one non-empty entry) sometimes needs only a single primitive relation in Allen's notation. As, however, coarser knowledge (as shown above with example (1b)) can often be processed more efficiently in our representation, this is rather a matter of trade-off than a shortcoming on either side. Third, not all interval relations can be represented by conjunctions of constraints. For instance, "interval X is disjoint with interval Y" must be represented by the array disjunction $\{[\top, \top, >, \top, \top, \top, \top, \top], [\top, \top, \top, \top, \top, >, \top, \top]\}$, which constitutes an *ambiguous* description. Again, since Allen's representation leads to a similar disjunction ($X\{<, >\}Y$) this is not an argument against either approach.

Finally, reasoning on interval boundary constraints is as easy as reasoning with Allen's relations. The axioms given in Table 1 state how an interval boundary is related to itself *(reflexivity)*, which constraints are contradictory *(contradiction)*, which constraints are weaker than others *(subsumption)*, and how constraints can be composed *(composition)*. For more elaborate complete reasoning with interval boundaries, cf., e.g., Vilain et al. (1989) and van Beek & Cohen (1990).

$\forall a, b, c \in B$, the set of interval boundaries:		
1.	$a \geq a$	(reflexivity)
2.	$a > a$	(contradiction)
3.	$(a > b \wedge a \geq b) \Leftrightarrow a > b$	(subsumption)
4.	$(a \geq b \wedge b \geq c) \Rightarrow a \geq c$	(composition 1)
5.	$(a \geq b \wedge b > c) \Rightarrow a > c$	(composition 2)
6.	$(a > b \wedge b \geq c) \Rightarrow a > c$	(composition 3)
7.	$(a > b \wedge b > c) \Rightarrow a > c$	(composition 4)

Table 1: Axioms for Qualitative Interval Boundary Constraints

Constraints with Distances

In order to accommodate to distances, too, we elaborate on the type of constraints that may hold between two interval boundaries. Instead of the common qualitative constraints (cf. Def. 1), we will here incorporate formal constructs for boundaries the semantics of which read as "*a boundary is at*

least/more than a distance x after another boundary" or "*a boundary is at most/less than a distance x before another boundary*". $\mathcal{D}^*$, the distance structure, is intentionally defined to incorporate only few restrictions such that it can easily be applied to different types of distances in the following section.

Definition 3 (Distance Structure) *The distance structure $\mathcal{D}^*$ is a triple $(D, >_{|D}, 0)$, which consists of a set of distances D, the elements of which are strictly partially ordered by $>_{|D}$, and a least element $0 \in D$.*

Given such a structure $\mathcal{D}^*$ we may define distance constraints as follows:

Definition 4 (Distance Constraints) *For all $x \in D$: "$\succ_x$", "$\succeq_x$", "$\succ_{-x}$", "$\succeq_{-x}$" and "$\top$" are distance constraints.*

The new constraints (cf. Fig. 2) are characterized as follows: $a \succeq_x b$ means that a is later[3] than b and the interval in between has at least the length x. $a \succ_x b$ is similar but requires for the temporal distance between a and b to be strictly larger than x. The occurrence of "$-$" in the index
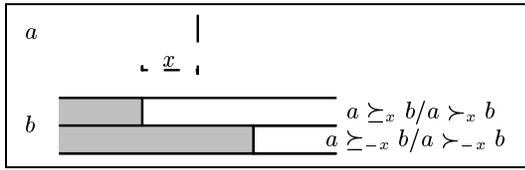
Figure 2: Distance Constraints
Assuming $x \in D$, the set of distances in the distance structure $\mathcal{D}^*$, the grey color indicates the regions to which b is restricted with respect to a by the constraints $\succeq_x$ and $\succ_x$ ($\succeq_{-x}$ and $\succ_{-x}$, respectively). In contrast to $\succ$, $\succeq$ allows b to lie on the borderline, too.

of such a constraint, e.g., $a \succeq_{-x} b$, indicates a slightly different semantics, namely that a is either before b with at most the distance x between a and b or a is after b. A corresponding proposition holds for $a \succ_{-x} b$. Since "$-$" results in weaker constraints and because we want to compare the strengths of constraints on the basis of their indices, we define "$-$" as an operator that is used to extend $\mathcal{D}^*$ to $\bar{\mathcal{D}}^*$:

Definition 5 (Operator $-$) $-$ *is a bijective function that maps $\bar{D} = D \cup \{-x | x \in D\}$ onto itself such that $-0 = 0$ and $x \in D \setminus \{0\} \Leftrightarrow -x \in \bar{D} \setminus D$.*

Definition 6 (Extended Distance Structure) *The extended distance structure $\bar{\mathcal{D}}^*$ is the quadruple $(\bar{D}, >, 0, -)$ derived from $\mathcal{D}^*$ (as defined in Def. 3) by extending D to $\bar{D} = D \cup \{-x | x \in D\}$ and $>_{|D}$ to $>$, using 0 and "$-$" as defined in Def. 5. Thereby, the strict partial ordering $>$ on $\bar{D}$ is defined as follows: $\forall x \in D, y \in \bar{D} \setminus D : x >_ y$, $\forall x, y \in D : x >_{|D} y \Leftrightarrow x > y$, and $\forall x, y \in \bar{D} : x > y \Leftrightarrow -y > -x$.*

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[3]Whenever we use a temporal expression, it is also valid for corresponding spatial and degree expressions and the reasoning on the expressions involved.

Most of the axioms from Table 1 can easily be adapted to the definition of the extended distance structure (cf. Table 2). Only the composition axioms depend crucially on the respective distance structure and their treatment is therefore deferred to the section on different distance types. The neutral element renders "$\succ_0$" and "$\succeq_0$" equivalent to the common relations "$>$" and "$\geq$", respectively. Also, the partial ordering $>$ on $\bar{D}$ very often allows to compare the strength of two constraints $a \succ_x b$ and $a \succ_y b$ and determine the one that subsumes the other. These considerations are reflected in the reflexivity, contradiction and subsumption axioms of Table 2.

	$\forall a, b \in B, \forall x, y \in \bar{D}$:	
1.	$x \leq 0 \Rightarrow (a \succeq_x a)$	(reflexivity 1)
2.	$x < 0 \Rightarrow (a \succ_x a)$	(reflexivity 2)
3.	$x > 0 \Rightarrow (a \succeq_x a \Rightarrow \bot)$	(contradiction 1)
4.	$x \geq 0 \Rightarrow (a \succ_x a \Rightarrow \bot)$	(contradiction 2)
5.	$(a \succ_x b \wedge a \succeq_x b) \Leftrightarrow a \succ_x b$	(subsumption 1)
6.	$(a \succeq_x b \wedge a \succeq_y b) \Leftrightarrow a \succeq_{\max(x,y)} b$	(subsumption 2)
7.	$(a \succ_x b \wedge a \succ_y b) \Leftrightarrow a \succ_{\max(x,y)} b$	(subsumption 3)

Table 2: Axioms for Interval Boundary Constraints with Distances

To facilitate the description, we also use several notational shortcuts that are equivalent to a conjunction of constraints $\succ$ and $\succeq$ (cf. Table 3).

	$\forall a, b \in B, \forall x \in \bar{D}$
A.	$a \prec_x b \Leftrightarrow b \succ_x a$
B.	$a \preceq_x b \Leftrightarrow b \succeq_x a$
C.	$a \hat{=}_x b \Leftrightarrow a \succeq_{-x} b \wedge b \succeq_{-x} a$
D.	$a \dot{=}_x b \Leftrightarrow a \succ_{-x} b \wedge b \succ_{-x} a$

Table 3: Logical Equivalences for Notational Shortcuts

Converting between Boundary Constraints and Interval Relations

As mentioned before, constraints on interval boundaries are on a rather low level of abstraction and, therefore, are often less convenient than interval relations. Though the conversion between qualitative boundary constraints and common qualitative interval relations — as illustrated in Fig. 1 — is almost trivial, the conversion between boundary constraints and interval relations with distances is, however, not that straightforward.

From Interval Relations to Boundary Constraints. While for qualitative interval relations commonly accepted standard relations exist (e.g., the ones given by Allen), this is not the case for interval relations with distances. We have the impression that the appropriateness of such interval relations is strongly influenced by the underlying domain and, thus, cannot fully be determined in such a canonical way. However, some exemplary interval relations with distances

$\forall X \in I$, the set of intervals, X_b being the lower and X_e being the upper boundary of X, and $\forall n \in D$:		
Relation	**Label**	**Constraints**
is at most n long[4]	$maxlo_n$	$X_b \succeq_{-n} X_e$
is at least n long	$minlo_n$	$X_e \succeq_n X_b$
is n long	$length_n$	$X_e \succeq_n X_b \land X_b \succeq_{-n} X_e$

Table 4: Binary Interval Relations Including Length Constraints

$\forall X, Y \in I$, the set of intervals, X_b and Y_b being the lower and X_e and Y_e being the upper boundaries of X and Y, respectively, and $\forall n \in D$:		
Relation	**Label**	**Constraints**
n older	$older_n$	$X_b \preceq_n Y_b \land Y_b \preceq_{-n} X_b$
survives at least with *n*	$svmin_n$	$X_e \succeq_n Y_e$
survives, but less than *n*	$svless_n$	$X_e \succ_0 Y_e \land X_e \prec_{-n} Y_e$
precedes with more than *n*	$minpr_n$	$X_e \prec_n Y_b$
precedes, but less than *n*	$maxpr_n$	$X_e \prec_0 Y_b \land X_e \succ_{-n} Y_b$
head to head with a tolerance of *n*	hh_n	$X_b \hat{=}_n Y_b$
contemporary for more than *n*	$minct_n$	$X_b \prec_n Y_e \land X_e \succ_n Y_b$

Table 5: Ternary Interval Relations with one Parameter for Distances

that should be of general use are given in Tables 4 and 5. The binary relations allow to state propositions about interval lengths (cf. Table 4), while the ternary relations describe distance constraints between the two intervals (cf. Table 5).

From Boundary Constraints to Interval Relations. Given the definitions in Tables 4 and 5, converting from interval relations to boundary constraints boils down to a simple table lookup. Converting, however, conjoined constraints into interval relations is a more difficult problem: The goal of the conversion is to find a conjunction of interval relations that subsumes all the given boundary constraints. Furthermore, these interval relations when remapped to boundary constraints should equal the originally given ones. In general, this is neither easily realized nor need it be useful at all. It is hard to realize, since the number of possible constraint combinations increases drastically by the additional distance parameters and, thus, requires plenty of new interval relations to account for additional combinations. It may even not be useful, since such a bounty of interval relations would possibly obscure the essential distinctions to be made.

Consider, e.g., a simple domain where the only interval relation which incorporates a distance and which is available is a binary relation that describes an interval length. This allows one to infer, in principle, that one interval X is a distance d before another interval Y. If, however, no interval relation incorporating such a distance constraint is specified, a corresponding qualitative distance-neutral relation, e.g., "*precedes*", would be an adequate means to approximate the boundary constraints. This need not be considered a disadvantage at all, since approximations that are

valid but not necessarily complete descriptions often constitute the right level of abstraction (e.g., "Schiller died before Goethe" seems to be a reasonable abstraction for "Schiller died at least 27 years before Goethe").

Therefore, given a set of interval relations (e.g., Tables 4 and 5) and a set of boundary constraints on two intervals, we do not require the "reconverted" interval relations to equal the given boundary constraints. Instead, we just set up the requirement that they should approximate them as narrowly as possible. In order to define "approximation", we here introduce the new concepts of a "distance constraint array" and of "subsumption of constraint arrays".[5]

Definition 7 (Distance Constraint Array) *A distance constraint array for two intervals, X and Y, is an array* $[c_1, \ldots, c_{12}]$ *of twelve distance constraints (cf. Def. 4). These constraints describe all possible restrictions between the beginnings* (X_b, Y_b) *and endings* (X_e, Y_e) *of these intervals in the following ordering:* $X_b c_1 Y_b \land Y_b c_2 X_b \land X_b c_3 Y_e \land Y_e c_4 X_b \land X_e c_5 Y_b \land Y_b c_6 X_e \land X_e c_7 Y_e \land Y_e c_8 X_e \land X_e c_9 X_b \land Y_e c_{10} Y_b \land X_b c_{11} X_e \land Y_b c_{12} Y_e$.

Distance constraint arrays allow to represent all combinations of distance constraints in a unique way. Hence, they allow to compare the restriction range of different constraint combinations by way of subsumption.

Definition 8 (Subsumption of Constraint Arrays) *A constraint array* $[c_{1,1}, \ldots, c_{1,12}]$ *subsumes another constraint array* $[c_{2,1}, \ldots, c_{2,12}]$, *iff* $\forall i \in [1, 12] : c_{1,i}$ *subsumes* $c_{2,i}$ *(cf. Table 2 for subsumption axioms).*

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[4] $X_b \prec_0 X_e$ need not be added, because it is entailed by the general knowledge that the beginning of an interval is before its ending. This general knowledge must be made available at all times for all events.

[5] We here handle boundary constraints on two intervals with at most one parameter and the constraints $\succ_0$ and $\succeq_0$. Interval relations with more than one parameter (and/or more constant constraints) can be defined at the cost of a slightly more complicated definition of approximation.
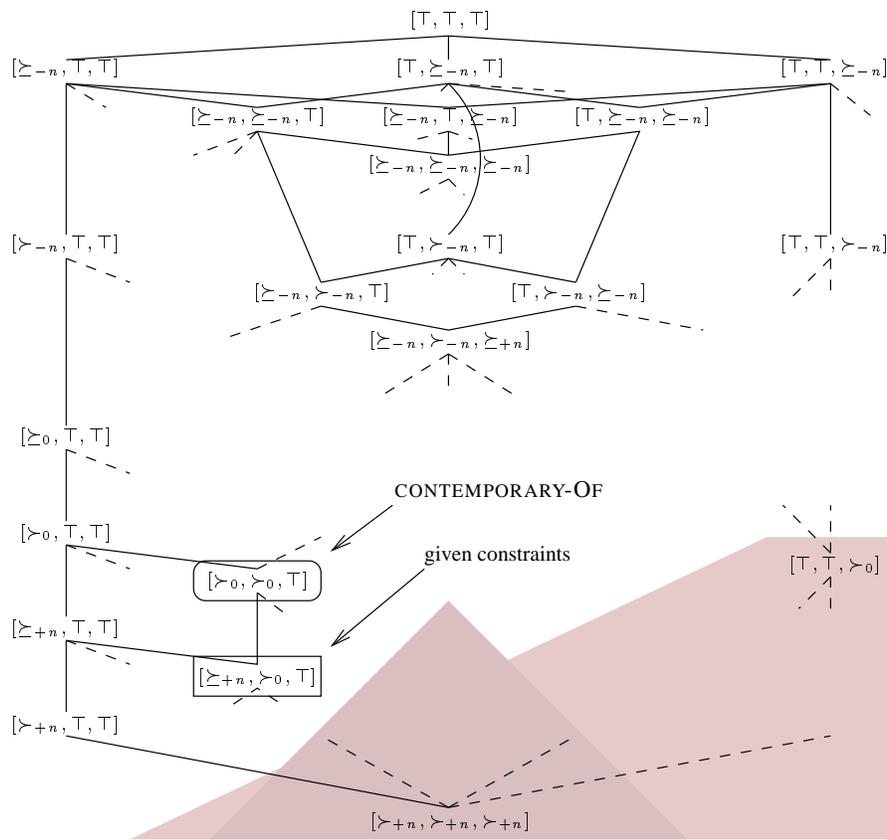
Assume the following order for the constraint array with at most one parameter n: $[X_e?Y_b, Y_e?X_b, X_b?X_e]$. The graph depicts three important aspects of this constraint array lattice:

1. The (transitive) subsumption relation between different arrays (esp. in the upper part of the graph).

2. Arrays that are subsumed by contradictory constraints ($X_b \succ_0 X_e$), and which are, as a consequence, contradictory, too. They are indicated by grey regions.
   Actually, for an array with twelve elements one special region would be at the top part of the lattice, too, since there are two constraints that are always true, namely $X_e \succ_0 X_b$ and $Y_e \succ_0 X_b$. Thus, all real constraint combinations are subsumed by the array that has exactly these two restrictions.

3. Due to the large number of possible constraint combinations an interval relation cannot be assumed to exist for all non-contradictory constraint combinations. Thus, a given constraint array must be abstracted by subsuming, best-approximating interval relations. For instance, in this example $[\succeq_{+n}, \succ_0, \top]$ is abstracted by the interval relation CONTEMPORARY-OF ($[\succ_0, \succ_0, \top]$).

Figure 3: The Lattice Structure of Constraint Combinations — a Simplified Version with 3 Constraints.

Subsumption of constraint arrays defines a partial ordering of a (semi-)lattice with $[\top, \ldots, \top]$ as its largest element[6].

Definition 9 (Best Approximation for One Parameter)
Let a set of boundary constraints on two intervals with at most one parameter n *be represented by a constraint array* C_1. *An interval relation represented by a constraint array* C_2 *with its possibly single parameter* n *fixed is called a*

best approximation for C_1, *iff* C_2 *subsumes* C_1 *and there is no other interval relation with one parameter (represented by a constraint array* C_3) *such that* C_2 *subsumes* C_3 *and* C_3 *subsumes* C_1 *and* $C_2 \neq C_3$ *(cf. Fig. 3).*[7]

This definition may yield several best-approximating interval relations, since it is based on a partial ordering. A unique best approximation is given by the conjunction of these interval relations. If the reconverted interval relations <u>must</u> equal the boundary constraints, one may provide an

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[6]If we postulate an infinite distance "∞" then $[\succeq_\infty, \ldots, \succeq_\infty] = [\bot, \ldots, \bot]$ can be considered the complement to $[\succ_{-\infty}, \ldots, \succ_{-\infty}] = [\top, \ldots, \top]$ and, thus, the least element of the lattice.

[7]When the constraints in C_1 come with parameters n_i then these parameters are appropriate candidates for *n*.

interval relation[8] for each of the 48 constraint combinations that cannot be described by a conjunction of constraint combinations, but which themselves can be used to describe any constraint combination ($1 : [\succeq_{-n}, \top, \ldots], 2 : [\succ_{-n}, \top, \ldots], 3 : [\succeq_{+n}, \top, \ldots], 4 : [\succ_{+n}, \top, \ldots], 5 : [\top, \succeq_{-n}, \top, \ldots], \ldots, 48 : [\top, \ldots, \top, \succ_{+n}]$).

Reasoning on Intervals with Distances

A sample inference on interval relations with distances proceeds as follows:

1. Use a lookup table to convert interval relations into disjunctions of constraint arrays (e.g., X {"precedes at least n","overlaps"} Y $\rightarrow$ $X\{C_1, C_2\}Y$; cf. Tables 4 and 5).

2. The composition axioms (cf. Table 1 for qualitative constraints and the following section for distance constraints) are applied to pairs of constraint arrays until no more inferences can be drawn (e.g. $X\{C_1, C_2\}Y \land Y\{C_3, C_4\}Z \Rightarrow X\{C_{1,3}, C_{1,4}, C_{2,3}, C_{2,4}\}Z$).

3. Subsumption tests (cf. Def. 8) eliminate redundant disjunctions (e.g., $X\{C_{1,3}, C_{1,4}, C_{2,3}, C_{2,4}\}Z \Leftrightarrow X\{C_{1,3}, C_{1,4}\}Z$, where $C_{1,3}$ and $C_{1,4}$ subsume $C_{2,3}$ and $C_{2,4}$, respectively).

4. Compute the best approximation (cf. Def. 9) from the boundary constraints.[9] Determine the best approximation for each array of the array set (e.g., $X\{C_{1,3}, C_{1,4}\}Z \rightarrow X$ {"contemporary-of", "overlaps with at least n"} Z; for an analogy consider Fig. 3, where given constraints are best approximated by "contemporary-of").

Thus, the subsumption criterion does not only yield the best approximating interval relation, but it also reduces unnecessary ambiguities.[10] Furthermore, under "natural" input conditions the mechanism is quite efficient.[11]

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[8]With one exception, all of Freksa's (1992) semi-interval relations are of this type, though, of course, without a mechanism to represent quantities.

[9]If there are interval relations that are defined by disjunctions of constraint arrays, one can adjust definitions to allow disjunctions to be elements of the lattice, too.

[10]A more elaborate redundancy avoidance mechanism might be based on the least common subsumer lcs of two constraint arrays, C_1 and C_2. If all constraint arrays subsumed by lcs do either subsume C_1 or C_2 or are subsumed by C_1 or C_2, then lcs is equivalent to the disjunction of C_1 and C_2. Still better, we can give an algorithm which determines the equivalence between subsets of a (disjunctive) distance constraint array set and single arrays. This allows to compute a minimal representation for n given constraint arrays in time $\mathcal{O}(n^{10})$. However, this procedure cannot be presented here due to its complexity.

[11]The general computational complexity of the inference process can be estimated as follows: The lookup in the table — which is fixed for a certain application — needs constant time. The computation complexity of the composition on boundary constraints (step 2) depends on the respective axioms. But given a constant time for each composition operation the constraint propagation (for a single alternative; e.g., combining $\{\{\ldots\}\}$ with $\{\{\ldots\}\{\ldots\}\}$ yields two alternatives) is straightforward, since it

Concluding this section, it should be noted that the representation by boundary constraints does not require the existence of abstract time points. Though it may be convenient to actually employ time points at one granularity level, the boundaries may themselves be considered events at a finer level of granularity. Then, a natural interpretation for $\succeq$ could be the minimal and maximal distance between the "centers" (e.g., midpoints) of the respective boundary intervals.

Treatment of Different Types of Distances

So far, we have said nothing about the types of distances that are allowed in our approach. An almost trivial one is the restriction of the distance structure to the single distance, 0, such that $D = \{0\}$. This exactly allows to represent Allen's relations, since "$\succ_0$" and "$\succeq_0$" are equivalent to the common relations "$>$" and "$\geq$", respectively. Much more interesting are the restrictions to quantitative or qualitative lengths or their combination. These considerations and the strategies to handle them are discussed subsequently.

Quantitative Distances

Quantitative distances are treated equivalently to non-negative real numbers. All the axioms and equivalence rules from Tables 2 and 3 do still apply. Furthermore, composition axioms for quantitative distances can be postulated (cf. Table 6). These are very simple, and even though distances marked with "$-$" embody a slightly different meaning, their composition with unmarked distances simply boils down to addition with negative reals. For the number zero the composition is simply equivalent to the transitivity property of "$>$" and "$\geq$".

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can be reduced to an interval label propagation problem with linear inequalities (Davis, 1987) with $m(m \perp 1)(m \perp 2)$ constraints (one constraint for each triple $a \succ_x b \land b \succ_y c \Rightarrow a \,\text{Compose}(\succ_x, \succ_y)\, c$) on $m(m \perp 1)$ nodes (one for each possible distance), where m is the number of interval boundaries. However, exponentially many alternatives may be concluded. Consider the distance series $d_{1,2} = \{1 \lor 3\}, d_{2,3} = \{5 \lor 10\}, d_{3,4} = \{20 \lor 40\}, \ldots$ Here, exponentially many alternatives must be concluded for $d_{1,i}$, since the later distances are always larger than the sum of all previous distances and no redundant distances emerge. It appears to us that this results from the non-convexity (cf. Vilain et al. (1989)) of the input conditions. Under more natural conditions we expect a much lower magnitude of runtime complexity.

A subsumption test between two given arrays requires constant time, since it involves twelve comparisons, at most. Step 3 is quadratic in the number of alternatives, at worst (compare each alternative with each other and do not find any comparability). The ordering of the interval relations can be precomputed. Thus, determining where to place a certain constraint combination needs only time linear to the number of interval relations.

Computations are valid, but not complete. This is analogous to results concerning 3-consistency in Allen's calculus (van Beek & Cohen, 1990).

	$\forall a, b, c \in B, \forall x, y \in \bar{D}$	
8.	$(a \succeq_x b \land b \succeq_y c) \Rightarrow a \succeq_{x+y} c$	(composition 1)
9.	$(a \succeq_x b \land b \succ_y c) \Rightarrow a \succ_{x+y} c$	(composition 2)
10.	$(a \succ_x b \land b \succeq_y c) \Rightarrow a \succ_{x+y} c$	(composition 3)
11.	$(a \succ_x b \land b \succ_y c) \Rightarrow a \succ_{x+y} c$	(composition 4)

Table 6: Composition Axioms for Quantitative Distances

Qualitative Distances

Let us now consider qualitative distances (cf. example (1c)) and their treatment within our calculus and representation schema. Basically, we adapt the composition rules described by Hernández et al. (1995) and Clementini et al. (1995) to our needs, while we keep the representation by distance constraint arrays and the axioms and equivalences from Tables 2 and 3, the new interval relations described in Tables 4 and 5, and the conversion mechanism described in the previous section.

Clementini et al. (1995) assume a totally ordered set of qualitative distances, *viz.* $\{\Delta_i | i \in [1, n] \land \Delta_i > 0\}$ (with $\Delta_n = \infty$). Depending on the availability of further restrictions, different composition rules are given. What is especially remarkable in this context is that Clementini et al. always compute *upper* and *lower bounds*. Nevertheless, they do not directly represent these bounds, instead they choose a disjunction of distance regions as representation. For some small n (i.e., few different distance regions), their representation schema is as good as ours. For larger numbers, however, the computational costs become unnecessarily large in their approach.

For illustrative purposes, we give some of the composition rules for qualitative distances described by Clementini et al. in our notation[12] in Table 7. Also, the more complicated ones for heterogenous structures or the absorption rule they define can be directly translated into our approach, if the proper conditions are fulfilled. Composition for a "positive" with a "negative distance" can be derived from the rules given by Clementini et al. for opposite directions.

Unlike the proposal made by Clementini et al., our approach allows a partial ordering for qualitative distances. This reflects a requirement that can be traced to the use of qualitative distances in natural language expressions. These expressions do often not constitute a total ordering, but only a partial one. Consider, e.g., the expressions *"somewhat later"*, *"a little later"* and *"much later"*. The precedence between *"somewhat"* and *"little"* is not clearly drawn, while both expressions are certainly ordered with respect to *"much"*.

With the composition rules described so far, the derivation of conclusions relating to two distances which are not

ordered with respect to each other is not supported. However, a simple scheme which is easily illustrated by the following small example allows to do exactly this. Assume four distances $\Delta_1, \Delta_{2a}, \Delta_{2b}, \Delta_3$, with $\Delta_1 < \Delta_{2a} < \Delta_3$ and $\Delta_1 < \Delta_{2b} < \Delta_3$, and the knowledge $a \succ_{-\Delta_{2a}} b \land b \succ_{-\Delta_{2b}} c$. The subsumption axioms from Table 2 require — and therefore allow to infer — that $a \succ_{-\Delta_3} b \land b \succ_{-\Delta_{2b}} c$ and $a \succ_{-\Delta_{2a}} b \land b \succ_{-\Delta_3} c$ to which the total ordering composition rules can be applied. Depending on the circumstances, the result may be a conjunction of two non-comparable constraints on an interval boundary pair, which requires a revision of the definitions 7 and 8, respectively.

Definition 10 (Multiple Distance Constraints Array) *A multiple distance constraints array for two intervals, X and Y, is an array $[s_1, \ldots, s_{12}]$ of twelve distance constraint sets s_i, where all $c_j \in s_i$ are distance constraints such that for all $i \in [1, 12]$ no constraint $c_j \in s_i$ subsumes another one $c_k \in s_i, k \neq j$ (i.e., s_i is a minimal representation).*

Definition 11 (Subsumption of Multiple Distance Constraints Arrays). *A multiple distance constraints array $C_1 = [s_{1,1}, \ldots, s_{1,12}]$ subsumes another one $C_2 = [s_{2,1}, \ldots, s_{2,12}]$, iff $\forall i \in [1, 12] : (\forall c_k \in s_{2,i} : \exists c_j \in s_{1,i} : c_j$ subsumes c_k).*

Consider an array like $[\{\succ_{-\Delta_{2a}}, \succ_{-\Delta_{2b}}\}, \{\top\}, \ldots]$, which describes a conjunction of constraints $X_b \succ_{-\Delta_{2a}} Y_b \land X_b \succ_{-\Delta_{2b}} Y_b$. Thus, composition rules must be applied to both constraints and the usual subsumption rules must enforce the minimality of entries in the arrays. However, the main ideas of computing compositions for quantitative and qualitative distances (cf. Tables 6 and 7) and computing a best approximation (cf. Def. 9) remain unchanged.

Combining Quantitative and Qualitative Distances

The major advantage of our reformulation is that it allows inferencing with either Allen's relations, or interval relations with quantitative distances, or interval relations with qualitative distances in an integrated framework.

While the latter two modes both subsume Allen's calculus, they really are complementary. Moreover, they also interact. This interaction can be described on the basis of a partial ordering between quantitative and qualitative distances. For instance, world knowledge may specify that "rather short" describes a temporal length less than three hours.[13] Composition rules for mixed quantitative/qualitative measures can then be handled analogously to partially ordered qualitative distances, namely by referring to common subsuming constraints. Of course, a fundamental aspect of mapping "rather short" onto "less than three hours" is the context in which the qualitative description is made. Staab & Hahn (1997) give an algorithm to

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[12]We assume $\delta_i = \Delta_i \perp \Delta_{i-1}$ and $\Delta_0 = 0$. The composition rules are applicable, if the respective triggering conditions on the boundaries ("lower" and "upper bound") and the respective structural restrictions on the distance system $\mathcal{D}^*$ ("monotonicity", "range restriction") are fulfilled. The names "monotonicity" and "range restriction" are taken from Clementini et al. (1995).

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[13]In general, this might also be a two-sided restriction like *"between one and three hours"*, but for the sake of simplicity we here avoid the second parameter that merely complicates approximation.

$\forall a, b, c \in B, \forall \Delta_i, \Delta_j \in D:$		
Application Conditions		**Composition Rule**
"Lower Bound" $\Delta_i, \Delta_j \geq 0$		$a \succ_{\Delta_i} b \wedge b \succ_{\Delta_j} c$ $\Rightarrow a \succ_{\Delta_{\max(i,j)}} c$
"Upper Bound" $\perp\Delta_i, \perp\Delta_j \leq 0$	"monotonicity": $\forall i \in [1, n] : \delta_{i+1} > \delta_i$	$a \succ_{-\Delta_i} b \wedge b \succ_{-\Delta_j} c$ $\Rightarrow a \succ_{-\Delta_{\min(i+j,n)}} c$
	"range restriction": $\forall i \in [1, n] : \delta_{i+1} > \Delta_i$	$a \succ_{-\Delta_i} b \wedge b \succ_{-\Delta_j} c$ $\Rightarrow a \succ_{-\Delta_{\min(\max(i,j)+1,n)}} c$

Table 7: Exemplary Composition Rules for Constraints with Qualitative Distances

deduce "comparison classes"[14] which is sensitive towards contextual criteria, and Hernández et al. (1995) sketch an articulation rule mechanism that is designed to find the correct mapping. However, the general problem still needs further research.

What is even more interesting is that this reformulation allows for the expression of new interval relations with qualities or quantities, ones that are cognitively plausible in that they involve a single act of perception. For instance, we can now introduce the relation *"roughly meets"* or the relation *"meets with a measuring tolerance of 100ms"*, which can be defined by the boundary constraints[15] $\{[X_e \stackrel{\simeq}{=}_{\Delta(\text{roughly})} Y_b, X_b \prec_0 Y_b]\} = X\{[\{\top\}, \{\succ_0\}, \{\top\}, \{\top\}, \{\succ_{-\Delta(\text{roughly})}\}, \{\succ_{-\Delta(\text{roughly})}\}, \{\top\}, \ldots]\}Y$ or $\{[X_e \stackrel{\simeq}{=}_{-\Delta(100\text{ms})} Y_b, X_b \prec_0 Y_b]\} = X\{[\{\top\}, \{\succ_0\}, \{\top\}, \{\top\}, \{\succ_{-\Delta(100\text{ms})}\}, \{\succ_{-\Delta(100\text{ms})}\}, \{\top\}, \ldots]\}Y$. Fig. 1 illustrates the possible scope of "roughly meets" by the region between the two dotted lines. Naturally, "roughly meets" subsumes "meets", but it also subsumes some parts of neighboring relations. Thus, its scope is a new kind of "conceptual neighborhood" that arises when only one parameter at a time is varied (cf. Freksa (1992)).

An Example of Dimensional Reasoning with Qualitative and Quantitative Distances

To illustrate the basic mechanisms we have introduced so far, let us return to example (1) in more technical detail. We assume H and J to denote the time intervals for Heinrich's and John's flight, respectively. Then, the sentences can be assigned the following representation structures:[16]

(2) a. $length_{\Delta(8h)}(H) = \{[H_e \succeq_{\Delta(8h)} H_b, H_b \succeq_{-\Delta(8h)} H_e]\} = H\{[\{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\succeq_{\Delta(8h)}\}, \{\succeq_{\Delta(8h)}\}, \{\succeq_{-\Delta(8h)}\}, \{\succeq_{-\Delta(8h)}\}]\}H$

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[14] The notion of "comparison class" in the natural language understanding community is roughly equivalent to "frame of reference" in the spatial reasoning community.

[15] We here assume that the function Δ maps expressions like "100ms" or "roughly" onto elements of the distance structure $\mathcal{D}^*$ using the proper frame of reference.

[16] We here formally capture binary interval relations $R_n(X)$ by ternary relations $\bar{R}_n(X, X)$ such that composition rules can be applied on the left and on the right side of other relations $R'_m(X, Y)$ and $R''_k(Z, X)$.

b. $older_{\Delta(1h)}(H, J) = \{[J_b \succeq_{\Delta(1h)} H_b, H_b \succeq_{-\Delta(1h)} J_b]\} = H\{[\{\succeq_{-\Delta(1h)}\}, \{\succeq_{\Delta(1h)}\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\succ_0\}, \{\succ_0\}, \{\top\}, \{\top\}]\}J$

c. $maxlo_{\Delta(\text{rather short})}(J) = \{[J_b \succeq_{-\Delta(\text{rather short})} J_e, J_e \succ_0 J_b]\} = J\{[\{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\top\}, \{\succ_0\}, \{\succ_0\}, \{\succeq_{-\Delta(\text{rather short})}\}, \{\succeq_{-\Delta(\text{rather short})}\}]\}J$

For the reasoning process, these constraints together with the assumption that *"rather short"* in the intended frame of reference (cf. previous subsection) means a distance which is not exactly specified but which is certainly less than three hours are taken for granted. We may then conclude the following additional constraints by propagation and application of composition rules: $H_b \succ_{-\Delta(4h)} J_e, J_e \succ_{\Delta(1h)} H_b, H_e \succeq_{\Delta(7h)} J_b, J_b \succeq_{-\Delta(7h)} H_e, H_e \succ_{\Delta(4h)} J_e, J_e \succ_{-\Delta(7h)} H_e$.

Combining the entire knowledge available from the initial data and the results of dimensional reasoning, we get: $H\{[\{\succeq_{-\Delta(1h)}\}, \{\succeq_{\Delta(1h)}\}, \{\succ_{-\Delta(4h)}\}, \{\succ_{\Delta(1h)}\}, \{\succeq_{\Delta(7h)}\}, \{\succeq_{-\Delta(7h)}\}, \{\succ_{\Delta(4h)}\}, \{\succ_{-\Delta(7h)}\}, \{\succeq_{\Delta(8h)}\}, \{\succ_0\}, \{\succeq_{-\Delta(8h)}\}, \{\succeq_{-\Delta(\text{rather short})}\}]\}J$. Since we employ only a trivial qualitative distance system with one distance $\Delta(\text{rather short})$, no composition rules can be applied to yield qualitative distance constraints (and, thus, non-singleton constraint sets), too.

Step 3 of the reasoning procedure need not be applied, since no ambiguities exist. Given exactly the interval relations from Tables 4 and 5 in step 4 the subsumption test would recognize that the relations "*maxlo*", "*minlo*", and "*length*" could be applied to H and "*maxlo*" to J, respectively. However, for H "*length*$_{\Delta(8h)}$" approximates better than either "*maxlo*" or "*minlo*" — this is just the information given in (2a). Furthermore "*older*", "*svmin*", "*svless*", "*hh*" and "*minct*" also subsume the constraints, but "*older*" is subsumed by "*hh*" and, therefore, the conjunction "*length*$_{\Delta(8h)}(H) \wedge maxlo_{\Delta(\text{rather short})}(J) \wedge older_{\Delta(1h)}(H, J) \wedge svmin_{\Delta(4h)}(H, J) \wedge svless_{\Delta(7h)}(H, J) \wedge minct_0(H, J)$" is returned as result. This is not a minimal result (an optimization could proceed by testing whether a conjunction is subsumed by a single interval relation), but it contains the essential information from a dimensional reasoning problem at a considerable level of abstraction, *viz.* that "*minct*$_0(H, J)$" can be inferred from "*older*$_{\Delta(1h)}(H, J) \wedge svmin_{\Delta(4h)}(H, J)$".

Related Work

Since this paper aims at the integration rather than the substitution of existing quantitative and qualitative approaches, it is similar to them with respect to several aspects. Still, our proposal achieves considerable extensions and an advanced level of conceptual abstraction in comparison to all other approaches.

The quantitative part of our approach closely relates to work done by Badaloni & Berati (1996) and Zimmermann (1995). Badaloni & Berati use a relational primitive which consists of upper and lower bounds for distances. However, they do not give a mechanism to abstract from these rather unwieldy primitives. Zimmermann uses the primitive relation $a(>, d)b \Leftrightarrow a = b + d$, which is quite similar to our constraints on interval boundaries. However, his formalism is not that convenient to formulate distance regions which are often necessary for coarse reasoning. Moreover, we elaborated extensively on the partial ordering he uses to allow for easy embedding of qualitative distances in the sense of Hernández et al. (1995). Though we did not touch on this issue, benefits of his mechanism like reasoning on proportions can be transferred to our mechanism, too. Earlier studies on duration reasoning, such as (Allen, 1983) or (Kautz & Ladkin, 1991), do not tightly integrate quantitative reasoning and reasoning on Allen's relations, but rather combine two different networks. Dechter et al. (1991) consider quantitative temporal constraint networks. These are extended to include Allen's relations in a single network by Mairi (1991). Nevertheless, this approach still does not allow for more abstract relations like *"overlap at least n"*.

As for qualitative distances, Hernández et al. (1995) and Clementini et al. (1995) present the most elaborate work (cf. Cohn (1996)) and thus serve as a blueprint for our qualitative distances reasoning part. We extend their mechanism to account for partially ordered distance systems, too.

Common interval label propagation networks that represent interval boundaries by *restrictions on time points* (cf. Davis (1987)) do not offer the same level of expressibility as our proposal, since simple transitive conclusions, e.g., a later b and b later c implies a later c, cannot be inferred. However, our approach can be reduced to such a network with *constraints on distances*. But then the implications are not structured in a way that is easily accessible from outside the reasoning system.

We build on Freksa's consideration who favors conjoined partial specifications (similar to the notion of "convex relation" by Vilain et al. (1989)) to allow for lucid reasoning about knowledge at a coarser level of specification. We do, however, not subscribe to his point of view when he introduces new labels for semi-interval relationships instead of simply using the constraints between the interval boundaries. His newly introduced labels are equivalent to single constraints on interval boundaries.[17]

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[17]One exception, *"is contemporary of"*, requires a conjunction of two constraints.

Conclusion

In this paper, we presented a mechanism for dimensional reasoning with qualitative and quantitative distances that integrates proposals by Allen (1983), Hernández et al. (1995) and Badaloni & Berati (1996). Given such an integration, our proposal considerably increases the reasoning power of the underlying calculus compared with either of these.

Our approach builds on *distance constraint arrays* as a new way of uniquely representing interval relations with different types of distances. Central to the reasoning scheme are the notions of *subsumption* and *approximation* which allow to infer interval relations that are much more lucid from a human designer point of view, since they achieve a high degree of conceptual abstraction from the underlying interval boundary constraints.

We were not able to consider three major points in detail in this paper. First, for knowledge like "A's flight is much shorter than B's" meta-reasoning on the different lengths of flights is required. This can easily be achieved by considering the distances, D, as being bounded by meta-intervals to which our proposal is applied, too. Second, though our approach requires exponential time at worst, it remains efficient when the input is restricted to input conditions that describe convex relations (cf. Vilain et al. (1989)). Third, we abstracted from how to actually assign numerical bounds to qualitative distances. This is a highly context-dependent problem for which first solutions exist (cf. the section on "Combining Quantitative and Qualitative Distances"), but which still needs further investigation.

To the best of our knowledge, no other proposal offers a similarly tight integration of qualitative and quantitative knowledge for dimensional reasoning or a similarly high level of interaction of constraints as the proposal we have outlined in this paper.

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