MODELLING DYNAMIC PORTFOLIO CREDIT RISK

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ABSTRACT. In this paper we present a model to price and hedge basket credit derivatives and collateralised loan obligation. Based upon the copula-approach by Schönbucher and Schubert (2001) the model allows a specification of the joint dynamics of credit spreads and default intensities, including a specification of the infection dynamics which cause credit spreads to widen at defaults of other obligors. Because of a high degree of analytical tractability, joint default and survival probabilities and also sensitivities can be given in closed-form which facilitates the development of hedging strategies based upon the model. The model uses a generalisation of the class of Archimedean copula functions which gives rise to more realistic credit spread dynamics than the Gaussian copula or the Student-t-copula which are usually chosen in practice. An example specification using Gamma-distributed factors is provided.

1. Introduction

While the arrival of a certain number of defaults over a given time period is to be expected during the normal course of business, major risks arise when either the *number* of defaults exceeds expectations or – even if the total number of defaults remains largely unaffected – when the *timing* of the defaults is such that several defaults occur closely after each other. In order to manage this risk a number of new financial instruments have been introduced (basket credit derivatives and collateralised debt obligations) which are explicitly designed to trade and manage the risks of default dependencies.

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In this paper we present a model to price and hedge these new instruments. Based upon the copula-approach the model allows a specification of the joint dynamics of credit spreads and default intensities, including a specification of the infection dynamics which cause credit spreads to widen at defaults of other obligors. Because of a high degree of analytical tractability, joint default and survival probabilities and also sensitivities can be given in closed-form which facilitates the development of hedging strategies based upon the model. The model is based upon a generalisation of the class of Archimedean copula functions which gives rise to much more realistic dynamics of the model variables than the Gaussian copula or the Student-t-copula which are usually chosen in practice.

Default correlation and (more generally) default dependency are a topic of high interest in the banking and investment community. This interest is further increased by other developments: First, the upcoming Basel II capital accord allows internally developed credit risk models to be used for regulatory capital allocation purposes. But also internally, the paradigm of the handling of credit risk in modern banking has changed significantly. While only a few years ago the only possibility to manage the credit risk of a large bank was by managing the origination process (i.e. the acception/rejection of new business), now credit risks can be managed directly by the use of credit derivatives and securitisation with loans and bonds as collateral assets: *collateralised loan obligations* (CLOs) *collateralised bond obligations* (CBOs) or more generally, *collateralised debt obligations* (CDOs). In short, credit risk management has evolved from a passive measurement and monitoring function into the *active* management of the credit risk exposure of a bank which uses the new possibilities to buy and sell exposures in order to optimize the risk-return profile of the credit book. Given the advantages of active credit portfolio management it is not surprising that the market for the instruments which make credit risks tradeable, the market for credit derivatives, is in full stride and still growing strongly. According to the latest survey by Risk magazine (Patel (2003)), the volume of the credit derivatives market has doubled again in 2002 reaching an outstanding notional of more than 2.3 trillion USD in February 2003.

The development towards active trading of credit risks has several consequences: With growing liquidity of single-name credit default swaps (CDS), a reliable marking-to-market of individual credit risks becomes possible. This means, that the *market risk* of a credit portfolio is now measurable, and should therefore be managed – it cannot be ignored any more. Insofar as credit spreads and CDS-spreads contain the market's opinion on the default risk of the obligor in question, they provide a new objective, market-based early-warning instrument for changes in the default risk of the obligors. In particular, it should be possible to calibrate the credit risk model to the prices of these instruments without much effort.
Secondly, a credit risk model that is to be used for trading must be much more accurate than a model that is just used to assess the overall risk of a portfolio or an institution: Prices must be found for both the bid and the offer side of the market, and these prices cannot be set too conservatively, or there will be no trading. On the other hand, prices that are too aggressive or any systematic deficiencies will be mercilessly picked off by the rest of the market.
Finally, to allow dynamic hedging and risk management, a quantitative model must be able to reflect not only the default risk, but also the market's *price dynamics* accurately. If – as is practice nowadays – single-name CDS are used for the hedging of portfolio credit derivatives, we need realistic price dynamics for these instruments and again require calibration, now to ensure that the model prices are arbitrage-free with respect to these hedge instruments.

Summing up, modern default risk models need not only to capture default dependency over a fixed time-horizon in a realistic manner, but also to capture the dynamics both of the timing of defaults as well as the dynamics of credit spreads and market prices (and thus actual and perceived default risk). Unfortunately, many quantitative models for portfolio credit risk have had difficulties to adapt to these new requirements. Standard models like Credit Metrics (Gupton et al. (1997)) or Credit Risk+ (Credit Suisse First Boston (1997)) are essentially static models which model only the default risk[1] of a defaultable portfolio over a fixed time horizon. Because of their fixed time-horizon these models are incapable of capturing the timing risk of defaults[2], and the lack of price dynamics makes them unsuitable for hedging purposes.

Li (1999) extended the Credit Metrics model to a Gauss copula model capturing the *timing risk* of defaults. The key contribution in this model is to shift the focus from modelling the dependency between default *events* up to a fixed time horizon (i.e. essentially discrete variables) to the dependency between *default times* which are continuous random variables and which do not depend on an arbitrarily chosen time-horizon. By keeping the dependency structure Gaussian, the fixed time-horizon default distribution of the Credit Metrics model is preserved and the copula-transform makes a calibration to a set of term structures of individual survival probabilities straightforward. These advantages made the Gaussian copula model (and its extension to a Student-t-copula model) one of standard models for the pricing of CDOs and basket credit derivatives today.

Nevertheless, the implicit price dynamics in the Gauss copula model remained unspecified in Li (1999), it was essentially a method to generate consistent default scenarios, but not scenarios for the development of spread curves. This gap was filled in Schönbucher and Schubert (2001) where the copula-approach was generalised to enable the use of general copula functions and a consistent specification of the dynamics of the individual default intensities (and thus credit spreads) was given. These dynamics involved *default contagion* in the sense that at default events, the credit spreads of the non-defaulted obligors would jump upwards.

The model proposed in this paper is in the tradition of the copula-approach as described in Schönbucher and Schubert (2001) but we propose not to use the Gaussian copula (as in Li (1999)) but a generalisation of the class of Archimedean copulae. We argue that the current standard choice in the industry, the Gaussian copula (and even more so the related Student-t-copula), imply an unrealistic term structure of default dependencies. If for example we measure the dependency between two defaults by the size of the default contagion that is active between the obligors at any given time, we can analyse this local dependency measure as a function of time. In the Gauss-copula model the dependency approaches infinity at $t = 0$ and decays strongly as time increases. This means that the model becomes strongly date-dependent, while usually there is no reason at all why $t = 0$ should be a date with special default dependency. Furthermore, it would mean that the model will give significantly different prices for the same credit derivative at different dates: Because of the concentration of dependency at $t = 0$, a five-year (spanning years 1 to 5) First-to-Default swap (FtD) priced at $t = 0$ would be much cheaper than a five-year FtD priced at $t = 1$ (now spanning years 2 to 6), even if the spreads of the underlying credits had not changed at all.

[1] Credit Metrics does include a first attempt to capture market risk by modelling rating transitions.

[2] Timing risk is very important, for example it is an essential risk in all cash-flow based debt securitisations.

It should be noted that this problem is *not* a weakness of the copula-approach in general, but only a weakness of the particular choice of the Gaussian copula or the t-copula as copula of the default times. These copulae simply do not seem to be a particularly well-suited model for dynamic default dependencies. If a different copula is chosen, these problems can be avoided. The class of copulae which we propose in this paper contains members which have a much more realistic dependency structure of defaults over time. Furthermore, the Gaussian copula and the Student-t-copula only exhibit a very limited degree of analytical tractability, while the copulae proposed in this paper can be evaluated in closed-form. In a related paper (Schönbucher (2002)) one can find closed-form loss distributions for a large homogeneous portfolio under Archimedean-copula default dependency. These simple formulae can be useful to assess the particular parametric specification of the dependency structures proposed in this paper.

Of course, the copula approach is not the only attempt to build a dynamic model of default dependency which can be easily calibrated and easily used in practice. In single-name default risk modelling, ease of calibration and a high degree of flexibility in a the specification of spread-dynamics are the hallmarks of the intensity-based approach, therefore we concentrate on a (very brief and incomplete) survey of extensions of this class of models.

The first, and most obvious way to introduce dependency between defaults in an intensity-based model is to introduce correlation between the default intensities of the obligors. Yet, if this done using only diffusion-based dynamics for the default intensities, the set of possible default correlations is strongly restricted[3]. Empirically, default correlations have been rather small so this may be a viable approach if default correlation is only to be captured broadly across the whole economy (as argued by Yu (2002)), but in cases of highly-dependent obligors with low individual default probabilities this approach may not be acceptable (e.g. to model default dependency *within* an industry sector or a specific region).

There are essentially two ways out of the low-correlation problem: Joint jumps in the default intensities, or joint defaults. The possibility of joint jumps in the default intensities allows a higher degree of dependency, in principle perfect correlation can be reached by letting both intensities jump to infinity at the same time. A good example of this approach are the affine jump-diffusion processes introduced by Duffie et al. (2001). Nevertheless, analytical tractability can be difficult in these models, in particular when it comes to calibration and to the analysis of the distribution of joint defaults that is implied by the model. As shown by Schönbucher and Schubert (2001), any copula-model can be written as an intensity-based model in which the default intensities of the non-defaulted obligors have a joint jump at default events, so there is no fundamental difference between copula-models and intensity-based models with a rich enough dynamic specification.

Default-event triggers which cause joint defaults of several obligors at the same time were used in Duffie (1998), Kijima (2000); Kijima and Muromachi (2000)): Again we do not have any restriction on the default dependency any more but the problem of the concrete specification of

[3]A back-of-the-envelope calculation would proceed as follows: Choose two obligors A and B with perfectly correlated default intensities $\lambda(t) = \lambda_A(t) = \lambda_B(t)$. Call $\Lambda(T) := \int_0^T \lambda(t)dt$, and assume for simplicity that $\Lambda(T)$ is normally $\Phi(m, s^2)$ distributed. Then the individual default probabilities are $p = e^{-m+s^2/2}$, and the default correlation between A and B is $\rho = \frac{p}{1-p}(e^{s^2} - 1)$. This is essentially of the same order of magnitude as the individual default probability p, unless the intensity volatility is extremely high.

the intensities of the joint default events and their intensities is still largely unresolved: For a small portfolio of just 10 obligors we already have 2^{10} possible joint default events. As it is not feasible to fully enumerate the joint default events, we essentially need another model in order to specify the intensities of these joint default events. Another problem is that the dynamics that are implied by this model are not quite realistic, either: Defaults do cluster, but they do not occur at exactly the same time. Furthermore, *after* a joint default event the dynamics of the non-defaulted obligors are unchanged, another feature which seems unrealistic. It should be noted that this modelling approach can be cast into the copula-framework using the so-called Marshall-Olkin copula.

An interesting modelling approach which yields rather realistic dynamics and default dependency structures is default infection or default contagion. Davis and Lo (2000, 2001), Jarrow and Yu (2001) and Giesecke and Weber (2002, 2003). The basic idea in these models is that the default intensity of the non-defaulted obligors is caused to jump upwards if another, related obligor defaults. This phenomenon is frequently observed in credit markets (see e.g. the emerging markets crises in the late 1990s or the explosion of US corporate spreads after the Enron and WorldCom defaults), yet if the jump at default is directly specified, the models become very hard to calibrate because of the cyclical dependence of between default intensities and default arrivals of all obligors: Essentially, every obligors default intensity depends on every other obligors' survival and thus also on the other obligors' default intensities, which in turn again depend on the first obligor's survival. Jarrow and Yu (2001) are therefore forced to model only one-way dependency, and Davis and Lo (2000, 2001) choose an extremely simplifed model. As a very similar type of default contagion arises endogenously in the copula-based models *without* incurring the same calibration problems, it seems that a an easier way to reach these dynamics is to use a copula model.

The rest of the paper is structured as follows:
In the next section we recapitulate the main results of Schönbucher and Schubert (2001) which sets the general framework for the analysis of the dynamic default risk model. This is followed by a section introducing the generalised Archimedean copula functions which we are going to use to model the dependency between the default events. We give the joint distribution function in closed-form and also provide an efficient algorithm to generate random vectors with this distribution function. In the next section we analyse the *conditional* distribution of the factors and the corresponding conditional copula, where the conditioning is done on default and survival of any given set of obligors in the future. The conditional distributions are reached by a simple change of measure from the original probability distribution and can be given in closed-form. In particular it turns out that the conditional distributions of the factor variables are of the same exponential family as the original factor variables. This distribution is central in deriving the dynamics of the default intensities in this model. It also allows to price any contingent claim in this framework. In the following section we characterise the dynamics of the default hazard rates using the conditional distributions of the previous sections. It turns out that the conditional jump sizes at defaults (the size of the contagion between two obligors) can be expressed as a covariance of the corresponding factors. As we can give the conditional contagion-jumps of the default intensities in closed-form, it may be desirable to fit a model to a given set of jump sizes. This question is also addressed in this section. Usually, this calibration is possible, and we give a possible choice of the factor loadings that will result in a given matrix of joint default influences. While all results of the previous sections are valid for any specification of the factors driving the default influences (as long as they are positive a.s. and their Laplace transform is easily

available), we give a concrete implementation example in the final section of the paper. This example is based upon Gamma-distributed driving factors which yield a generalisation of the Clayton copula as dependency structure.

2. Model Setup

This section gives a short overview over the Schönbucher and Schubert (2001) copula modelling framework. More details and proofs can be found in the original article.

2.1. **Preliminaries.** The model is set in a filtered probability space $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, P)$. All filtrations in this paper are assumed to satisfy the usual conditions and are augmented, the probability measure P need not necessarily be a martingale measure, but it is helpful to consider it as a martingale measure. For a stochastic processes like $\lambda(\omega, t)$ we only write $\lambda(t)$, suppressing the dependence on ω, and we assume that all stochastic processes are continuous from the right with left limits (càdlàg).

Vectors are written in **boldface** $\mathbf{x} = (x_1, \ldots, x_I)^T$. Vectors of functions $F_i : \mathbb{R} \to \mathbb{R}$ are written as

$$\boldsymbol{F}(\boldsymbol{x}) := (F_1(x_1), F_2(x_2) \ldots, F_I(x_I)))^T. \tag{2.1}$$

Standard arithmetical functions of vectors (except multiplication where we use matrix multiplication) and comparisons between vectors are meant by component, i.e. $\ln(\boldsymbol{u}) = (\ln(u_1), \ldots, \ln(u_I))^T$, and also $\boldsymbol{u}/\boldsymbol{v} = (u_1/v_1, \ldots, u_I/v_I)^T$. We use the following notation if we replace the i-th component of $\mathbf{x}$ with y:

$$(\mathbf{x}_{-i}, y) := (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_N)^T. \tag{2.2}$$

$\mathbf{1}$ is the vector $(1, \ldots, 1)^T$ and $\mathbf{0} = (0, \ldots, 0)^T$. Frequently, partial derivatives are written in index notation, i.e. $\frac{\partial}{\partial x_i} C() = C_{x_i}()$.

The connection between default intensities and credit spreads is by now well-understood, for example in the fractional recovery /multiple default model the default intensity times the local loss quota gives the short-term credit spread (see Duffie and Singleton (1997) or Schönbucher (1998) for more details). In Schönbucher (1999) it is also shown that the CDS spread of an obligor can be viewed as "expected loss in default times an average of default hazard rates" if the recovery-of-par model is used. We therefore restrict ourselves to modelling default intensities and leave the choice of the recovery model to the reader. All results for default intensities will directly carry over to corresponding results on credit spreads.

2.2. **Default Risk Modelling with Copula Functions.** Copula-based default risk models grew out of the need to extend univariate default-risk models for the individual obligors to a multivariate framework which keeps all salient features of the individual default-risk models while incorporating a realistic dependency structure between the defaults of the obligors. "Keeping the salient features" means in this context that the model reduces to the original one-obligor default risk model, *if only that obligor's default and survival behaviour is observed* (and some general background information).

In order to make precise what is meant we need to define the information sets that would obtain if only one obligor $i \leq I$ was observed. But first we define the *background process:*

Definition 1. *The* background process $X(t)$ *is a* m*-dimensional stochastic process. We denote the filtration generated by* $X(t)$ *with* $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$*, and* $\mathcal{G} := \sigma(\bigcup_{t\in[0,\overline{T}]} \mathcal{G}_t)$.

The background process is the process driving all non-default dynamics in the model, i.e. the default intensity dynamics, default-free interest-rate dynamics and any other state variables that may be relevant for pricing purposes.

Individual defaults in this model are triggered as follows:

Assumption 1 (Default Mechanism)**.**
We consider joint defaults and survivals of a set of I *individual obligors. We define:*

(i) *The* default trigger variables U_i, $i = 1, \ldots, I$ *are random variables taking values on the unit interval* $[0, 1]$.
(ii) *The* pseudo default-intensity $\lambda_i(t)$ *is a nonnegative càdlàg stochastic process which is adapted to the filtration* $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$ *of the background process.*
(iii) *The* default countdown process $\gamma_i(t)$ *is defined as the solution to* $d\gamma_i(t) = -\lambda_i(t)\gamma_i(t-)dt$ *with* $\gamma_i(0) = 1$. *The solution is*

$$\gamma_i(t) := \exp\{-\int_0^t \lambda_i(u)\, du\}. \tag{2.3}$$

We denote by τ_i *the* time of default *of obligor* $i = 1, \ldots, I$*, and denote the* default and survival indicator processes *as* $N_i(t) := \mathbf{1}_{\{\tau\leq t\}}$ *and* $I_i(t) := \mathbf{1}_{\{\tau_i>t\}}$.

The time of default is the first time, when the default countdown process $\gamma_i(t)$ *reaches the level of the trigger variable* U_i*:*

$$\tau_i := \inf\{t : \gamma_i(t) \leq U_i\}. \tag{2.4}$$

Note that the assumption of the existence of a default-intensity in equation (2.3) can be replaced by specifying $\gamma_i(t)$ as the survival probability function of obligor i, i.e. $\gamma_i(t) := \mathbf{P}[\, \tau_i > t \,]$. Thus, the copula-approach can also be used for models with default times that do not have an intensity.

Definition 2 (Filtrations)**.**
For all $i \leq I$ *we define the following filtrations:*

(i) *Filtration* $(\mathcal{F}^i_t)_{t\in[0,\overline{T}]}$ *contains only information on default and survival of obligor* i*. Thus, it is the augmented filtration that is generated by* $N_i(t)$.
(ii) *In addition to this, filtration* $(\mathcal{H}^i_t)_{t\in[0,\overline{T}]}$ *contains information about the background process up to time* t

$$\mathcal{H}^i_t := \sigma\left(\mathcal{F}^i_t \cup \mathcal{G}_t\right).$$

(iii) In addition to this, filtration $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ contains information about the defaults of all *obligors until t, (and still information about the background process until time t)*

$$\mathcal{H}_t = \sigma\left(\bigcup_{i=1}^{I} \mathcal{H}_t^i\right).$$

We can now make precise what was meant with "keeping the salient features" of an individual default risk model: Conditional on $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$, the model is supposed to reduce to the original one-dimensional default risk model for obligor i. This is achieved as follows:

Assumption 2.
For all $i = 1, \dots, I$, the default threshold U_i is uniformly distributed on $[0,1]$ under $\left(P, \mathcal{H}_0^i\right)$, and $\mathcal{U}_i$ is independent from $\mathcal{G}_\infty$ under P.

Then, by proposition 3.4 of Schönbucher and Schubert (2001), we have the univariate survival probabilities (given $\tau_i > t$) as

$$P_i'(t,T) = \mathbf{E}^P\left[\left.\frac{\gamma_i(T)}{\gamma_i(t)}\,\right|\mathcal{H}_t^i\right] = \mathbf{E}^P\left[\left.e^{-\int_t^T \lambda_i(s)ds}\,\right|\mathcal{H}_t^i\right], \tag{2.5}$$

Equation (2.5) is exactly the expression that gives the survival probabilities in an intensity-based default risk model with default intensity process $\lambda_i(t)$. Not surprisingly, the default intensity of obligor i under $\mathcal{H}_t^i$ is indeed $\mathbf{1}_{\{\tau_i>t\}}\lambda_i(t)$ (Schönbucher and Schubert (2001), prop. 3.5).

Note that in assumption 2, only the *marginal* distribution of the trigger levels U_i was specified, because only the distribution under $\mathcal{H}_t^i$ was given and $\mathcal{H}_t^i$ does not contain any information on the other $U_j, j \neq i$. This leaves us enough freedom to specify a rich structure of dependency between the defaults of the obligors using the *joint* distribution of the random variables $U_1, U_2, \dots, U_I$:

Assumption 3.
Under $(\mathcal{H}_0, P)$ the I-dimensional vector $\boldsymbol{U} = (U_1, \dots, U_I)^T$ is distributed according to the I-dimensional copula

$$C(\boldsymbol{u}).$$

$\boldsymbol{U}$ is independent from $\mathcal{G}_\infty$. Furthermore, C is I times continuously differentiable.

A particular focus of this paper (and of Schönbucher and Schubert (2001)) are the dynamics of the default and survival probabilities, and this involves in particular the distribution of the default times $\boldsymbol{\tau}$ conditional on the information that may be available at a later time $t > 0$.

Definition 3 (Conditioning Information)**.**

(i) The conditioning information is summarized in a pair

$$(\overline{\boldsymbol{u}}, \boldsymbol{d}) = ((\overline{u}_1, \dots, \overline{u}_I)^T, \{d_1, \dots, d_D\}), \tag{2.6}$$

of a vector $\overline{\boldsymbol{u}} \in [0,1]^I$ of observed countdown levels and a set $\boldsymbol{d} \subset \{1, \dots, I\}$ of defaulted obligors. The corresponding σ-algebra is

$$\sigma\left(\bigcup_{i\notin \boldsymbol{d}}\{U_i < \overline{u}_i\} \quad \cup \quad \bigcup_{j\in \boldsymbol{d}}\{U_j = \overline{u}_j\}\right). \tag{2.7}$$

(ii) *We call the measure that reflects this new information $P(\overline{\boldsymbol{u}}, \boldsymbol{d})$.*

(iii) *The distribution function of the $\boldsymbol{U}$, conditioned on this information, is denoted with*

$$C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d}) = \mathbf{P}\left[\, \boldsymbol{U} \le \boldsymbol{u} \mid (\overline{\boldsymbol{u}}, \boldsymbol{d}) \,\right].$$

We interpret the conditioning information as follows:

- Levels $\overline{\boldsymbol{u}}$.
  In most cases, the levels $\overline{u}_i$ can be identified with the *current* state of the countdowns at the current time time t, i.e. $\overline{u}_i = \gamma_i(t)$. Alternatively, one can also identify $\overline{u}_i = \gamma_i(T_i)$ with the levels of the countdowns at times T_i which may be different for each obligor. This is useful for the determination of default and survival likelihoods *conditional on survival of individual obligors up to a later date.* In the latter case one should bear in mind that $\gamma_i(T_i)$ will be stochastic if λ_i is stochastic. In any case, for defaulted obligors, $\overline{u}_i$ is the level of the countdown at the time of default, i.e. $\overline{u}_i = \gamma_i(\tau_i)$.
- Survival of the obligors.
  All obligors $i \le I$ have survived until just before $\overline{u}_i$, i.e.
  $$U_i \le \overline{u}_i.$$
- Defaults of obligors $\{d_1, \ldots, d_D\}$.
  For all $k = 1, \ldots, D$, obligor d_k defaults at countdown level $\overline{u}_{d_k}$:
  $$U_{d_k} = \overline{u}_{d_k}.$$
  Note that we assume that default takes place exactly at the trigger level $\overline{u}_{d_k}$. This makes sense, as $\overline{u}_{d_k}$ can be viewed as "the last time obligor d_k was seen alife". Relaxing this assumption is trivial but would mess up the notation even more.
- All other obligors are still alive at t, i.e. for all $i \notin \{d_1, \ldots, d_D\}$
  $$U_i < \overline{u}_i.$$

The connection to the filtrations $\mathcal{H}$ and $\mathcal{H}^i$ is the following:

$$\mathcal{H}_t \leftrightarrow \sigma\left((\overline{\boldsymbol{u}}, \boldsymbol{d}) \cup \mathcal{G}_t\right), \quad \text{where for all } i \begin{cases} \overline{u}_i = \gamma_i(t), & \text{if} \quad \tau_i > t \\ \overline{u}_i = \gamma_i(\tau_i), & \text{if} \quad \tau_i \le t \\ i \in \boldsymbol{d} & \text{if} \quad \tau_i \le t. \end{cases} \tag{2.8}$$

$$\mathcal{H}^i_t \leftrightarrow \sigma\left((\overline{\boldsymbol{u}}, \boldsymbol{d}) \cup \mathcal{G}_t\right), \quad \text{with} \begin{cases} \overline{u}_j = 1, & \text{for} \quad j \ne i \\ \overline{u}_i = \gamma_i(t), & \text{if} \quad \tau_i > t \\ \overline{u}_i = \gamma_i(\tau_i), & \text{if} \quad \tau_i \le t \\ \boldsymbol{d} = \{i\} & \text{if} \quad \tau_i \le t, \\ \boldsymbol{d} = \emptyset & \text{if} \quad \tau_i > t. \end{cases} \tag{2.9}$$

When we write expressions like "let $(\overline{\boldsymbol{u}}, \boldsymbol{d})$ represent the information until t" we mean that $(\overline{\boldsymbol{u}}, \boldsymbol{d})$ are given as in equation (2.8). While this is the most common interpretation we give to $(\overline{\boldsymbol{u}}, \boldsymbol{d})$, we chose a different notation for $\overline{u}$ and $\gamma(t)$ because $\overline{\boldsymbol{u}}$ need not always have the interpretation of the countdown levels at a given time t. Note that almost always (unless C was the independence copula) $C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d})$ will *not* have uniform marginal distributions, so it will not be a copula any more.

The joint survival probabilities at some future date $t \geq 0$ are now given by the following lemma:

Lemma 4 (Conditional Distributions)**.**
If C is sufficiently differentiable, the distribution of the $\boldsymbol{U}$ conditional on $(\overline{\boldsymbol{u}}, \boldsymbol{d})$ is for all $\boldsymbol{u} \leq \overline{\boldsymbol{u}}$

$$C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d}) = \frac{\frac{\partial^D}{\partial x_{d_1} \cdots \partial x_{d_D}} C(\boldsymbol{u})}{\frac{\partial^D}{\partial x_{d_1} \cdots \partial x_{d_D}} C(\overline{\boldsymbol{u}})}. \tag{2.10}$$

Let $\overline{u}_i = \gamma_i(t)$ for all $i \notin \boldsymbol{d}$ and $\overline{u}_i = \gamma_i(\tau_i)$ for all $i \in \boldsymbol{d}$. The joint distribution function of the default times $\boldsymbol{\tau}$, conditional on $(\overline{\boldsymbol{u}}, \boldsymbol{d})$, is given through the joint survival function $\overline{F}(\boldsymbol{t}, \boldsymbol{T})$

$$\overline{F}(t, \boldsymbol{T}) = \mathbf{P}\left[\, \boldsymbol{\tau} \geq \boldsymbol{T} \mid (\overline{\boldsymbol{u}}, \boldsymbol{d}) \,\right] = \mathbf{E}^P\left[\, C(\boldsymbol{\gamma}(\boldsymbol{T}); \overline{\boldsymbol{u}}, \boldsymbol{d}) \mid \mathcal{G}_t \,\right]. \tag{2.11}$$

where $T_i \geq t_i$ for $i \notin \boldsymbol{d}$ and $T_i = 0$ for $i \in \boldsymbol{d}$.

The joint survival function $\overline{F}(t, \boldsymbol{T})$ gives the probability of survival of all obligors until T_i, given information $\mathcal{H}_t$ at time t. Essentially, the initial survival function is

$$\overline{F}(\boldsymbol{0}, \boldsymbol{T}) = \mathbf{E}^P\left[\, C(\boldsymbol{\gamma}(\boldsymbol{T})) \mid \mathcal{H}_0 \,\right]. \tag{2.12}$$

If no defaults happen until t, it is updated to

$$\overline{F}(t, \boldsymbol{T}) = \frac{\mathbf{E}^P\left[\, C(\boldsymbol{\gamma}(\boldsymbol{T})) \mid \mathcal{H}_t \,\right]}{C(\boldsymbol{\gamma}(t))}, \tag{2.13}$$

and whenever a default happens (of obligor j, say), we must take a partial derivative of the copula function with respect to the defaulted obligor, and fix the value of γ_j at $\gamma_j(\tau_j)$. These operations reflect the updating of the survival function with respect to the information that keeps arriving in form of defaults and survivals of the obligors.

By lemma 4 we can give a full term structure of survival probabilities at all times for every obligor. In particular, we can also derive the default hazard rates and their respective dynamics. The default hazard rates are defined as follows:

Definition 5 (Hazard Rates)**.** *For each obligor $i \in I$ with $\tau_i > t$ we define*

(i) the survival probability $P_i(t, T) = \mathbf{P}\left[\, \tau_i > T \mid \mathcal{H}_t \,\right]$.
(ii) the default intensity $h_i(t) := -\frac{\partial}{\partial T} P_i(t, t)$
(iii) the default hazard rate

$$h_i(t, T) = -\frac{\frac{\partial}{\partial T} P_i(t, T)}{P_i(t, T)}$$

(iv) the survival probability of i, given $j \neq i$ defaults at t as $P_i^{-j}(t, T) = \mathbf{E}\left[\, \tau_i > T \mid \mathcal{H}_t \wedge \{\tau_j = t\} \,\right]$.
(v) the default hazard rate of i, given $j \neq i$ defaults at t

$$h_i^{-j}(t, T) = -\frac{\frac{\partial}{\partial T} P_i^{-j}(t, T)}{P_i^{-j}(t, T)}.$$

The dynamics of the default intensities are derived in proposition 4.7 of Schönbucher and Schubert (2001), which is reproduced here, adapted to the conditioning information $(\overline{\boldsymbol{u}}, \boldsymbol{d})$ (Schönbucher and Schubert (2001) only give the dynamics before and up to the first default $\boldsymbol{d} = \emptyset$.)

Proposition 6.
Let $(\overline{\boldsymbol{u}}, \boldsymbol{d})$ represent the information until time t. For each obligor $i \notin \boldsymbol{d}$, the dynamics of the default intensity h_i are given by

$$\frac{dh_i}{h_i} = \frac{d\lambda_i}{\lambda_i} - (h_i \Delta_{ii}^{(\overline{\boldsymbol{u}},\boldsymbol{d})} + \lambda_i)dt - dN_i + \sum_{j \notin \boldsymbol{d}, j \neq i} \Delta_{ij}^{(\overline{\boldsymbol{u}},\boldsymbol{d})} (dN_j - h_j dt), \tag{2.14}$$

where the matrix of the mutual default influences is given by

$$\Delta_{ij}^{(\overline{\boldsymbol{u}},\boldsymbol{d})} := \frac{C_{x_i x_j}(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{d})}{C_{x_i}(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{d})\, C_{x_j}(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{d})} - 1.$$

The matrix Δ contains all necessary information on the effects of a default of one obligor on the default risk of the other obligors. It governs the dynamics of the hazard rates: At a default of j, the hazard rate of obligor i jumps up to $1 + \Delta_{ij}$ times the pre-default hazard rate.

Δ depends almost exclusively on the specification of the copula function in this model (the hazard rates only enter by influencing *where* the copula is evaluated). This opens a new way of judging the appropriateness of a given copula specification: Does it imply realistic dynamics for the default intensities? Simultaneously, we may even want to take the converse route: For a given matrix Δ, what is the copula that recovers these mutual influences? In the following we are going to provide answers to these questions for the class of generalised Archimedean copula functions.

3. Generalised Archimedean Copulae

Essentially, there are two types of ingredients to the portfolio credit risk model for which we need a more concrete specification:

- The individual (pseudo-)default intensities $\lambda_i(t)$ and their dynamics, and
- the copula of the default thresholds.

In this section we propose to specify the default copula function using a generalisation of the well-known *Archimedean* copula functions.

Definition 7 (Archimedean Copula Function)**.**
An Archimedean Copula function $C(\boldsymbol{u})$ *is a copula which can be represented as*

$$C(\boldsymbol{u}) = \varphi\left(\sum_{i=1}^{I} \psi(u_i)\right), \qquad \textit{where} \qquad \psi = \varphi^{[-1]}. \tag{3.1}$$

$\varphi : \mathbb{R}_0^+ \to [0,1]$ *is known as the* generator function[4] *of the copula* $C(\cdot)$.

Not every function $\varphi()$ generates an Archimedean copula. In the following, we will require that $\varphi()$ *is the Laplace transform of a positive random variable* Y, i.e. there is a random variable

[4] A footnote for the Hellenically challenged: ψ is "psi", φ is "phi", and ϕ is also "phi", just in a different typeface.

1.	Name:	Clayton Copula
	$\psi(t) =$	$(t^{-\theta} - 1)$
	$\varphi(s) = \psi^{[-1]}(s) =$	$(1+s)^{-1/\theta}$
	Parameter:	$\theta \geq 0$, independence for $\theta = 0$
	Y-Distribution:	Gamma $(1/\theta)$
	Density of Y:	$\frac{1}{\Gamma(1/\theta)} e^{-y} y^{(1-\theta)/\theta}$
2.	Name:	Gumbel Copula
	$\psi(t) =$	$(-\ln t)^{\theta}$
	$\varphi(t) = \psi^{[-1]}(t) =$	$e^{(-s^{1/\theta})}$
	Parameter:	$\theta \geq 1$, independence for $\theta = 1$
	Y-Distribution:	α-stable, $\alpha = 1/\theta$
	Density of Y:	(no closed-form is known)
3.	Name:	Frank Copula
	$\psi(t) =$	$-\ln \frac{e^{-\theta t}-1}{e^{-\theta}-1}$
	$\varphi(s) = \psi^{[-1]}(t) =$	$-\frac{1}{\theta} \ln[1 - e^{-s}(1 - e^{-\theta})]$
	Parameter:	$\theta \in \mathbb{R}\backslash\{0\}$
	Y-Distribution:	Logarithmic series on $\mathbb{N}_+$ with $\alpha = (1 - e^{-\theta})$
	Distribution of Y:	$\mathbf{P}\left[\, Y = k \,\right] = \frac{-1}{\ln(1-\alpha)} \frac{\alpha^k}{k}$

TABLE 1. Some generators for Archimedean copulas, their inverses and their Laplace transforms. Source: Marshall and Olkin (1988).

$Y > 0$ such that

$$\varphi(s) = \mathcal{L}_Y(s) = \mathbf{E}\left[\, e^{-sY} \,\right]. \tag{3.2}$$

In particular, φ is strictly monotonically decreasing and invertible. Table 3 gives a number of possible specifications for the distribution of Y and the corresponding Laplace transforms.

Requiring a representation as a Laplace transform seems rather unrelated to definition 7 but the following algorithm will show that this representation actually is at the core of a simulation algorithm to generate random variates with joint distribution function 3.1.

Proposition 8 (Marshall and Olkin (1988))**.**
Follow the following algorithm:

1. *Generate I independent random variates X_i, $i = 1, \dots, I$ with uniform distribution on* $[0, 1]$.
2. *Generate one random variate Y such that Y is independent of the X_i and such that it satisfies 3.2 (i.e. such that the Laplace transform of Y is $\varphi(s)$).*

3. Form

$$U_i := \varphi\left(\frac{1}{Y}(-\ln X_i)\right). \tag{3.3}$$

Then the joint distribution function of the U_i is the Archimedean copula with generator $\psi(\cdot) = \varphi^{[-1]}(\cdot)$, i.e.

$$\mathbf{P}\left[\, \boldsymbol{U} \le \boldsymbol{u} \,\right] = C(\boldsymbol{u}) = \varphi\left(\sum_{i=1}^{I} \psi(u_i)\right). \tag{3.1}$$

As we are going to generalize this algorithm, the proof is postponed to the proof of proposition 9.

Archimedean copula functions are the first step to break out of the straightjacket imposed by the normal distribution and the Gaussian copula function. We already have made some progress on the analytical front because the joint distribution function of the random vector $\boldsymbol{U}$ is given in closed-form even for very high-dimensional problems, which is not the case for Gaussian copula functions. Furthermore, algorithm 8 shows that the generation of random variates with a given Archimedean copula function is rather easy and also not numerically more expensive than the generation of a similar number of correlated normally distributed random variates.

The remaining disadvantage of the Archimedean copula functions is the fact that they impose too much structure on the dependency. In particular, all random variates U_i are *exchangeable*, i.e. the distribution of any permutation of the U_i is still the same as the original distribution because we can interchange the order of summation of the $\psi(U_i)$ in (3.1) as we like. For default risk this means that we cannot (yet) have some groups (or pairs) of obligors with higher dependency, and others with less dependency. This restriction is lifted in the following generalisation of proposition 8.

Proposition 9 (Generalised Archimedean Copula Functions)**.**
Let $\boldsymbol{Y} = (Y_1, \dots, Y_N)^T$ be a vector of positive random variables. Let a_{in} be the components of a $(I \times N)$-matrix A of factor weights. Define for all $i \le I$, $n \le N$ and $s \ge 0$

$$\tilde{Y}_i := \sum_{n=1}^{N} a_{in} Y_n \qquad \tilde{\boldsymbol{Y}} = A\boldsymbol{Y} \tag{3.4}$$

$$\tilde{\varphi}_i(s) := \mathcal{L}_{\tilde{Y}_i}(s) = \mathbf{E}\left[\, e^{-s\sum_{n=1}^{N} a_{in} Y_n} \,\right], \qquad \tilde{\psi}_i(t) := \tilde{\varphi}_i^{[-1]}(t) \tag{3.5}$$

$$\varphi(s_1, \dots, s_N) := \mathbf{E}\left[\, e^{-\sum_{n=1}^{N} s_n Y_n} \,\right] \qquad \varphi(\boldsymbol{s}) := \mathcal{L}_{\boldsymbol{Y}}(\boldsymbol{s}) = \mathbf{E}\left[\, e^{-\boldsymbol{s}^T \boldsymbol{Y}} \,\right]. \tag{3.6}$$

$$\varphi_n(s) := \mathcal{L}_{Y_n}(s) = \mathbf{E}\left[\, e^{-sY_n} \,\right]. \tag{3.7}$$

Follow the following algorithm:

1. Generate I random variates X_i, $1 \le i \le I$ i.i.d. uniform on $[0, 1]$.
2. Generate Y_n $1 \le n \le N$ as above.

3. Define U_i as follows for $1 \leq i \leq I$

$$U_i := \tilde{\varphi}_i \left(\frac{1}{\sum_{n=1}^N a_{in} Y_n} \cdot (-\ln X_i) \right) = \tilde{\varphi}_i \left(\frac{1}{\tilde{Y}_i} \cdot (-\ln X_i) \right) \tag{3.8}$$

or simply

$$\boldsymbol{U} = \tilde{\boldsymbol{\varphi}} \left(-\ln(\boldsymbol{X}) / \tilde{\boldsymbol{Y}} \right). \tag{3.9}$$

Then the joint distribution function of the U_i is given by

$$C(\boldsymbol{u}) := \mathbf{P}\left[\, U_i \leq u_i, \ \ \forall\, i \leq I \,\right] = \mathbf{E}\left[\prod_{i=1}^I \exp\left\{ -\tilde{Y}_i \tilde{\psi}_i(u_i) \right\} \right] \tag{3.10}$$

$$= \mathbf{E}\left[\exp\left\{ -\sum_{i=1}^I \sum_{n=1}^N a_{in} \tilde{\psi}_i(u_i) Y_n \right\} \right] \tag{3.11}$$

$$= \varphi \left(\sum_{i=1}^I a_{i1} \tilde{\psi}_i(u_i), \ldots, \sum_{i=1}^I a_{iN} \tilde{\psi}_i(u_i) \right) \tag{3.12}$$

or in vector notation

$$C(\boldsymbol{u}) = \mathbf{E}\left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u})} \right] = \mathbf{E}\left[e^{-\boldsymbol{Y}^T A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u})} \right] = \varphi \left(A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u}) \right). \tag{3.13}$$

Furthermore, the U_i are distributed on $[0,1]^I$ and have uniform marginal distributions. Thus the joint distribution function of the U_i is a copula function.

Note that if the factors Y_n are independent, the multivariate Laplace transform $\varphi(\cdot)$ reduces to a product of univariate Laplace transforms and we have

$$C(\boldsymbol{u}) = \prod_{n=1}^N \mathcal{L}_{Y_n} \left(\sum_{i=1}^I a_{in} \tilde{\psi}_i(u_i) \right) = \prod_{n=1}^N \varphi_n \left(\sum_{i=1}^I a_{in} \tilde{\psi}_i(u_i) \right). \tag{3.14}$$

We give the proof here in the main text as it shows clearly why the algorithm works.

Proof. By construction, the event $U_i \leq u_i$ is equivalent to

$$X_i \leq \exp\{-\tilde{Y}_i \tilde{\psi}_i(u_i)\}.$$

Therefore

$$\begin{aligned}
\mathbf{P}\left[\, U_i \le u_i, \ \ \forall\, i \le I \,\right] &= \mathbf{P}\left[\, X_i \le \exp\{-\tilde{Y}_i \tilde{\psi}_i(u_i)\}, \ \ \forall\, i \le I \,\right] \\
&= \mathbf{E}\left[\, \mathbf{P}\left[\, X_i \le \exp\{-\tilde{Y}_i \tilde{\psi}_i(u_i)\}, \ \ \forall\, i \le I \,\middle|\, \tilde{Y}_1, \dots, \tilde{Y}_I \,\right] \,\right] \\
&= \mathbf{E}\left[\, \exp\{-\sum_{i=1}^{I} \tilde{Y}_i \tilde{\psi}_i(u_i)\} \,\right] \\
&= \mathbf{E}\left[\, \exp\{-\sum_{n=1}^{N} \left(\sum_{i=1}^{I} a_{in} \tilde{\psi}_i(u_i) \right) Y_n\} \,\right] \\
&= \varphi\left(\sum_{i=1}^{I} a_{i1} \tilde{\psi}_i(u_i), \dots, \sum_{i=1}^{I} a_{iN} \tilde{\psi}_i(u_i) \right)
\end{aligned}$$

which proves the form of the distribution function (3.10), (3.11) and (3.12).

It remains to show that the distribution indeed has uniform marginals. For this we need for all $u_i \in [0,1]$ that $u_i = \mathbf{P}[\, U_i \le u_i \,]$. Using the same iterated condititional expectations as above wer reach

$$\mathbf{P}\left[\, U_i \le u_i \,\right] = \mathbf{E}\left[\, \exp\{-\tilde{Y}_i \tilde{\psi}_i(u_i)\} \,\right] = \mathcal{L}_{\tilde{Y}_i}(\tilde{\psi}_i(u_i)) = u_i,$$

because (3.5), $\tilde{\psi}_i()$ was chosen in such a way that it is exactly the inverse function of the LT of $\tilde{Y}$. Thus the marginal distributions of the u_i are uniform on $[0,1]$, and the joint distribution function is indeed a copula. □

Although independence of the factor variables Y_n will be the rule rather than the exception, (3.12) holds also if the factors are not independent. We only use independence to simplify the expressions in (3.14). The setup of proposition 9 reduces to the "classical" Archimedean copula of proposition 8 if there is only one driving factor ($N = 1$), and all factor weights are equal.

Obviously, to be practically useful, the distribution of the Y_n should be chosen such that the Laplace transforms (3.5) and (3.6) can be easily evaluated and inverted. A particularly simple case is reached if a set of independent factors from the same summation-stable family of distributions is used (e.g. Gamma or positive α-stable). Stability ensures that the weighted sums $\tilde{Y}$ of the factors Y_n are still within the same class of distributions, and thus that the corresponding Laplace transform is easy to calculate.

If the driving factors Y_n are independent, they need not belong to the same family of distributions, as long as their Laplace transforms are available and easily invertible. The Laplace transforms of the weighted factor sums $\tilde{\varphi}()$ is still easily calculated, because the LT of a sum of independent random variables is the product of the individual LTs. We have in this case

$$\tilde{\varphi}_i(s) = \mathbf{E}\left[\, e^{-s\sum_{n=1}^{N} a_{in} Y_n} \,\right] = \prod_{n=1}^{N} \mathbf{E}\left[\, e^{-s a_{in} Y_n} \,\right] = \prod_{n=1}^{N} \varphi_n(a_{in} s).$$

In general, an analytical inversion of $\tilde{\varphi}_i(s)$ will not be possible if the factors do not come from the same family of distributions, and numerical inversions will be necessary to evaluate the distribution function (3.12). Because $\tilde{\varphi}_i(s)$ is a particularly well-behaved function[5], this numerical inversion is not an implementation obstacle, even if it has to be done I times for high dimensions I.

Using factors with different distributions can be useful if one wants to use an Archimedean copula, but would like to perform a *specification test* on the "right" Archimedean copula function. To each Archimedean copula there is a corresponding factor distribution, so by incorporating one factor of each possible distribution class we can build a model which nests these Archimedean copulae. The "large" model can be estimated by maximum likelihood, and using standard tests it can be determined which specification fits the data best.

3.1. **The Density.** For maximum likelihood estimation, the density of the distribution function is necessary. To calculate the density, we need the partial derivatives of the Copula function. The first derivative is:

$$\frac{\partial C}{\partial u_j} = \frac{\partial}{\partial u_j}\mathbf{E}\left[\prod_{i=1}^{I} e^{-\tilde{Y}_i\tilde{\psi}_i(u_i)}\right] = -\tilde{\psi}_j{}'(u_j)\cdot\mathbf{E}\left[\tilde{Y}_j\prod_{i=1}^{I} e^{-\tilde{Y}_i\tilde{\psi}_i(u_i)}\right] \tag{3.15}$$

Second and higher cross-derivatives (never twice the same index) work exactly the same. Eventually, we reach the density of the copula as

$$c(\boldsymbol{u}) = \frac{\partial}{\partial u_1\cdots\partial u_I}C(\boldsymbol{u}) = \left[\prod_{j=1}^{I}(-\tilde{\psi}_j{}'(u_j))\right]\cdot\mathbf{E}\left[\prod_{i=1}^{I}\tilde{Y}_i e^{-\tilde{Y}_i\tilde{\psi}_i(u_i)}\right] \tag{3.16}$$

In the case of the Gaussian copula the density had a singularity at $t = 0$ (or equivalently $\boldsymbol{u} = \mathbf{1}$) which was the reason for the unrealistic jump sizes close to $t = 0$ and a front-loading of joint defaults for this copula specification. By proposition **??**, a sufficient condition for $\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})} < \infty$ is that $c(\boldsymbol{u})$ is finite. Note that – even if not all moments of Y_n may exist[6] under P, all moments of all Y_n will exist under $P(\overline{\boldsymbol{u}},\boldsymbol{d})$ if $\overline{\boldsymbol{u}} < \mathbf{1}$, i.e. as soon as $t > 0$. Thus, if $\tilde{\psi}_i(u_i) > 0$ i.e. $u_i < 1$ for all $i \leq I$ then the density exists and is finite.

The only possibly problematic points are therefore at $t = 0$ which corresponds to $u_i = 1$ or at $t = \infty$ which corresponds to $u_i = 0$. (We ignore $t = \infty$ for now as it is clearly less relevant.) At $t = 0$ the density is only finite if $\mathbf{E}\left[\prod_{i=1}^{I}\tilde{Y}_i\right] < \infty$ and if $|\tilde{\psi}_i'(1)| < \infty$ for all i where $u_i = 1$. We will show in section 6 that for our example implementation these conditions are satisfied.

4. Conditional Probability Measures

We aimed to build a model which is capable of reproducing the *dynamics* of the default probabilities and -intensities as time proceeds. As time proceeds, information about the state of the economy is revealed through the occurrence and absence of defaults of the obligors in the

[5]It is one-dimensional, monotone, concave, and all derivatives are available in closed-form.

[6]The α-stable random variables in the Gumbel copula are one case where moments may not exist under P.

portfolio. This information is reflected in an updated probability distribution on the times of default as described in lemma 4.

Proposition 10 (Conditional Measures)**.**
Let $\boldsymbol{u} \leq \overline{\boldsymbol{u}} < \mathbf{1}$*, and* $u_d = \overline{u}_d$ *for all* $d \in \boldsymbol{d}$*. We have*

$$C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d}) = \mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})} \left[e^{-\tilde{\boldsymbol{Y}}^T [\tilde{\boldsymbol{\psi}}(\boldsymbol{u}) - \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})]} \right],$$

where

$$\frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d})}{dP} = \frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d}} \tilde{Y}_d}{\mathbf{E}^P \left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d}} \tilde{Y}_d \right]} \tag{4.1}$$

and

$$\frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d} \cup \{j\})}{dP(\overline{\boldsymbol{u}}, \boldsymbol{d})} = \frac{\tilde{Y}_j}{\mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})} \left[\tilde{Y}_j \right]} \tag{4.2}$$

for all $j \notin \boldsymbol{d}$*.*

Proof. See appendix A. □

Given no defaults, the Radon-Nikodym density in proposition 10 is of the form of a negative exponential $e^{-\tilde{\psi} Y}$ in the factor variables Y. Such families of transformations are called members of the *exponential family* of the original distribution of $\boldsymbol{Y}$.

Economically, the exponential transformation *increases* the probability mass for *low* values of Y, while the probability mass for *high* values of Y is *decreased*. This is good news, because it is the *large* values of the factor variables Y that represent high default risk. (Remember that $X_i \leq e^{-\tilde{\psi}_i \tilde{Y}_i}$ is necessary for i to survive.) This effect is larger, the larger the factor $\tilde{\psi}$, and this factor in turn depends on the time spent in survival so far because it is $\tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})$.

It is desirable to directly characterize the distribution of the factor variables Y_n under the new, updated probability measure. This is done in the following proposition. Note that (4.5) allows the iterative construction of the conditional Laplace transforms under all measures $P(\boldsymbol{u}, \boldsymbol{d})$, starting from $\boldsymbol{d} = \emptyset$ (i.e. equation (4.4)) and iteratively adding defaulted obligors.

Proposition 11 (Conditional Factor Distribution)**.**
Let $\boldsymbol{u} < \mathbf{1}$ *and* $\boldsymbol{d} \subset \{1, \ldots, I\}$*. We write* $\varphi(\boldsymbol{s}; \boldsymbol{u}, \boldsymbol{d}) := \mathbf{E}^{P(\boldsymbol{u}, \boldsymbol{d})} \left[e^{-\boldsymbol{s}^T \boldsymbol{Y}} \right]$ *for the Laplace transform of the factor variables* $\boldsymbol{Y}$ *under* $P(\boldsymbol{u}, \boldsymbol{d})$*, and* $\varphi(\boldsymbol{s}; \boldsymbol{d})$ *for* $\varphi(\boldsymbol{s}; \mathbf{1}, \boldsymbol{d})$*. (Note that* $\tilde{\psi}_i(1) = 0$ *for all* $i \in I$*.) Then:*

$$\varphi(\boldsymbol{s}; \boldsymbol{u}, \boldsymbol{d}) = \frac{\varphi(\boldsymbol{s} + A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u}); \boldsymbol{d})}{\varphi(A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u}); \boldsymbol{d})} \tag{4.3}$$

If no defaults have occurred, i.e. $\boldsymbol{d} = \emptyset$*,*

$$\varphi(\boldsymbol{s}; \boldsymbol{u}, \emptyset) = \mathbf{E}^{P(\boldsymbol{u}, \emptyset)} \left[e^{-\boldsymbol{s}^T \boldsymbol{Y}} \right] = \frac{\varphi(\boldsymbol{s} + A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u}))}{\varphi(A^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u}))}. \tag{4.4}$$

Given $\varphi(\boldsymbol{s};\boldsymbol{u},\boldsymbol{d})$, the Laplace transform of $\boldsymbol{Y}$ after an additional default of obligor $j \notin \boldsymbol{d}$ is

$$\varphi(\boldsymbol{s};\boldsymbol{u},\boldsymbol{d}\cup\{j\}) = \frac{\sum_{n=1}^{N} a_{jn}\frac{\partial}{\partial x_n}\varphi(\boldsymbol{s};\boldsymbol{u},\boldsymbol{d})}{\sum_{n=1}^{N} a_{jn}\frac{\partial}{\partial x_n}\varphi(\boldsymbol{0};\boldsymbol{u},\boldsymbol{d})} = \frac{\boldsymbol{a}_j\nabla\varphi(\boldsymbol{s};\boldsymbol{u},\boldsymbol{d})}{\boldsymbol{a}_j\nabla\varphi(\boldsymbol{0};\boldsymbol{u},\boldsymbol{d})}, \tag{4.5}$$

where $\boldsymbol{a}_j = (a_{j1},\ldots,a_{jN})$ denotes the j-th row vector of A.

The distribution function of $\boldsymbol{U}$ is

$$C(\boldsymbol{u};\overline{\boldsymbol{u}},\boldsymbol{d}) = \varphi(A^T(\tilde{\boldsymbol{\psi}}(\boldsymbol{u})-\tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}}));\overline{\boldsymbol{u}},\boldsymbol{d}) = \frac{\varphi(A^T\tilde{\boldsymbol{\psi}}(\boldsymbol{u});\boldsymbol{d})}{\varphi(A^T\tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}});\boldsymbol{d})}. \tag{4.6}$$

where $u_i \le \overline{u}_i$ for $i \notin \boldsymbol{d}$, $u_i = \overline{u}_i$ otherwise.

The Laplace transform of $\tilde{Y}_i$ under $P(\overline{\boldsymbol{u}},\boldsymbol{d})$ is

$$\tilde{\varphi}_i(s;\overline{\boldsymbol{u}},\boldsymbol{d}) = \mathbf{E}^{P(\overline{\boldsymbol{u}},\boldsymbol{d})}\left[\, e^{-s\tilde{Y}_i}\,\right] = \varphi(s\,\boldsymbol{a}_i;\overline{\boldsymbol{u}},\boldsymbol{d}). \tag{4.7}$$

Proof. Substitute the Radon-Nikodym density (4.1) to reduce all expressions to expectations under P. The claims follow after elementary transformations. □

5. Dynamics of Hazard Rates

A particularly interesting question for the credit risk modelling application are the dynamics of the default intensities and hazard rates as time proceeds and in particular as a default occurs.

Let us consider the following situation: The current time $t > 0$ is uniquely described by the levels $\overline{\boldsymbol{u}} = \boldsymbol{\gamma}(t) < \mathbf{1}$ of the countdowns, obligors $\boldsymbol{d}$ have already defaulted. Now obligor $j \notin \boldsymbol{d}$ may default as well. We are interested in the influence this has on the survival probabilities and default hazard rates of another obligor $i \neq j$. To simplify the notation, all calculations are performed conditional on the realisations of the pseudo-hazard rates $\boldsymbol{\lambda}$, or equivalent conditional on $\mathcal{G}_\infty$.

In general, without a default of j, the survival probability of i is

$$P_i(t,T) = C((\overline{\boldsymbol{u}}_{-i},u_i(T));\overline{\boldsymbol{u}},\boldsymbol{d}) = \mathbf{E}^{P(\overline{\boldsymbol{u}},\boldsymbol{d})}\left[\, e^{-\tilde{Y}_i[\tilde{\psi}_i(u_i(T))-\tilde{\psi}_i(u_i(t))]}\,\right].$$

where $u_i(T) = \gamma_i(T)$ is chosen such that the survival horizon T is reached. We call $\boldsymbol{u} = (\overline{\boldsymbol{u}}_{-i},u_i(T))$. Then the default hazard rate $h_i(t,T)$ is

$$h_i(t,T) = \frac{\partial u_i(T)}{\partial T}\tilde{\psi}_i'(u_i)\,\mathbf{E}^{P(\overline{\boldsymbol{u}},\boldsymbol{d})}\left[\,\tilde{Y}_i\frac{e^{-\tilde{Y}_i[\tilde{\psi}_i(u_i(T))-\tilde{\psi}_i(u_i(t))]}}{\mathbf{E}^{P(\overline{\boldsymbol{u}},\boldsymbol{d})}\left[\, e^{-\tilde{Y}_i[\tilde{\psi}_i(u_i(T))-\tilde{\psi}_i(u_i(t))]}\,\right]}\,\right] \tag{5.1}$$

$$= \frac{\partial u_i(T)}{\partial T}\tilde{\psi}_i'(u_i)\,\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\,\right] \tag{5.2}$$

With a default of j, the definitions are the same, but under the measure $P(\boldsymbol{u},\boldsymbol{d'}) = P((\overline{\boldsymbol{u}}_{-i}, u_i), (\boldsymbol{d}\cup \{j\}))$ that includes the default of j at $\overline{u}_j$. The Radon-Nikodym density of this measure was given in proposition 10 from which follows that for any random variable X, we have

$$\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d'})}\left[\,X\,\right] = \frac{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,X\tilde{Y}_j\,\right]}{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_j\,\right]}.$$

This helps us to reach the hazard rate of obligor i, if in addition obligor j has defaulted at $\overline{u}_j$:

$$\begin{aligned} h_i^{-j}(t,T) &= -\frac{\partial u_i(T)}{\partial T}\tilde{\psi}_i'(u_i)\,\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d'})}\left[\,\tilde{Y}_i\,\right] \\ &= -\frac{\partial u_i(T)}{\partial T}\tilde{\psi}_i'(u_i)\frac{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\tilde{Y}_j\,\right]}{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_j\,\right]}. \end{aligned}$$

Now we can go and compare the hazard rates of default of i with and without a default of j. The relative increase in the default hazard rate of i, given a default of j, is

$$\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})} = \frac{h_i^{-j}(t,T)}{h_i(t,T)} - 1 = \frac{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\tilde{Y}_j\,\right]}{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_j\,\right]\,\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\,\right]} - 1, \tag{5.3}$$

where $h_i^{-j}(t,T)$ denotes the default hazard rate of obligor i given that obligor j defaults at time t. This can be interpreted as the the *covariance* of $\tilde{Y}_i$ and $\tilde{Y}_j$ *given that* i *survives until* T, where $\tilde{Y}_i$ and $\tilde{Y}_j$ have been normalized to unit means.

The proportional jump does not directly depend on the level of the unconditional hazard rates $\boldsymbol{\lambda}$. The dependence is only indirect through the change of measure and the connection between the time of maturity T and the threshold level u_i.

The default influences can also be given in terms of the factor matrix A and the factors $\boldsymbol{Y}$. Using

$$\tilde{Y}_i\tilde{Y}_j = (A\boldsymbol{Y}\boldsymbol{Y}^T A^T)_{ij}$$

the relative jump size is

$$\begin{aligned} \frac{h_i^{-j}(t,T)}{h_i(t,T)} - 1 &= \frac{\left(A\left[\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\boldsymbol{Y}^T\,\right] - \mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}^T\,\right]\right]A^T\right)_{ij}}{\left(A\;\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}^T\,\right]A^T\right)_{ij}} \\ &= \frac{\left(A\,\mathrm{cov}^{P(\boldsymbol{u},\boldsymbol{d})}(\boldsymbol{Y},\boldsymbol{Y})A^T\right)_{ij}}{\left(A\;\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}^T\,\right]A^T\right)_{ij}} \end{aligned}$$

Summing up the results of this section, we reach

Proposition 12 (Default Influences).
The matrix $\Delta^{(\boldsymbol{u},\boldsymbol{d})}$ of mutual default influences at time and state $(\boldsymbol{u},\boldsymbol{d})$ is

$$\begin{aligned}\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})} &= \frac{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\tilde{Y}_j\,\right]}{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_j\,\right]\,\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\tilde{Y}_i\,\right]} - 1\\ &= \frac{(A\,\Sigma^{(\boldsymbol{u},\boldsymbol{d})}A^T)_{ij}}{(A\,\mu^{(\boldsymbol{u},\boldsymbol{d})}\mu^{(\boldsymbol{u},\boldsymbol{d})T}A^T)_{ij}}\end{aligned}$$

where $\Sigma^{(\boldsymbol{u},\boldsymbol{d})}$ is the covariance matrix of the factor vector $\boldsymbol{Y}$ under $P(\boldsymbol{u},\boldsymbol{d})$, and $\mu^{(\boldsymbol{u},\boldsymbol{d})}$ the mean. In terms of the Laplace transform of the factor variables the default influences are reached by substituting

$$\begin{aligned}\mu^{(\boldsymbol{u},\boldsymbol{d})} &= \mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\,\right] = -\nabla\varphi(\boldsymbol{u};\boldsymbol{u},\boldsymbol{d})\\ \Sigma^{(\boldsymbol{u},\boldsymbol{d})} &= \mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\boldsymbol{Y}^T\,\right] - (\,\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\,\boldsymbol{Y}\,\right])^2 = H\varphi(\boldsymbol{u};\boldsymbol{u},\boldsymbol{d}) - \mu^{(\boldsymbol{u},\boldsymbol{d})}\mu^{(\boldsymbol{u},\boldsymbol{d})T}\end{aligned}$$

where H is the Hessian matrix of the cross-derivatives, and ∇ is the gradient (here as a column vector).

5.1. **Calibration to Jump Sizes.** In the previous subsection we were able to completely characterize the joint relative jump sizes of the hazard rates for every pair (i,j) of obligors. As they are intimately connected to credit spreads, hazard rates and their dynamics are quantities that are relatively easy to observe, that have a direct P&L effect and for which the modeller will find it easier to build an intuition than for abstract model parameters whose effects are very difficult to estimate. Thus, a natural question is: How can we calibrate such a model?

I.e. for a given matrix of jump influences, can we find an underlying model (a specification of $\boldsymbol{Y}$ and A) that reproduces these dynamics? If the matrix of jump influences can be written as MM^T with a nonnegative matrix $M \geq 0$, the answer is yes.

Formally, the calibration problem can be stated as follows: For a given matrix $M \geq 0$ with I rows, find a matrix A, and factor variables $\boldsymbol{Y}$, such that

$$(MM^T)_{ij} = \frac{(A\Sigma A^T)_{ij}}{((A\mu)(A\mu)^T)_{ij}}, \qquad \text{for } i \neq j\text{, and} \qquad A \geq 0, \tag{5.4}$$

where Σ and μ are the covariance matrix and means of the factors. MM^T represents the prespecified jump sizes in the hazard rates at defaults. We only need to require a fit for $i \neq j$ because we can only specify influences between *distinct* pairs of obligors. We require that $A \geq 0$ because we want to keep $\tilde{Y} \geq 0$.

In general, there are many possible solutions for the problem (5.4), for example any positive scaling of the rows of a solution A again yields a solution. Here we present just one possibility to show the existence. It may be not parsimonious, but it is relatively simple:

Lemma 13 (Calibration of Jump Sizes).
A solution to the calibration problem (5.4) *is:*

(i) *Choose* $N := I +$ *number of rows of* M*.*
(ii) *Let* D *be a* $I \times I$ *diagonal matrix with diagonal elements* $D_{ii} = \alpha - \sum_{n=1}^{N} M_{in}$*, where* $\alpha = \max_{i \leq I}\{\sum_{n=1}^{N} M_{in}\}$*.*
(iii) *Choose* A *as concatenation of* M *and* D

$$A = (M, D) \tag{5.5}$$

(iv) *Choose* Y_n *independent and identically distributed with means* μ *and variances* σ^2*, such that* $\mu/\sigma = \alpha$*.*

The diagonal matrix was added to the factor matrix A in order to ensure that we have $A\mathbf{1} = \alpha\mathbf{1}$, which will simplify the denominator in (5.4) to a simple scalar. After that was achieved, the factors Y_n could be chosen i.i.d., and only the ratio of mean over standard deviation was needed in order to re-scale the solution to the right magnitude.

In special cases, the number of factors in the model can be reduced significantly, it is desirable to have $N \ll I$. For example if there exists a vector $\boldsymbol{\mu} > \mathbf{0}$ such that

$$M\boldsymbol{\mu} = \mathbf{1},$$

then $\boldsymbol{\mu}$ can be used as mean vector for the factors $\boldsymbol{Y}$, and $A = M$ with $\sigma = 1$ would be a valid model specification.

If M has N_M columns, it can be written as concatenation of M_N column vectors v_n, i.e. $M = (v_1, \ldots, v_{N_M})$. Then

$$MM^T = \sum_{n=1}^{N_M} v_n v_n^T. \tag{5.6}$$

Thus, every column vector v of M contributes vv^T to the full joint influence matrix MM^T. A first consequence of this is that we can always write any symmetric nonnegative influence matrix C as $C = MM^T$ by choosing a large number ($I(I-1)/2$, to be precise) of column vectors v_n, where each v_n contributes exactly one off-diagonal element to MM^T: Simply choose v_n to be zero, except for a value of $\sqrt{c_{ij}}$ at positions i and j. Then $(v_n v_n^T)_{ij} = c_{ij}$, and $(v_n v_n^T)_{i'j'} = 0$ for $(i', j') \neq (i, j)$ and $i \neq j$. (Of course v_n will also contribute to the diagonal of MM^T, but we are not interested in this.)

As second consequence, (5.6) suggests the following strategy to specify the influence matrix in a practical application, by hierarchically building up the columns of M directly, following a similar strategy as the industry group allocation in the CreditMetrics model. Here, every industry group n will correspond to a column vector v_n (and thus to a risk factor), and the i-th entry in this vector gives the participation of obligor i in industry n. Adding another column vector v to M increases the influence matrix by vv^T. Because $(vv^T)_{ij} \neq 0$ only where v_i and v_j are nonzero, one can start by adding the broad influences, e.g. $v_1 = c\mathbf{1}$ for an overall mutual influence, and refine the matrix MM^T by adding column vectors with more and more zeros.

6. A Concrete Implementation: Generalized Clayton Copula

6.1. **Model Setup.** In this section we analyse a concrete specification of the generalised Archimedean copula model, based upon the Clayton-copula, or (equivalently), factor variables that are Gamma $\Gamma(\alpha, \beta)$ distributed.

Definition 14.
The random variable $X \in \mathbb{R}_+$ has a Gamma $\Gamma(\alpha, \beta)$ distribution with shape parameter α and scale parameter β if one of the following equivalent conditions holds

(i) Its density function is for $x > 0$

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \tag{6.1}$$

(ii) Its Laplace Transform is

$$\mathcal{L}_X(s) = \mathbf{E}\left[\, e^{-sX} \,\right] = (1 + \beta s)^{-\alpha}. \tag{6.2}$$

The following lemma recalls some useful facts about gamma-distributed random variables which can be found in any good textbook on statistics or probability.

Lemma 15. *Let X be $\Gamma(\alpha, \beta)$ distributed under the measure P.*

- *Mean and variance:*
  $\mathbf{E}^P[\, X \,] = \alpha\beta$, *the variance is* $\alpha\beta^2$.
- *Scaling:*
  cX *is* $\Gamma(\alpha, c\beta)$*for* $c > 0$.
- *Adding:*
  If $X_{1,2}$ *are independent* $\Gamma(\alpha_{1,2}, \beta)$ *RVs, then* $Y := X_1 + X_2$ *is* $\Gamma(\alpha_1 + \alpha_2, \beta)$ *distributed.*
- *Exponential family:*
  If $dP'/dP = e^{-\delta X} / \mathbf{E}^P\left[\, e^{-\delta X} \,\right]$, *then* X *is* $\Gamma(\alpha, (\delta + 1/\beta)^{-1})$ *- distributed under* P'.
- *Product family:*
  If $dP'/dP = X / \mathbf{E}^P[\, X \,]$, *then* X *is* $\Gamma(\alpha + 1, \beta)$ *- distributed under* P'.
- *Higher moments:*
  $\mathbf{E}^P\left[\, X^k \,\right] = \beta^k \alpha(\alpha + 1) \cdots (\alpha + (k-1))$

We set up the model as follows:

Assumption 4.
The factor variables Y_n, $n \leq N$ are independent and Y_n is $\Gamma(\alpha_n, \beta_n)$-distributed. Without loss of generality[7] *we set $\beta_n = 1$ for all $n \leq N$.*

Hence we have the following consequences: The Laplace transform of $\boldsymbol{Y}$ is

$$\varphi(\boldsymbol{s}) = \prod_{n=1}^{N} (1 + s_n \beta_n)^{-\alpha_n} =: \varphi(\boldsymbol{s}; \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{6.3}$$

[7] By the scaling property we can absorb β_n in the factor matrix A.

The partial derivatives are

$$\frac{\partial}{\partial s_n}\varphi(\boldsymbol{s}) = -\alpha_n\beta_n(1-\beta_n s_n)^{-1}\varphi(\boldsymbol{s}). \tag{6.4}$$

The Laplace transform of $\tilde{Y}_i$ is for $i \leq I$

$$\tilde{\varphi}_i(s) = \varphi(s\boldsymbol{a_i}) = \prod_{n=1}^{N}(1+sa_{in}\beta_n)^{-\alpha_n}. \tag{6.5}$$

In general there is no closed-form solution for $\tilde{\psi}_i(t)$, but numerical inversion is highly efficient.

6.2. **The Development of the Distribution of the Factor Variables.** For the dynamics of the model we first consider the case of no defaults until time t: Let $(\overline{\boldsymbol{u}}, \boldsymbol{d} = \emptyset)$ describe the default information at time t. Then, using equation (4.1) and the lemma, the distribution of the factor variables under $\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}$ is

$$Y_n \sim \Gamma(\alpha_n, \beta_n^{\overline{\boldsymbol{u}}}) \qquad \text{under} \qquad \boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}, \tag{6.6}$$

where

$$\beta_n^{\overline{\boldsymbol{u}}} = \left(\frac{1}{\beta_n} + (A^T\tilde{\psi}(\overline{\boldsymbol{u}}))_n\right)^{-1} = \left(1 + (A^T\tilde{\psi}(\overline{\boldsymbol{u}}))_n\right)^{-1}. \tag{6.7}$$

Thus, as time proceeds, the β_n parameters of the Gamma-distributions for the mixing variables change: At $t = 0$ we have $\overline{u} = 1$ and $\tilde{\psi}(\overline{u}) = 0$, so initially we will have no change $\beta_n^{\boldsymbol{1}} = \beta_n$. As t increases, $\tilde{\psi}(\overline{\boldsymbol{u}})$ will increase, too, and so will $A^T\tilde{\psi}(\overline{\boldsymbol{u}})$ because $A \geq 0$. Thus, we expect $\beta_n^{\overline{\boldsymbol{u}}}$ top *decrease* as time proceeds. In particular we do not leave the parametric family of the distribution upon conditioning. The model is stable with respect to survival-events.

We next show that the change of the parameters is slow, we therefore have the desirable property of *time-stability* of the model: By equation (3.13), the *surival probability* of all obligors until the point in time given by countdown levels $\boldsymbol{u}$ is

$$p := C(\boldsymbol{u}) = \prod_{n=1}^{N}\varphi_n((A^T\tilde{\psi}(\boldsymbol{u}))_n).$$

As $\varphi_n(s) \leq 1$ this implies

$$p \leq \varphi_n((A^T\tilde{\psi}(\boldsymbol{u}))_n) = (1 + (A^T\tilde{\psi}(\boldsymbol{u}))_n)^{-\alpha_n}$$
$$p^{1/\alpha_n} \leq (1 + (A^T\tilde{\psi}(\boldsymbol{u}))_n)^{-1} = \beta_n^{\boldsymbol{u}} \leq 1.$$

Thus, the parameter $\beta_n^{\boldsymbol{u}}$ of the n-th factor decreases from 1 as time proceeds, but it decreases by less than the decrease of the joint survival probability over the same horizon to the power of $1/\alpha_n$. If the portfolio is not too large and the portfolio quality not too bad, the joint survival probability remains quite positive.

The conditional mean of the factor variable Y_n is $\alpha_n\beta_n^{\overline{\boldsymbol{u}}}$. As time proceeds (without defaults), $\beta_n^{\overline{\boldsymbol{u}}}$ will decrease, so the mean of Y_n will also decrease and so will the variance of Y_n. This is a reflection of the interpretation of "no defaults" as "good news": We update the default hazard rates downwards.

As an additional check of the regularity of the dynamics in this model setup, we check that the density of the copula is finite even at $t = 0$ using the criteria developed in section 3.1. By the inverse function theorem, the derivatives of $\tilde{\psi}_i$ are zero for all i at $u = 1$

$$|\tilde{\psi}_i'(1)| = |1/\tilde{\varphi}_i'(0)| = |(-\tilde{\varphi}_i(0)\sum_{n=1}^{N} \alpha_n A_{in})^{-1}| = (\sum_{n=1}^{N} \alpha_n A_{in})^{-1} < \infty.$$

Secondly, we need to show that $\mathbf{E}\left[\prod_{i=1}^{I} \tilde{Y}_i\right] < \infty$. This follows from the fact that $\prod_{i=1}^{I} \tilde{Y}_i$ is a polynomial of finite order in the Y_n and all moments of the Y_n are finite. Thus the density of the copula proposed here is finite and therefore the jump sizes $\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})}$ will always be finite.

6.3. **The Default Hazard Rates as Time Proceeds.** The updating upon defaults takes a more complicated form if several defaults have already happened and if we are using a large number of factors. But the parameters that we are really interested in are the default hazard rates and their dynamics.

Closer inspection of equations (5.2) and (5.3) yields that the expressions that have to be evaluated in order to be able to specify the joint dynamics of default hazard rates are always of the same form: The default hazard rates are given by (5.2)

$$h_i(t,T) = \frac{\partial u_i(T)}{\partial T} \tilde{\psi}_i'(u_i)\, \mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\tilde{Y}_i\right] \tag{5.2}$$

and the only unknown parameter in the joint dynamics of the hazard rates is $\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})}$ given by (5.3)

$$\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})} = \frac{h_i^{-j}(t,T)}{h_i(t,T)} - 1 = \frac{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\tilde{Y}_i\tilde{Y}_j\right]}{\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\tilde{Y}_j\right]\mathbf{E}^{P(\boldsymbol{u},\boldsymbol{d})}\left[\tilde{Y}_i\right]} - 1, \tag{5.3}$$

We always have to evaluate an expectation of a particular $\tilde{Y}_i$, or an expectation of a product $\tilde{Y}_i\tilde{Y}_j$ under the measure $\boldsymbol{P}^{(\overline{\boldsymbol{u}},\boldsymbol{d}}$. By equation (4.1) the density of $\boldsymbol{P}^{(\overline{\boldsymbol{u}},\boldsymbol{d})}$ with respect to $\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}$ is

$$\frac{d\boldsymbol{P}^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}{d\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}} = \frac{1}{\mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[\prod_{d\in\boldsymbol{d}} \tilde{Y}_d\right]} \prod_{d\in\boldsymbol{d}} \tilde{Y}_d.$$

Using the relationship above we therefore propose to evaluate the following three sets of parameters which then allow the construction of the hazard rates and their dynamics:

$$c := \mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[\prod_{d\in\boldsymbol{d}} \tilde{Y}_d\right]$$

$$c_n := \mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[Y_n\prod_{d\in\boldsymbol{d}} \tilde{Y}_d\right]$$

$$c_{nn} := \mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[Y_n^2\prod_{d\in\boldsymbol{d}} \tilde{Y}_d\right]$$

The values of these expressions can be found by expanding the products in the expectation operators. We know the parameters of the distribution of the factor variables Y_n from the results of the previous section, and lemma 15 gives the higher-order moments for the variables

Y_n. Expressions like $\mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\tilde{Y}_i\,\right]$ or $\mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\tilde{Y}_i\tilde{Y}_j\,\right]$ which determine the default hazard rates in this setup, are then only linear combinations of these parameters, so that the model dynamics are now fully specified.

For example, initially, i.e. while no defaults have occurred so far, i.e. $\boldsymbol{d} = \emptyset$. In this case, the parameters take the following values:

$$c = 1 \tag{6.8}$$

$$c_n = \mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[\,Y_n\,\right] = \alpha_n\beta_n^{\overline{\boldsymbol{u}}} \tag{6.9}$$

$$c_{nn} = \mathbf{E}^{\boldsymbol{P}^{(\overline{\boldsymbol{u}},\emptyset)}}\left[\,Y_n^2\,\right] = \alpha_n(1+\alpha_n)(\beta_n^{\overline{\boldsymbol{u}}})^2. \tag{6.10}$$

Later on, when several defaults have already happened, the expressions become more involved. The degree of complexity is governed by the number of defaults and the number of factors which drive the model. The simplest case would be the one-factor case which reduces to the *Clayton* copula function. Here, the number of evaluations would not grow with an increasing number of defaults.

Fortunately, it is usually not necessary to evaluate these expressions at all times and for all possible default scenarios. They are usually only needed to analyse the dynamics of the default hazard rates at the current point in time in order to derive hedging strategies. Nevertheless, here is a point where it would be desirable to find an efficient strategy to evaluate these expressions.

As an example let us calculate the jump sizes explicitly

$$\left(\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}\boldsymbol{Y}^T\,\right] - \mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}^T\,\right]\right)_{mn} = \begin{cases} 0 & \text{if} \quad m \neq n \\ (c_{nn} - c_n^2)/c^2 & \text{if} \quad m = n \end{cases}$$

After multiplication with matrix A we reach

$$\left(A\left[\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}\boldsymbol{Y}^T\,\right] - \mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}^T\,\right]\right]A^T\right)_{ij} = \frac{1}{c^2}\sum_{n=1}^{N}(c_{nn} - c_n^2)A_{in}A_{jn}$$

$$\left(A\,\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}\,\right]\mathbf{E}^{P^{(\overline{\boldsymbol{u}},\boldsymbol{d})}}\left[\,\boldsymbol{Y}^T\,\right]A^T\right)_{ij} = \frac{1}{c^2}\left(\sum_{n=1}^{N}A_{in}c_n\right)\left(\sum_{m=1}^{N}A_{jm}c_m\right).$$

yielding

$$\Delta_{ij}^{(\boldsymbol{u},\boldsymbol{d})} = \frac{\sum_{n=1}^{N}(c_{nn} - c_n^2)A_{in}A_{jn}}{\left(\sum_{n=1}^{N}A_{in}c_n\right)\left(\sum_{m=1}^{N}A_{jm}c_m\right)}. \tag{6.11}$$

According to equation (5.2) the default hazard rates themselves are given by

$$h_i(t,T) = \frac{\partial u_i(T)}{\partial T}\tilde{\psi}_i'(u_i)\sum_{n=1}^{N}A_{in}\frac{c_n}{c}. \tag{6.12}$$

This completes the representation of the model dynamics using the "c"-parameters. The only difficulty is now the efficient evaluation of (6.8) to (6.10). This may be a problem if the number of defaults grows too large because then the products will involve too many summands. In many typical applications of this model we do not expect difficulties, though. These applications include the derivation of hedge strategies (which means the hedge ratios for the *current* state of the world, later on we can re-calculate the hedge as time proceeds), the valuation and hedging of First-to-Default Swaps (obviously no more than one default is of interest here), the valuation of option-like payoffs like options on FtD-swaps or options to enter CDO tranches, or active risk-management: as long as the horizon is not too long, not too many defaults can occur.

Finally, we compare this model to the current market standard, the Gaussian (or Student-t) copula model: The model parameters are much more stable over time than in the Gauss copula, in particular we have avoided the front-loading of default dependency that is implicit in the Gauss copula. Secondly, we have much more analytical tractability than in a Gaussian model where a multivariate cumulative normal distribution function must be evaluated. The analytical tractability allows us to explicitly specify the matrix of default contagion influences Δ_{ij} and even to calibrate the model to this matrix.

Appendix A. Proof of proposition 10

Proof. We prove by induction over $D = |\boldsymbol{d}|$. Sufficient differentiability is given through the assumption of $\overline{\boldsymbol{u}} < \boldsymbol{1}$.

$D = 0$:
Here $\boldsymbol{d} = \emptyset$. Thus

$$C(\boldsymbol{u}, \overline{\boldsymbol{u}}, \emptyset) = \mathbf{P}\left[\, \boldsymbol{U} \le \boldsymbol{u} \,|\, \boldsymbol{U} \le \overline{\boldsymbol{u}} \,\right] = \frac{\mathbf{P}\left[\, \boldsymbol{U} \le \boldsymbol{u} \,\right]}{\mathbf{P}\left[\, \boldsymbol{U} \le \overline{\boldsymbol{u}} \,\right]}$$

because $\{\boldsymbol{U} \le \boldsymbol{u}\} \subseteq \{\boldsymbol{U} \le \overline{\boldsymbol{u}}\}$

$$\begin{aligned} &= \frac{C(\boldsymbol{u})}{C(\overline{\boldsymbol{u}})} = \frac{\mathbf{E}^P\left[\, e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\boldsymbol{u})} \,\right]}{\mathbf{E}^P\left[\, e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \,\right]} = \mathbf{E}^P\left[\, e^{-\tilde{\boldsymbol{Y}}^T(\tilde{\boldsymbol{\psi}}(\boldsymbol{u}) - \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}}))} \frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})}}{\mathbf{E}^P\left[\, e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \,\right]} \,\right] \\ &= \mathbf{E}^{P(\overline{\boldsymbol{u}}, \emptyset)}\left[\, e^{-\tilde{\boldsymbol{Y}}^T[\tilde{\boldsymbol{\psi}}(\boldsymbol{u}) - \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})]} \,\right]. \end{aligned}$$

$D \to D + 1$:
We assume $\boldsymbol{d}$ is given with $|\boldsymbol{d}| = D$, and $j \notin \boldsymbol{d}$. Set $\boldsymbol{d'} := \boldsymbol{d} \cup \{j\}$. First note that

$$C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d} \cup \{j\}) = \frac{\frac{\partial}{\partial u_j}\Big|_{u_j = \overline{u}_j} C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d})}{\frac{\partial}{\partial u_j} C(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{d})}$$

For the numerator we have (setting $u_j := \overline{u}_j$)

$$\frac{\partial}{\partial u_j}\Big|_{u_j = \overline{u}_j} C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d}) = -\tilde{\psi}'_j(\overline{u}_j)\, \mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})}\left[\, \tilde{Y}_j e^{-\tilde{\boldsymbol{Y}}^T(\tilde{\boldsymbol{\psi}}(\boldsymbol{u}) - \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}}))} \,\right]$$

while for the denominator we reach

$$\frac{\partial}{\partial u_j} C(\overline{\boldsymbol{u}}; \overline{\boldsymbol{u}}, \boldsymbol{d}) = -\tilde{\psi}'_j(\overline{u}_j)\, \mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})}\left[\, \tilde{Y}_j \,\right].$$

Thus, we can define the measure $P(\overline{\boldsymbol{u}}, \boldsymbol{d'})$ via

$$\frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d'})}{dP(\overline{\boldsymbol{u}}, \boldsymbol{d})} = \frac{\tilde{Y}_j}{\mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})}\left[\tilde{Y}_j\right]},$$

and reach

$$C(\boldsymbol{u}; \overline{\boldsymbol{u}}, \boldsymbol{d} \cup \{j\}) = \mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d'})}\left[e^{-\tilde{\boldsymbol{Y}}^T(\tilde{\boldsymbol{\psi}}(\boldsymbol{u}) - \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}}))}\right].$$

It remains to show that $dP(\overline{\boldsymbol{u}}, \boldsymbol{d'})/dP$ has the form claimed. This follows through

$$\begin{aligned}
\frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d'})}{dP} &= \frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d'})}{dP(\overline{\boldsymbol{u}}, \boldsymbol{d})} \frac{dP(\overline{\boldsymbol{u}}, \boldsymbol{d})}{dP} \\
&= \frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d'}} \tilde{Y}_d}{\mathbf{E}^{P(\overline{\boldsymbol{u}}, \boldsymbol{d})}\left[\tilde{Y}_j\right] \mathbf{E}^P\left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d}} \tilde{Y}_d\right]} \\
&= \frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d'}} \tilde{Y}_d}{\mathbf{E}^P\left[\frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d'}} \tilde{Y}_d}{\mathbf{E}^P\left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d}} \tilde{Y}_d\right]}\right] \mathbf{E}^P\left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d}} \tilde{Y}_d\right]} \\
&= \frac{e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d'}} \tilde{Y}_d}{\mathbf{E}^P\left[e^{-\tilde{\boldsymbol{Y}}^T \tilde{\boldsymbol{\psi}}(\overline{\boldsymbol{u}})} \prod_{d \in \boldsymbol{d'}} \tilde{Y}_d\right]}.
\end{aligned}$$

□

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