FACTORIZATION THEOREMS ON SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. We prove the analogs of the Khinchin factorization theorems for K-invariant probability measures on symmetric spaces $X = G/K$ with G semisimple noncompact. We use the Kendall theory of delphic semigroups and some properties of the spherical Fourier transform and spherical functions on X.

1. Introduction. — The study of K-invariant probability measures on symmetric spaces, for example on the spaces of positive definite symmetric or hermitian matrices, has been developed in recent years by many authors ([Gr1-3],[GrL1-2], [R],[Te],[Zh]). In this short paper we deal with the decomposability properties of these measures.

The fundamental facts of the arithmetic of probability distributions on the real line were proved by Khinchin([Kh]). The arithmetic of measures on locally compact abelian groups has been intensively studied, especially by Fel'dman (see his monograph [F]). The factorization problems on real hyperbolic spaces $SO(n,1)/SO(n)$ were considered by Trukhina([Tr]) who proved, using explicit formulae for spherical functions in this case, that the fundamental results of Khinchin are true on $SO(n,1)/SO(n)$. We prove the two principal theorems of Khinchin for K-invariant measures on any symmetric space of non-compact type. The main result we prove is that the semigroup of K-invariant probability measures on a symmetric space of non-compact type is delphic. To do this, we use the spherical Fourier transform.

In the next section we recall the basic facts about K-invariant measures, spherical Fourier transform and spherical functions used in the sequel. In Section 3, we formulate the results of the paper, we recall the definition of a delphic semigroup, the crucial notion of the Kendall theory([Ke]) of factorization of probability measures and we prove the main result.

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2. K-invariant probability measures and spherical Fourier transform. Let G be a semisimple noncompact Lie group with finite center and K a maximal compact subgroup of G.

A probability measure μ on $X = G/K$ is called *K-invariant* if μ is invariant with respect to the action of K on X. Then we write $\mu \in M^\natural(X)$. One may identify such measures with K-biinvariant measures on G. We denote the corresponding measures on G in the same way as on X. This makes it possible to define the convolution on $M^\natural(X)$. This convolution is commutative.

We recall now some basic facts concerning the spherical functions and transforms on symmetric spaces, the principal tools of spherical harmonic analysis (see e.g. Helgason[H]).

A K-invariant $\mathcal{C}^\infty$ function ϕ on X is said to be *spherical* if $\phi(eK) = 1$ and ϕ is an eigenfunction of all G-invariant differential operators on X. On the level of the group G all the spherical functions are given by the Harish-Chandra formula:

$$\phi_\lambda(g) = \int_K e^{<i\lambda-\rho,\mathcal{H}(gk)>} dk, \quad g \in G,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}} \tag{1}$$

where $\mathfrak{a}$ is the Lie algebra of the Abelian group A in the Cartan decomposition of G, $g = k \exp \mathcal{H}(g) n$ is the Iwasawa decomposition of $g \in G$, $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$ where Σ^+ denotes the set of the positive roots of multiplicity m_α and $< \cdot, \cdot >$ denotes the scalar product on $\mathfrak{a}$ and $\mathfrak{a}^*$ induced from the Killing form of the Lie algebra of G. Throughout all this paper we use the same notation for a $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and the dual element of $\mathfrak{a}_{\mathbb{C}}$, i.e. we write $\lambda(a) =< \lambda, a >$.

The spherical function $\phi_\lambda(g)$ is holomorphic in λ for every $g \in G$ fixed; it is invariant with respect to the Weyl group W acting on $\mathfrak{a}$:

$$\phi_\lambda \equiv \phi_{w\lambda}, \quad w \in W.$$

By the Helgason-Johnson theorem, the spherical functions are bounded if and only if λ belongs to the tube $T_\rho = \mathfrak{a}^* + iC(\rho)$, where $C(\rho)$ is the convex hull of the set $\{w\rho,\ w \in W\}$.

The *spherical Fourier transform* of $\mu \in M^\natural(X)$ is defined by

$$\hat{\mu}(\lambda) = \int_X \phi_\lambda(x) d\mu(x)$$

for $\lambda \in T_\rho$. It is holomorphic in the interior of the tube, continuous on the whole closed tube and W-invariant.

By the Lévy - Gangolli continuity theorem([Ga]), if $\mu_n, \mu \in M^\natural(X)$ and $\hat{\mu}_n(\lambda) \to \hat{\mu}(\lambda)$ on $\mathfrak{a}^*$ then $\mu_n \Rightarrow \mu$, that is

$$\int f d\mu_n \to \int f d\mu, \quad n \to \infty$$

for each bounded continuous K-invariant function f on X (we say that μ_n converges weekly to μ.)

The Gaussian measures on X are defined by means of generators. A measure $\kappa \in M^\natural(X)$ is called *Gaussian* if it belongs to a continuous semigroup of probability measures whose generator is a second order G-invariant elliptic differential operator on X annihilating constants. If X is irreducible then all such operators are given by positive multiples of the Laplace-Beltrami operator Δ on X (see [Gr2]).

We will say that a measure $\mu \in M^\natural(X)$ is *indecomposable* if

$$\mu = \mu_1 * \mu_2, \quad \mu_1, \mu_2 \in M^\natural(X) \quad \Rightarrow \mu_1 = \delta_{eK} \text{ or } \mu_2 = \delta_{eK}.$$

Example. Let $\mu^\natural$ denote the K-invariant probability measure corresponding to a probability measure μ on G:

$$\mu^\natural(B) = \int\int_{K\times K} \mu(k_1 B k_2) dk_1 dk_2, \quad B \in \mathcal{B}_G.$$

Then all the measures $\delta_g^\natural$, $g \in G$ are indecomposable in $M^\natural(X)$ as they are supported by the K-orbit of gK in X.

We say that a measure $\mu \in M^\natural(X)$ is *without indecomposable factors* if μ itself and each of its non-trivial (i.e. different from δ_{eK}) factors is decomposable into non-trivial factors in $M^\natural(X)$. We then write $\mu \in I_o$.

Example. All Gaussian measures on X are without indecomposable factors. In fact, the Gaussian measures are decomposable non-trivially because they belong to the heat semigroup. On the other hand, by the Cramér theorem on symmetric spaces of non-compact type ([Gr3]) any non-trivial factor of a Gaussian measure is again a Gaussian measure, so it is decomposable.

A measure $\mu \in M^\natural(X)$ is called *infinitely divisible* if for any $n \in \mathbb{N}$ there exists a measure $\mu_n \in M^\natural(X)$ such that $\mu = \mu_n^{*n}$. Gaussian measures are infinitely divisible. The spherical Fourier transform of an infinitely divisible K-invariant measure on X is given by the Lévy-Khinchin-Gangolli formula and the infinitely divisible measures are limits of infinitesimal triangular arrays of measures in $M^\natural(X)$([Ga]).

3. Factorization theorems and delphic semigroups. — The main results of our work are the following two theorems, analogs of theorems of Khinchin([Kh]) in the case of the real line.

THEOREM 1. — *Every measure $\mu \in M^\natural(X)$ may be decomposed*

$$\mu = (\nu_1 * \nu_2 * \ldots) * \omega$$

where $\nu_1, \nu_2 \ldots$ is a countable or finite family of indecomposable K-invariant measures on X (their convolution product converging weekly if infinite) and $\omega \in M^\natural(X)$ is a measure without indecomposable factors.

THEOREM 2. — *If $\omega \in M^\natural(X)$ is a measure without indecomposable factors then ω is infinitely divisible.*

The main tool of proof of these two theorems is the Kendall theory of delphic semigroups([Ke]) that we recall briefly now.

Let $\mathcal{S}$ be a topological Hausdorff commutative semigroup with a unique unit element e. $\mathcal{S}$ is called *delphic* if there exists a continuous semigroup homomorphism

$$\begin{aligned} \theta : \mathcal{S} &\rightarrow ([0,\infty),+) \\ \theta(uv) &= \theta(u)+\theta(v) \end{aligned}$$

and if $\mathcal{S}$ and θ verify the three following postulates:

(i) $\theta(u)=0$ if and only if $u=e$.

(ii) For each $u \in \mathcal{S}$ the set of factors of u is compact.

(iii) Let

$$\begin{array}{l} u_{11} \\ u_{21}\ u_{22} \\ \dots\ \dots\dots \\ u_{n1}\ \dots\ u_{nn} \\ \dots\dots\dots\dots \end{array}$$

be a triangular array of elements of $\mathcal{S}$ which is θ-infinitesimal, that is

$$\max_{1\le j\le n} \theta(u_{nj}) \rightarrow 0, \quad n\rightarrow\infty. \tag{2}$$

Suppose that the sequence $u_n := u_{n1}u_{n2}\dots u_{nn}$ converges to $u \in \mathcal{S}$. Then u is infinitely divisible.

By the results of Kendall([Ke]), the Theorems 1 and 2 will be proved if we show the following theorem

THEOREM 3. — *The semigroup $M^\natural(X)$ is delphic.*

Proof. — The semigroup $M^\natural(X)$ is a topological Hausdorff commutative semigroup with the only unit δ_{eK}. We define the Kendall homomorphism θ by

$$\theta(\mu) = -\ln\hat{\mu}(0).$$

By (1), the spherical function ϕ_o is given by the formula

$$\phi_o(g) = \int_K e^{-\langle\rho,\mathcal{H}(gk)\rangle}dk \tag{3}$$

so $\phi_o > 0$, $\hat{\mu}(0) > 0$ for any measure $\mu \in M^\natural(X)$ and θ is well defined. On the other hand, as $\phi_{i\rho} \equiv 1$ and ϕ_λ is W-invariant in λ, by the strict convexity of the exponential function and by (1) and (3) we infer that

$$\phi_o \leq 1 \quad \text{and} \quad \phi_o(x) = 1,\ x \in X \Leftrightarrow x = eK.$$

It follows that for any $\mu \in M^\natural(X)$ different from δ_{eK} one has $\hat{\mu}(0) < 1$, so $\theta(\mu) > 0$ and θ verifies the postulate (i) of delphic semigroup. The homomorphism propriety of θ follows from the convolution property of the spherical Fourier transform. θ is continuous by continuity of the spherical Fourier transform.

Proof of (ii). — The compactness of the factor set of a measure $\mu \in M^\natural(X)$ may be deduced from results of Dani, McCrudden and Raja ([D-MC],[D-R]). Let us give an independent simple proof of this property based on spherical Fourier transform.

We should show that if $\mu = \nu_n * \gamma_n$ with $\nu_n, \gamma_n \in M^\natural(X), n \in \mathbb{N}$, then the sequence ν_n is relatively compact in $M^\natural(X)$. Let $\tilde{\mu}$ denote the Abel transform of μ, that is the subprobability measure on $\mathfrak{a} \cong \mathbb{R}^r$ (r being the rank of X) such that

$$\hat{\mu}(\lambda) = \mathcal{F}(\tilde{\mu})(\lambda), \quad \lambda \in \mathfrak{a}^*_{\mathbb{R}}$$

where $\mathcal{F}$ denotes the ordinary Fourier transform on $\mathfrak{a} \cong \mathbb{R}^r$. We have then

$$\tilde{\mu} = \tilde{\nu}_n * \tilde{\gamma}_n$$

where $*$ is the standard convolution on $\mathfrak{a} \cong \mathbb{R}^r$.

It is easy to see, using the holomorphic and continuous extension from $\mathfrak{a}^*_{\mathbb{R}}$ to the closed tube T_ρ, that one may suppose that $\tilde{\mu}, \tilde{\nu}_n$ and $\tilde{\gamma}_n$ are probability measures. By the continuity theorem of Lévy-Gangolli it is then sufficient to prove that the sequence $\tilde{\nu}_n$ is relatively compact on $\mathfrak{a}$.

By classical compactness properties of the convolution on $\mathbb{R}^r$ we deduce that $\tilde{\nu}_n$ is shift-relatively compact. Let x_n be a sequence of elements of $\mathfrak{a}$ such that $\tilde{\nu}_n * \delta_{x_n}$ is relatively compact. The measures $\tilde{\nu}_n$ are W-invariant, so for each $w \in W$ the sequence $\tilde{\nu}_n * \delta_{wx_n}$ is relatively compact. Taking the convolution of all these sequences and using the fact that the only W-invariant vector in $\mathfrak{a}$ is 0 we see that the sequence $\tilde{\nu}_n^{*|W|}$ is relatively compact.

On the other hand, the sequence $\tilde{\nu}_n^{*|W|} * \delta_{|W|x_n}$ is also relatively compact. It follows that the sequence x_n is relatively compact in $\mathfrak{a}$ which gives the relative compactness of $\tilde{\nu}_n$.

Proof of (iii). — In order to prove (iii) we are going to use the central limit theorem of Gangolli([Ga]) saying that if (μ_{nj}) is a triangular array of measures from $M^\natural(X)$ verifying

(4) $$\max_{1 \leq n} |\hat{\mu}_{nj}(\lambda) - 1| \to 0,\ n \to \infty, \qquad \text{uniformly for } \|\lambda\| \leq a, \forall a > 0$$

and if $\mu_n = \mu_{n1} * \ldots * \mu_{nn}$ converge weekly to μ then μ is infinitely divisible.

We must therefore show that a θ-infinitesimal array (μ_{nj}) verifies (4). To do this we formulate and prove the following properties of spherical functions which are of interest independently of their application in the proof of the Theorem 3.

We denote $\gamma_\Delta(\lambda) = -(\|\lambda\|^2 + \|\rho\|^2)$ the eigenvalue of Δ acting on ϕ_λ.

PROPOSITION 4. — *(i) Let $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$. Then*

$$\lim_{H \to 0, H \in \mathfrak{a}} \frac{1 - \phi_\lambda(\exp H)}{1 - \phi_o(\exp H)} = \frac{\gamma_\Delta(\lambda)}{\gamma_\Delta(0)} = \frac{\|\lambda\|^2 + \|\rho\|^2}{\|\rho\|^2}$$

and for all $a > 0$ the convergence is uniform with respect to $\lambda \in \{\|\cdot\| \leq a\}$.

(ii) Let $a > 0$. Then there exists a constant $C > 0$ such that

$$|1 - \phi_\lambda(\exp H)| \leq C|1 - \phi_o(\exp H)|$$

for all $H \in \mathfrak{a}$ and $\|\lambda\| \leq a, \lambda \in \mathfrak{a}^*_{\mathbb{R}}$.

Proof of the Proposition 4. — (i) By hypothesis the space X does not have an Euclidean component. Then the Taylor expansion of $\phi_\lambda(\exp H)$ at $H = 0$ writes([Gr2])

$$\phi_\lambda(\exp H) = 1 + b(\lambda)\|H\|^2 + r(\lambda, H)$$

where $b(\lambda) = \dfrac{\gamma_\Delta(\lambda)}{\Delta\|H\|^2\big|_{H=0}}$ and the rest may be expressed by

$$r(\lambda, H) = \sum_{i \in I} p_i(H) F_i(\lambda, H)$$

where p_i are homogeneous polynomials on $\mathfrak{a}$ of degree 3 (not necessarily W-invariant), I is a finite set and $F_i(\lambda, H)$ are $\mathcal{C}^\infty$ functions of (λ, H). Then

$$\frac{r(\lambda, H)}{\|H\|^2} \to 0, \quad H \to 0$$

uniformly for $\|\lambda\| \leq a$, for any fixed $a > 0$. This gives the first part of the Proposition.

(ii) Fix an $a > 0$. By the part (i) of the Proposition there exists $\epsilon > 0$ such that if $\|H\| < \epsilon$ and $\|\lambda\| \leq a$ then

$$\left|\frac{1 - \phi_\lambda(\exp H)}{1 - \phi_o(\exp H)}\right| \leq 2\frac{a^2 + \|\rho\|^2}{\|\rho\|^2} = C_1.$$

Now we show that there exists $\eta < 1$ such that for all H with $\|H\| \geq \epsilon$

$$\phi_o(\exp H) \leq \eta. \tag{5}$$

If it was not true, we could find a sequence $H_n \in \mathfrak{a}^+$, the positive Weyl chamber, such that $\|H_n\| \to \infty$ and $\phi_o(\exp H_n) \to 1, n \to \infty$.

By a Harish-Chandra estimate of the function ϕ_o (see [H], page 483) it follows that

$$0 < \phi_o(\exp H_n) \leq c(1 + \|H_n\|)^d e^{-\langle \rho, H_n \rangle}$$

for some positive constants c and d.

The absolute value of the angle between ρ and a vector $H \in \bar{\mathfrak{a}}^+$ is inferior to an angle $\beta < \pi/2$, so there exists $A > 0$ such that $-\langle \rho, H_n \rangle \leq -A\|H_n\|$. Then

$$\phi_o(\exp H_n) \leq c(1 + \|H_n\|)^d e^{-A\|H_n\|} \to 0$$

when $\|H_n\| \to \infty$, which gives a contradiction. This proves (5).

Denoting $\delta = 1 - \eta > 0$ we obtain for $\|H\| \geq \epsilon$ and for $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$

$$\left| \frac{1 - \phi_\lambda(\exp H)}{1 - \phi_o(\exp H)} \right| \leq \frac{2}{\delta} = C_2.$$

One obtains in this way the part (ii) of the Proposition with $C = \max(C_1, C_2)$. $\square$

End of the proof of the Theorem 4. — From the Proposition 4(ii), using the same notations, it follows that for $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$ such that $\|\lambda\| \leq a$ we have

$$|\hat{\mu}_{nj}(\lambda) - 1| \leq C|\hat{\mu}_{nj}(0) - 1|$$

which shows that on the semigroup $M^\natural(X)$ (2) implies (4) and the third postulate of delphic semigroup is verified. $\square$

REFERENCES

[D-MC] S.G. Dani and M. McCrudden, On factor sets of measures and local tightness of convolution semigroups over Lie groups, *J. Theoretical Probability* **1** (1988), 357-370.

[D-R] S.G. Dani and C.R.E. Raja, Asymptotics of measures under group automorphisms and an application to factor sets, Preprint (to appear in Proceedings of 'International Colloquium on Lie Groups and Ergodic Theory'- Tata Institute of Fundamental Research, Bombay 1996).

[Ga] R. Gangolli, Isotropic infinitely divisible measures on symmetric spaces, *Acta Math.* **111**(1964), 213-246.

[Gr1] P. Graczyk, A central limit theorem on the space of positive definite symmetric matrices, *Ann.Inst.Fourier* **42**(1992), 857-874.

[Gr2] P. Graczyk, Dispersions and a central limit theorem on symmetric spaces, *Bull.Sciences Math.* **118**(1994), 105-116.

[Gr3] P. Graczyk, Cramer Theorem on Symmetric Spaces of Noncompact Type, *J.Theor.Prob.* **7**(1994),609-613.

[GrL1] P. Graczyk and J.J. Loeb, Bochner and Schoenberg theorems on symmetric spaces in the complex case, *Bull.Soc.Math.France* **122**(1994), 571-590.

[GrL2] P. Graczyk and J.J. Loeb, Spherical analysis and central limit theorems on symmetric spaces, Probability Measures on Groups and Related Structures XI, Proceedings Oberwolfach 1994, World Scientific, 146-166.

[F] G.M. Fel'dman, Arithmetic of Probability Distributions and Characterization Problems on Abelian Groups, Mathematical Monographs Vol.116, AMS.

[H] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.

[Ke] D.G. Kendall, Delphic Semi-Groups, Infinitely Divisible Regenerative Phenomena, and the Arithmetic of p-Functions, Z. Wahr. verw. Geb. **9** (1968), 163-195.

[Kh] A. Khinchin, Contribution à l'arithmétique des lois de distribution, Bull.Math.Univ.Moscou **1**(1937)

[R] D.St.P. Richards, The central limit theorem on spaces of positive definite matrices, *J.Multivariate Anal.* **29**(1989), 326-332.

[Te] A. Terras, Harmonic Analysis on Symmetric Spaces and Applications I,II, Springer Verlag, New York,1985,1988.

[Tr] I.P. Trukhina, Arithmetic of spherically symmetric measures on Lobatchevsky space(in Russian), Teor. Fun'kcii,Funkc.Anal.Pril.**34**(1980)

[Zh] G. Zhang, The Asymptotics of Spherical Functions and the Central Limit Theorem on Symmetric Cones, preprint, University of Odense 1993.