Palindromes and two-dimensional Sturmian sequences

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Abstract This paper introduces a two-dimensional notion of palindrome for rectangular factors of double sequences: these palindromes are defined as centrosymmetric factors. This notion provides a characterization of two-dimensional Sturmian sequences in terms of two-dimensional palindromes, generalizing to double sequences the results in [13].

Keywords Palindromes, double sequences, generalized Sturmian sequences, symbolic dynamics, combinatorics on words.

1 Introduction

This paper studies some properties of symmetry for the rectangular factors of a family of two-dimensional sequences obtained as a binary coding of a $\mathbb{Z}^2$-action defined on the one-dimensional torus $\mathbb{T}^1(=\mathbb{R}/\mathbb{Z})$ by two irrational rotations. More precisely, such a sequence $(U_{m,n})_{(m,n)\in\mathbb{Z}^2}$ is defined on the alphabet $\{0,1\}$ as follows: consider a partition of the unit circle into two half-open intervals I_0 and I_1; let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \notin \mathbb{Q}$; we have

$$\forall (m,n) \in \mathbb{Z}^2,\ (U_{m,n} = 0 \Longleftrightarrow m\alpha + n\beta + \gamma \in I_0 \text{ modulo } 1).$$

We will consider in particular the case where I_0 has length α and $1, \alpha, \beta$ are rationally independent. Such sequences can be considered as a generalization of Sturmian sequences. Recall that Sturmian sequences code the approximation of a line by a discrete line made of horizontal and vertical segments with integer vertices (see for instance [8] and the surveys [9, 17]). This two-dimensional generalization of Sturmian sequences has been introduced in [23, 6] and is closely connected (via a letter-to-letter projection) to the sequences defined on a three-letter alphabet which code discrete planes as follows. One first constructs a discrete plane defined as a plane approximation by three kinds of square faces oriented according to the three coordinate planes. Then, after projection, one obtains a tiling of the plane by three kinds of diamonds, being the projections of the square faces. This tiling is associated with a $\mathbb{Z}^2$-lattice. Then one codes this tiling over a double sequence defined on a three-letter alphabet.

The property of symmetry we consider is the following. We define the centrosymmetric image of a rectangular factor $W = [w_{i,j}]$ as the factor $\tilde{W} = [\tilde{w}_{i,j}]$ of same size, defined by

$$\forall (i,j),\ \tilde{w}_{i,j} = w_{m-i+1,n-j+1}.$$

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Let U be a double sequence of the family of double sequences we consider. We prove in Section 3 that the set of rectangular factors of given size (m,n) is stable by centrosymmetry and that there are exactly one or two centrosymmetric factors of size (m,n), according to the parity of m and n. These properties generalize well-known properties in the one-dimensional case: factors of Sturmian sequences (or more generally codings of irrational rotations) are known to have some remarkable palindrome properties (see for instance [10, 11, 12, 13, 14, 17, 25]).

In particular, Droubay and Pirillo prove in [13] that the set of factors of any Sturmian sequence contains either two palindromes of length n if n is odd, or only one palindrome of length n, if n is even. They prove conversely that this property characterizes Sturmian sequences. We use here this result to prove in Section 4 the following two-dimensional analogue.

Theorem *Let U be a uniformly recurrent sequence defined on $\{0,1\}$. The sequence U has exactly one centrosymmetric factor of size (m,n), if m is even and n odd, and two otherwise, if and only if there exist $\alpha,\beta,\gamma\in\mathbb{R}$, with $1,\alpha,\beta$ rationally independent such that we either have*

$$\forall(m,n),\ (U_{m,n}=0 \Longleftrightarrow m\alpha+n\beta+\gamma\in[0,\alpha[\ \textit{modulo}\ 1),$$

or

$$\forall(m,n),\ (U_{m,n}=0 \Longleftrightarrow m\alpha+n\beta+\gamma\in\,]0,\alpha]\ \textit{modulo}\ 1).$$

Note that one can also characterize these sequences in terms of rectangle complexity function: these are exactly the uniformly recurrent sequences of complexity $mn+n$, for m large enough [7]; the corresponding property for Sturmian sequences (defined over $\mathbb{N}$) is that they are exactly the sequences of complexity function $n+1$ [19].

Rote introduces in [21] an additional symmetry property for one-dimensional sequences, namely the complementation of symbols, i.e., the interchange of zeros and ones for binary sequences: he proves that any sequence with complexity $2n$ which fulfills this complementation-symmetry condition is a coding of an irrational rotation with respect to a partition of the unit circle into two intervals of the same size. The second author of this paper has introduced in [24] the two-dimensional analogues of these sequences in order to study the local configurations in a discrete plane, in connection with plane partitions. They belong to the family of sequences we consider with $|I_0|=|I_1|=1/2$. We introduce in Section 5 a notion of (rectangular) three-dimensional palindrome for the local configurations in a discrete plane and we apply the previous study to the determination of the number of palindromic local configurations of given size. See also [16, 20, 22] for similar notions.

2 Definitions

2.1 Factors and centrosymmetry

Consider the finite two-letter alphabet $\{0,1\}$. A **rectangular word** or **block** (also called picture) $W=[w_{i,j}]$ over the alphabet $\{0,1\}$ is defined as a rectangular finite array of letters:

$$\begin{matrix} w_{1,n} & \cdots & w_{m,n} \\ \vdots & & \vdots \\ w_{1,1} & \cdots & w_{m,1}, \end{matrix}$$

i.e., as a map from $\{1,\cdots,m\}\times\{1,\cdots,n\}$ to $\{0,1\}$. We thus say that the word W has **size** (m,n). Note that here we do not use the usual matrix indexing.

Consider a two-dimensional sequence $U=(U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined over the finite alphabet $\{0,1\}$. We shall use the usual representation for two-dimensional sequences:

the first index indicates the column number from left to right, whereas the second index n denotes the row number, from bottom to top. We call **rectangular factor** of the infinite sequence U a block $W = [w_{i,j}]_{1 \le i \le m, 1 \le j \le n}$ of consecutive letters of U such that there exist k, l satisfying $w_{i,j} = U_{k+i-1, l+j-1}$, with $1 \le i \le m,\ 1 \le j \le n$. Let $\mathcal{L}(m,n)$ denote the language of rectangular factors of size (m,n) of the sequence U.

Let us associate to the double sequence U a measure of its "complexity" as follows: let $P(m,n)$ denote the number of distinct rectangular factors of size (m,n) of the sequence U, i.e., $P(m,n) = \text{ Card } \mathcal{L}(m,n)$; the function $(m,n) \mapsto P(m,n)$ is called **rectangle complexity** of the sequence U. Recall that in the one-dimensional case, the complexity $p(n)$ counts the number of factors of length n. For a survey on the notion of complexity, see [3, 15].

The **frequency** $f(W)$ of a factor W of the sequence U is defined as the limit, if it exists, of the number of occurrences of this block in the "central" square factor

$$\begin{matrix} U_{-n,n} & \dots & U_{n,n} \\ \vdots & & \vdots \\ U_{-n,-n} & \dots & U_{n,-n}, \end{matrix}$$

of the sequence divided by $(2n+1)^2$.

Definition 1 *A double sequence is said* **uniformly recurrent** *if for every integer n, there exists an integer N such that every square factor of size (N,N) contains every square factor of size (n,n).*

Definition 2 *Let $W = [w_{i,j}]$, with $1 \le i \le m,\ 1 \le j \le n$ be a rectangular word of size (m,n). The* **centrosymmetric image** *or* **reverse image** *of W is defined as the factor $\tilde{W} = [\tilde{w}_{i,j}]$ $(1 \le i \le m,\ 1 \le j \le n)$, with*

$$\forall (i,j),\ \tilde{w}_{i,j} = w_{m-i+1, n-j+1},$$

i.e.,

$$\tilde{W} = \begin{matrix} w_{m,1} & \dots & w_{1,1} \\ \vdots & & \vdots \\ w_{m,n} & \dots & w_{1,n} \end{matrix}.$$

If W is equal to its reverse image $\tilde{W}$, then W is called a **centrosymmetric factor** *or a* **two-dimensional palindrome**.

This property is a generalization of the notion of palindrome in the one-dimensional case, although the one-dimensional words (in line or in column) which appear in a centrosymmetric factor are not necessarily palindromes, as shown by the following example: $\begin{matrix} 001 \\ 100 \end{matrix}$. But, if $m = 1$ or $n = 1$, then a centrosymmetric factor is a palindrome.

Definition 3 *Let $W = [w_{i,j}]$, with $1 \le i \le m,\ 1 \le j \le n$ be a rectangular word of size (m,n) defined over the alphabet $\{0,1\}$. The* **complement** *of $W = [w_{i,j}]_{1 \le i \le m, 1 \le j \le n}$ is the rectangular factor $\overline{W}$ of same size obtained from W by interchanging the letters 0 and 1:*

$$\overline{W} = [\overline{w}_{i,j}],\ \text{with } \overline{w}_{i,j} = 1 - w_{i,j},\ \text{for } 1 \le i \le m,\ 1 \le j \le n.$$

If W is equal to its complement $\overline{W}$, then W is called a **complementation-symmetric factor**.

We say that a double sequence is **complementation-symmetric** *if the set of its rectangular factors is stable by complementation-symmetry.*

We call **centrosymmetric complementation** *the (commutative) composition of the operations of centrosymmetry and complementation.*

Note that we follow here the terminology of [21], where the corresponding notions for one-dimensional sequences are introduced.

2.2 Definition of the two-dimensional binary codings

In all that follows, R_α denotes the rotation of angle α defined on the unit circle, i.e, the circle of perimeter 1, identified with the one-dimensional torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ by:

$$R_\alpha(x) = x + \alpha \ (modulo\ 1).$$

Definition 4 *Let $\alpha, \beta, \gamma, \varphi$ be real numbers, with $\alpha \notin \mathbb{Q}$, and $0 < \varphi < 1$. Let $I_0 = [0, \varphi[$, $I_1 = [\varphi, 1[$. We call binary coding of the $\mathbb{Z}^2$-action defined by (R_α, R_β) the two-dimensional sequence $U = (U_{m,n})_{(m,n)\in\mathbb{Z}^2}$ with values in $\{0,1\}$ defined as follows:*

$$\forall (m,n) \in \mathbb{Z}^2,\ \forall l \in \{0,1\},\ (U_{m,n} = l \Longleftrightarrow m\alpha + n\beta + \gamma \in I_l \ \ modulo\ 1).$$

Note that one works here with left-closed right-open intervals. One could choose equivalently to work with a partition into left-open right-closed intervals.

Recall that in the one-dimensional case, a Sturmian sequence is a coding of a rotation R_α, with $\varphi = \alpha$ [19]. More precisely, Sturmian sequences are usually indexed by $\mathbb{N}$ and are defined as the sequences of complexity $n+1$. Let us extend the definition of Sturmian sequences to sequences indexed by $\mathbb{Z}$. Note that the following sequences have complexity $n+1$, but cannot be defined as codings of rotations:

$$\ldots 000 \ldots 0001000 \ldots 000 \ldots$$

$$\ldots 000 \ldots 000111 \ldots 111 \ldots$$

Nevertheless if one adds the assumption of recurrence, then a sequence over $\mathbb{Z}$ of complexity $n+1$ is a coding of rotation [19]. Recall that a sequence is said to be **recurrent** if its factors have infinitely many occurrences. Hence we will call **two-sided Sturmian sequence** a recurrent sequence indexed by $\mathbb{Z}$ of complexity $n+1$. We have the following description of two-sided Sturmian sequences in terms of codings of rotations.

Theorem 1 (Morse, Hedlund) *Let u be a Sturmian sequence with values in $\{0,1\}$. There exists $\alpha \notin \mathbb{Q}$ and $\rho \in \mathbb{R}$ such that one either has*

$$\forall n \in \mathbb{Z},\ (u_n = 0 \Longleftrightarrow R_\alpha^n(\rho) = \rho + n\alpha \in [0, \alpha[\ modulo\ 1),$$

or

$$\forall n \in \mathbb{Z},\ (u_n = 0 \Longleftrightarrow R_\alpha^n(\rho) = \rho + n\alpha \in]0, \alpha]\ modulo\ 1).$$

We call **angle** *or* **slope** *of a Sturmian sequence the real number α which is thus associated. We note (I_0, I_1) the corresponding partition, i.e., I_0 is either equal to $[0, \alpha[$ or $]0, \alpha]$ and $I_1 = \mathbb{T}^1 - I_0$.*

Let us consider two particular cases for the sequence U, according to the value of φ in Definition 4.

Suppose $\varphi = \alpha$. Then the sequence U can be considered as a two-dimensional generalization of Sturmian sequences (see [6]). Indeed, the sequences in line of the two-dimensional sequence U are Sturmian sequences, whereas the sequences in columns are binary codings of the irrational rotation R_β of angle β.

Suppose $\varphi = 1/2$. The sequence U can be considered as a generalized Rote sequence. Recall that Rote sequences are exactly the one-dimensional sequences which are complementation-symmetric with complexity $2n$ (see [21]). This two-dimensional generalization has been introduced in [24] to study local configurations in a discrete plane in connection with plane partitions. The sequences in lines and columns have complexity $P(n) = 2n$, for all n. The language of rectangular factors is stable by complementation [24].

2.3 Factors and intervals

The set of rectangular factors of given size of a sequence U defined as in Definition 4 can be characterized in terms of a finite partition into intervals of the unit circle. This characterization is classical in the one-dimensional case for Sturmian sequences or more generally for codings of rotations (see for instance [1, 2]).

Lemma 1 *Let U be a sequence defined as in Definition 4. A block W ($W = [w_{i,j}]_{1\le i\le m, 1\le j\le n}$) with values in the alphabet $\{0,1\}$ is a factor of the sequence U if and only if*

$$I(W) := \bigcap_{1\le i\le m, 1\le j\le n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}} \neq \emptyset.$$

The frequency of every rectangular factor W of U exists and is equal to the length of $I(W)$.

Suppose furthermore that $0 < \varphi \le \sup(\alpha, 1-\alpha)$ and $0 < \varphi \le \sup(\beta, 1-\beta)$, then the set $I(W)$ is connected.

Proof By definition, a block $W = [w_{i,j}]$ defined on $\{0,1\}$ of size (m,n) is a factor of the sequence U if and only if there exist two integers k, l such that for $1 \le i \le m, 1 \le j \le n$:

$$\gamma + k\alpha + l\beta + (i-1)\alpha + (j-1)\beta \in I_{w_{i,j}} \text{ modulo } 1$$

i.e.,

$$\gamma + k\alpha + l\beta \in I(W) := \bigcap_{1\le i\le m, 1\le j\le n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}} \text{ modulo } 1.$$

This implies that, if W is a factor, then $I(W) \neq \emptyset$.

Conversely, suppose that $I(W) \neq \emptyset$. Then the interior of $I(W)$ is not empty, for $I(W)$ is defined as the intersection of left-closed right-open intervals. Hence for any fixed integer l, there exists k such that $\gamma + k\alpha + l\beta \in I(W)$ modulo 1, since the sequence $(k\alpha)_{k\in\mathbb{Z}}$ is dense on the unit circle, from the assumption $\alpha \notin \mathbb{Q}$.

Given any interval I of the unit circle, the convergence when n tends towards $+\infty$ of

$$\frac{\text{Card}\{-n \le i \le n,\ i\alpha + \gamma \in I\}}{2n+1}$$

towards the length of I is uniform in γ (in other words, an irrational rotation is uniquely ergodic). Hence the frequency of every factor W of U exists and is equal to the length of $I(W)$.

The proof of the connectedness can be found in [6] (see also [1]). This property can be proved by induction and is based on the following remark: if I and J are two left-closed and right-open intervals of the unit circle whose intersection is not connected, then the sum of their lengths is strictly larger than 1 and the ends of I (respectively J) belong to the interior of J (respectively I). ■

We thus deduce from Lemma 1, when $0 < \varphi \leq \sup(\alpha, 1-\alpha)$ and $0 < \varphi \leq \sup(\beta, 1-\beta)$, that the intervals $I(W)$, associated to the rectangular factors of size (m,n), are in one-to-one correspondence with the intervals of endpoints in the set $\mathcal{P}_{m,n}$

$$\mathcal{P}_{m,n} = \{-i\alpha - j\beta + l\varphi,\ 0 \leq i \leq m-1,\ 0 \leq j \leq n-1,\ l = 0,1\}.$$

We can deduce for instance from this property the expression of the rectangle complexity function. One just has to count the points of $\mathcal{P}_{m,n}$, as expressed in the following proposition.

Proposition 1 *Let U be a sequence defined as in Definition 4. Suppose $0 < \varphi \leq \sup(\alpha, 1-\alpha), \sup(\beta, 1-\beta)$.*

- *Assume that $1, \alpha, \beta$ are rationally independent.*

  If $\varphi = \alpha$, then the complexity of the sequence U satisfies:

  $$\forall (m,n),\ P(m,n) = mn + n.$$

  If $\varphi \notin \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ (this is the case in particular if $\varphi = 1/2$), then

  $$\forall (m,n),\ P(m,n) = 2mn.$$

- *Suppose $\beta = \alpha$.*

  If $\varphi = \alpha$, then

  $$\forall (m,n),\ P(m,n) = m + n,\ .$$

  If $\varphi \notin \mathbb{Z} + \alpha\mathbb{Z}$, then

  $$\forall (m,n),\ P(m,n) = 2(m + n - 1).$$

Remark Any sequence of complexity $mn + n$ is not necessarily of the previous form. Consider for instance the case of a double sequence corresponding to a "degenerated" discrete plane:

$$\begin{array}{c}
\ldots 0000000010000000 \ldots \\
\ldots 0000000010000000 \ldots \\
\ldots 0000000010000000 \ldots \\
\ldots 0000000001000000 \ldots \\
\ldots 0000000001000000 \ldots \\
\ldots 0000000001000000 \ldots \\
\ldots 0000000001000000 \ldots
\end{array}$$

It is easily seen that such a sequence has complexity $mn + n$. But if one adds the assumption of uniform recurrence (Definition 1), then one obtains that the sequences of complexity $mn + n$, for m large enough, are exactly the ones obtained with $\varphi = \alpha$ [7].

3 Properties of centrosymmetry

Theorem 2 *Let U be a sequence defined as in Definition 4. Suppose $0 < \varphi \leq \sup(\alpha, 1-\alpha), \sup(\beta, 1-\beta)$. The language of U is stable by centrosymmetry. Furthermore the frequency of any factor W is equal to the frequency of its reverse $\tilde{W}$. The sequence U has infinitely many two-dimensional palindromes of arbitrarily large size. More precisely, we have the following.*

- *Assume that $1, \alpha, \beta$ are rationally independent.*
  - *Suppose $\varphi = \alpha$. If m is even and n is odd, then there exists exactly one centrosymmetric factor of size (m,n), otherwise there are exactly two centrosymmetric factors of size (m,n).*
  - *Suppose $\varphi \notin \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$. For any (m,n), there exist two centrosymmetric factors of size (m,n).*
- *Suppose $\beta = \alpha$.*
  - *Suppose $\varphi = \alpha$. For any (m,n), there are two centrosymmetric factors of size (m,n), if m and n are both even or both odd. Otherwise, there is only one centrosymmetric factor of size (m,n).*
  - *Suppose $\varphi \notin \mathbb{Z} + \alpha\mathbb{Z}$. For any (m,n), there are two centrosymmetric factors of size (m,n).*

Proof We can suppose $\gamma = 0$ since the set of rectangular factors of given length does not depend on γ from Lemma 1. Suppose $0 < \varphi \leq \sup(\alpha, 1-\alpha), \sup(\beta, 1-\beta)$. Consider the factors of the sequence U of size (m,n). Recall that we associate in a one-to-one correspondence to each factor $W = [w_{i,j}]$, with $1 \leq i \leq m$, $1 \leq j \leq n$, the interval

$$I(W) = \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}}$$

of the partition of the unit circle into intervals of endpoints in $\mathcal{P}_{m,n}$:

$$\mathcal{P}_{m,n} = \{-i\alpha - j\beta + k\varphi, \text{ with } 0 \leq i \leq m-1, \ 0 \leq j \leq n-1, \ l = 0,1\}.$$

Let

$$y_{m,n} = \frac{-(m-1)\alpha - (n-1)\beta + \varphi}{2} \in \mathbb{T}^1.$$

Let $S_{m,n}$ denote the symmetry of the unit circle with respect to $y_{m,n}$:

$$S_{m,n}: \quad \mathbb{T}^1 \to \mathbb{T}^1, \quad x \mapsto 2y_{m,n} - x.$$

We have for $l = 0,1$:

$$S_{m,n}(R_\alpha^{-i+1} R_\beta^{-j+1} \mathring{I}_l) = R_\alpha^{-m+i} R_\beta^{-n+j} \mathring{I}_l,$$

where $\mathring{I}$ denotes the closure of I. This implies that

$$S_{m,n}(\mathring{I}(W)) = \mathring{I}(\tilde{W}).$$

Hence the set of factors of size (m,n) is stable by centrosymmetry and a factor and its reverse image have the same frequency. Furthermore, a factor is centrosymmetric if and only if $S_{m,n}(\mathring{I}(W)) = \mathring{I}(\tilde{W})$; this is equivalent to the fact that either $y_{m,n}$ or (in an exclusive way) $y_{m,n} + \frac{1}{2}$ belongs to the interior of $I(W)$, and it is thus the middle-point of the interval $I(W)$.

Suppose $1, \alpha, \beta$ rationally independent.

- Suppose $\varphi = \alpha$. The point $y_{m,n}$ belongs to the set of bounds of the intervals $I(W)$ if and only if m is even and n is odd. In this case $y_{m,n}$ does not belong to the interior of any $I(W)$ but there exists one interval $I(W)$ which contains $y_{m,n} + \frac{1}{2}$ in its interior. We thus have only one centrosymmetric factor. Otherwise, if m is odd or n is even, then $y_{m,n}$ belongs to the interior of an interval $I(W)$ (and is even the middle-point of $I(W)$). The same holds for $y_{m,n} + \frac{1}{2}$. We thus have two centrosymmetric factors.

- Suppose $\varphi \notin \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$. Then the point $y_{m,n}$ never belongs to the set of bounds of the intervals $I(W)$ but always belongs to the interior of one and only one interval $I(W)$. The same holds for $y_{m,n} + \frac{1}{2}$. Hence we have two centrosymmetic factors.

The same reasoning applies to the remaining cases. ■

4 A characterization of two-dimensional Sturmian sequences

The aim of this section is to provide a characterization of the two-dimensional Sturmian sequences ($\varphi = \alpha$ in Definition 4) in terms of palindromes. This result extends to the two-dimensional case the characterization by Droubay and Pirillo of Sturmian sequences [13].

Theorem 3 *Let U be a two-dimensional uniformly recurrent sequence defined on $\{0, 1\}$. The sequence U has exactly one centrosymmetric factor of size (m, n), if m is even and n odd, and two otherwise, if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}$, with $1, \alpha, \beta$ rationally independent, and $0 < \alpha < 1$, such that we either have*

$$\forall (m,n),\ (U_{m,n} = 0 \Longleftrightarrow m\alpha + n\beta + \gamma \in [0, \alpha[\text{ modulo } 1),$$

or

$$\forall (m,n),\ (U_{m,n} = 0 \Longleftrightarrow m\alpha + n\beta + \gamma \in]0, \alpha] \text{ modulo } 1).$$

Note that there is no restriction in assuming that the sequence U is defined on a two-letter alphabet, since this condition is implicit in the fact that there are two palindromes of size $(1, 1)$.

The same counter-example as previously shows that one cannot avoid the assumption of uniform recurrence:

...0000000010000000...
...0000000010000000...
...0000000010000000...
...0000000001000000...
...0000000001000000...
...0000000001000000...
...0000000001000000...

Proof Note that the proof of this result follows the same scheme as the one of Theorem 3 in [7] (by noticing that we partition here the unit circle by the point α whereas it is partitioned by the point $1 - \alpha$ in [7]).

Let U be a uniformly recurrent sequence defined on $\{0, 1\}$ such that the sequence U has exactly one centrosymmetric factor of size (m, n), if m is even and n odd, and two otherwise.

Let $\mathcal{L}_l$ denote the set of factors in line of the two-dimensional sequence U. By assumption ($n = 1$), there is one (one-dimensional) palindrome of any even length and two palindromes of any odd length in the set $\mathcal{L}_l$. We thus deduce from the study of Droubay and Pirillo [13] that the complexity of this set equals $n+1$, for every integer n. Note that this study was held in the framework of sequences defined over $\mathbb{N}$. The proof can be extended to factorial and two-sided extendable languages. Recall that a language L is called factorial if every factor of a word of L still belongs to L; L is said to be two-sided extendable if for every word $w \in L$, there exist two letters x, y such that xwy belong to L.

Hence we have: $\forall m,\ P(m,1) = m+1$. It is not difficult to prove that if the complexity of a factorial and two-sided extendable language L equals $n+1$, then there exists a sequence u (which has thus complexity $n+1$) such that L is exactly the set of its factors ([7], Lemma 2). Let us apply this to the language $\mathcal{L}_l$. Let u be a (one-dimensional) sequence such that $\mathcal{L}_l$ is exactly the set of its factors. Let us show that the sequence u is recurrent. Otherwise, there exists W_m factor of size $(m,1)$ such that $W_m X W_m \notin \mathcal{L}_l$, for every word X. As the two-dimensional sequence U is uniformly recurrent, then there exists $N(m)$ such that every factor of size $(N(m), N(m))$ contains W_m. Consider the lines of index $1, 2, \ldots, N(m)$. There exists k_m such that W_m has no occurrence in the lines of index 1 to $N(m)$, for a column index larger than k_m. By considering a rectangular window of size $(N(m), N(m))$ located at line index 1 and at a column index larger than k_m, one gets a contradiction. Hence the sequence u is recurrent.

We have seen that one can describe explicitly the recurrent two-sided sequences of complexity $n+1$ (Theorem 1): there exists $\alpha \notin \mathbb{Q}$ such that u is a Sturmian sequence of angle α. As the Sturmian sequences of angle α have the same set of factors (see for instance [17]), we get that $\mathcal{L}_l$ is equal to the set of factors $\mathcal{L}_\alpha$ of any Sturmian sequence of angle α. More precisely, we have the following situation for each sequence in line of the double sequence U. For every integer $i \in \mathbb{Z}$, there exists an interval I_0^i which is either equal to $[0, \alpha[$ or $]0, \alpha]$, there exists $\rho_i \in \mathbb{R}$ such that

$$\forall m \in \mathbb{Z},\ (U(m,i) = 0 \Longleftrightarrow m\alpha + \rho_i \in I_0^i \text{ modulo } 1).$$

One thus defines I_1^i as the complement of I_0^i in the unit circle. For every $i \in \mathbb{Z}$, one puts furthermore $\beta_i = \rho_{i+1} - \rho_i$.

We will work with the lines taken two by two and consider factors of size $(m,2)$ which appear with line index i. Let $\mathcal{L}_i(m,2)$ denote the set of factors of size $(m,2)$ which appear with line index i. We note $\mathcal{L}_i'(m,2)$ the set of those factors in $\mathcal{L}_i(m,2)$ which have an infinite number of occurrences at index i.

As in Lemma 1, we will associate to the set of rectangular factors of given size a partition of the unit circle into intervals. Indeed, the same proof as the one of Lemma 1 works to prove the following (see also Lemma 6 in [7]). A rectangular factor $W = \begin{matrix} w_1 & \ldots & w_m \\ v_1 & \ldots & v_m \end{matrix}$ appears at index (i,k) in the sequence U if and only if

$$\rho_i + k\alpha \in I_i(W) := (\bigcap_{1 \le j \le m} R_\alpha^{-j+1} I_{v_j}^i) \bigcap (\bigcap_{1 \le j \le m} R_{-\beta_i} R_\alpha^{-j+1} I_{w_j}^{i+1}) \text{ modulo } 1.$$

Note that the sets $\bigcap_{0 \le j \le m-1} R_\alpha^{-j+1} I_{v_j}$ and $\bigcap_{0 \le j \le m-1} R_\alpha^{-j+1} I_{w_j}$ are connected, for every m (these intervals are in one-to-one correspondence with the factors of size m of the Sturmian sequence of angle α).

In the case where $I_0^i \neq I_0^{i+1}$, then one can have $I_i(W)$ reduced to a point. This can only happen if there exists $k_i \in \mathbb{Z}$ such that $\beta_i = k_i\alpha$ modulo 1 and if the half-open intervals $\bigcap_{1 \le j \le m} R_\alpha^{-j+1} I_{v_j}^i$ and $R_\alpha^{-k_i}(\bigcap_{1 \le j \le m} R_\alpha^{-j+1} I_{w_j}^{i+1})$ (which have two distinct opening directions) have a common endpoint. This endpoint has thus the form $-j_i\alpha$, with $-1 \le j_i \le m-1$ and $-1 \le j_i - k_i \le m-1$. Furthermore, this endpoint is associated with a factor if and only if $\rho_i \in \mathbb{Z} + \alpha\mathbb{Z}$;

this factor thus appears (with a line index i) at the unique column index l_i satisfying $\rho_i + l_i\alpha = -j_i\alpha$ modulo 1.

Consider now the factors in $\mathcal{L}'_i(m,2)$. Let $I_0 =]0,\alpha[$ and $I_1 =]\alpha, 1[$. A rectangular factor $W = \begin{matrix} w_1 & \dots & w_m \\ v_1 & \dots & v_m \end{matrix}$ appears infinitely often in the sequence U with an index of line equal to i if and only if

$$I'_i(W) := (\bigcap_{1\le j\le m} R_\alpha^{-j+1} I_{v_j}) \bigcap (\bigcap_{1\le j\le m} R_\alpha^{-j+1} R_{-\beta_i} I_{w_j}) \neq \emptyset.$$

There exists furthermore an integer m_0 which does only depend on α, such that the sets $I_i(W)$ and $I'_i(W)$ are connected for any i and for $m \ge m_0$ ([6], Lemma 3).

We thus deduce that the intervals $I'_i(W)$ (for $W \in \mathcal{L}'_i(m,2)$ and $m \ge m_0$) are in one-to-one correspondence with the intervals of endpoints in the set $\mathcal{P}^{(i)}_{m,2}$

$$\mathcal{P}^{(i)}_{m,2} = \{-j\alpha - l\beta_i,\ -1 \le j \le m-1,\ l = 0,1\}.$$

Let us distinguish two cases in the proof according to the numbers β_i. We will suppose furthermore $m \ge m_0$ in the sequel of this proof, so that the sets $I_i(W)$ and $I'_i(W)$ are connected, for any $W \in \mathcal{L}_i(m,2)$.

First case Suppose that $\beta_i \in \alpha\mathbb{Z} + \mathbb{Z}$, for every $i \in \mathbb{Z}$. For any given i, there exists a (unique) integer $k_i \in \mathbb{Z}$ such that $\beta_i = k_i\alpha$ (modulo 1). Let us prove that $k_i = k_j$, for every $(i,j) \in \mathbb{Z}^2$.

Suppose that there exists (i,j) such that $k_i \neq k_j$. Let

$$y^{(i)}_{m,2} = \frac{-(m-1)\alpha - k_i\alpha + \alpha}{2}.$$

Suppose k_i even (respectively odd). By using the same argument as in the proof of Theorem 2, we see that there are two centrosymmetric factors in $\mathcal{L}'_i(m,2)$ of size $(m,2)$, when m is odd (respectively even): these are the factors W such that the intervals $I'_i(W)$ contain $y^{(i)}_{m,2}$ or $y^{(i)}_{m,2} + 1/2$. But there also exists at least one centrosymmetric factor in $\mathcal{L}'_j(m,2)$ of size $(m,2)$. Let us prove that for m large enough, we have $\mathcal{L}'_i(m,2) \cap \mathcal{L}'_j(m,2) = \emptyset$. We will thus get three distinct centrosymmetric factors of same size, which leads to a contradiction.

Suppose $\mathcal{L}'_i(m,2) \cap \mathcal{L}'_j(m,2) \neq \emptyset$, with $m \ge \sup(|k_i|+1, |k_j|+1, m_0)$. (Recall that m_0 denotes the index from which on the sets $I'_i(W)$ and $I'_j(W)$ are connected.)

Let $W = \begin{matrix} w_1 & \dots & w_m \\ v_1 & \dots & v_m \end{matrix} \in \mathcal{L}'_i(m,2) \cap \mathcal{L}'_j(m,2)$.

Suppose for instance $k_i \ge 0$. We have

$$(\bigcap_{0\le l\le m-1} R_\alpha^{-l} I_{v_{l+1}}) \bigcap R_\alpha^{-k_i} (\bigcap_{0\le l\le m-1} R_\alpha^{-l} I_{w_{l+1}}) \neq \emptyset.$$

In particular, as $m \ge k_i + 1$:

$$R_\alpha^{-k_i} I_{v_{k_i+1}} \bigcap R_\alpha^{-k_i} I_{w_1} \neq \emptyset,$$

which implies

$$w_1 = v_{k_i+1},$$

and similarly

$$w_2 = v_{k_i+2}, \dots, w_{m-k_i} = v_m.$$

One can do the same work if $k_i \leq 0$. We thus get

$$v_1 = w_{-k_i+1}, \ldots, v_{m+k_i} = w_m.$$

Hence we have (for both cases $k_i \geq 0$ or $k_i \leq 0$)

$$v_{l+k_i} = w_l, \text{ for } 1 \leq l, l+k_i \leq m,$$

$$v_{l+k_j} = w_l, \text{ for } 1 \leq l, l+k_j \leq m,$$

and thus

$$v_{l+k_i} = v_{l+k_j}, \text{ for } 1 \leq l, l+k_j, l+k_i \leq m.$$

Let $k = |k_i - k_j|$. The factor $v_1 \ldots v_m$ is thus periodic, k being one period. The Sturmian language $\mathcal{L}_\alpha$ is uniformly recurrent (for every integer n, there exists an integer N such that every factor of size N contains every factor of size n). This implies that there do not exist arbitrarily long powers of a given factor and thus, that there do not exist arbitrarily long factors of a given period. Hence the condition

$$v_{l+k_i} = v_{l+k_j}, \text{ for } 1 \leq l, l+k_j, l+k_i \leq m$$

does not hold, for m large enough. We thus have $\mathcal{L}'_i(m,2) \cap \mathcal{L}'_j(m,2) = \emptyset$, for m large enough.

We thus have proved that $k_i = k_j$, for every $(i,j) \in \mathbb{Z}^2$. Let k be this common value. The sequence U satisfies:

$$\forall (i,j) \in \mathbb{Z}^2, \ (U(i,j) = l \Longleftrightarrow j\alpha + i(k\alpha) + \rho_i \in I_l^i \ \textit{modulo } 1.$$

Consider now the opening directions of the intervals of the partition (I_0^i, I_1^i). ¿From Theorem 2, we can neither have for every i, j, $I_0^i = I_0^j$, nor $\rho_i \notin \mathbb{Z} + \alpha\mathbb{Z}$.

Hence there exists an integer i such that $I_0^i \neq I_0^{i+1}$ and $\rho_i \in \mathbb{Z} + \alpha\mathbb{Z}$. Suppose k even (respectively odd). Let m be odd (respectively even). Then there are two centrosymmetric factor in $\mathcal{L}'_i(m,2)$. Let us exhibit a centrosymmetric factor in $\mathcal{L}_i(m,2) - \mathcal{L}'_i(m,2)$. Consider a factor $W = \begin{matrix} w_1 & \ldots & w_m \\ v_1 & \ldots & v_m \end{matrix}$ for which

$$I_i(W) = I_{v_1 \ldots v_n} \cap R_\alpha^{-k} I_{w_1 \ldots w_n}$$

is reduced to a point, where

$$I^i_{v_1 \ldots v_n} = \cap_{0 \leq j \leq m-1} R_\alpha^{-j} I^i_{v_j+1}, \text{ and } I^{i+1}_{w_1 \ldots w_n} = \cap_{0 \leq j \leq m-1} R_\alpha^{-j} I^{i+1}_{w_j+1}.$$

Let us note for $-1 \leq j \leq m-1$, $I^+_{-j\alpha}$ (respectively $^-_{-j\alpha}$) the unique interval of left (respectively right) endpoint $-j\alpha$ in the partition of the unit circle by the points $-j\alpha$, $-1 \leq j \leq m-1$, corresponding to the factors of the Sturmain sequences of length m. Suppose without loss of generality that $I_0^i = [0, \alpha[$. The set $I_i(W) = I^i_{v_1 \ldots v_n} \cap R_\alpha^{-k} I^{i+1}_{w_1 \ldots w_n}$ is reduced to a point if and only if there exists j such that

$$I^i_{v_1 \ldots v_n} = I^+_{-j\alpha} \text{ and } I^{i+1}_{w_1 \ldots w_n} = I^+_{(-j+k)\alpha},$$

with $-1 \leq j \leq m-1$ and $-1 \leq j-k \leq m-1$. It is now sufficient to find an integer m large enough such that $y_{m,2} = \frac{-(m-1)\alpha - k\alpha + \alpha}{2}$ equals (modulo 1) to the point $-j\alpha$ for the corresponding factor W to be centrosymmetric, following the proof of Theorem 2. This means that $y_{m,2}$ satisfies

$$y_{m,2} \in \{-j\alpha, \ -1 \leq j \leq m-1, \ -1 \leq j-k \leq m-1\},$$

i.e.,

$$-1 \leq \frac{(m-1)+k-1}{2} \leq m-1 \text{ and } -1 \leq \frac{(m-1)+k-1}{2} - k \leq m-1,$$

which holds for m large enough.

We thus get more than two centrosymmetric factors, which proves that this first case cannot occur.

Second case We thus have $\beta_i \notin \alpha\mathbb{Z}+\mathbb{Z}$, for every $i \in \mathbb{Z}$. Suppose that there exists $(i,j) \in \mathbb{Z}^2$ such that $\beta_i \neq \beta_j$. Consider the partition of the unit circle by the points of $\mathcal{P}^{(i)}(m,2) = \{-j\alpha + l\beta_i,\ -1 \leq j \leq m-1,\ l = 0,1\}$. The intervals of endpoints in $\mathcal{P}^{(i)}(m,2)$ are in one-to-one correspondence with the factors in $\mathcal{L}_i(m,2)$. It is not difficult to prove as previously that there are two centrosymmetric factors in $\mathcal{L}_i(m,2)$, whatever the parity of m: let V_i and W_i be the centrosymmetric factors which correspond respectively to the intervals which contain $y^{(i)}_{m,2} = \frac{-(m-1)\alpha-\beta_i+\alpha}{2}$ and $y^{(i)}_{m,2}+1/2$; we similarly define V_j and W_j. Let us prove that for m large enough, neither $y^{(i)}_{m,2}$ nor $y^{(i)}_{m,2}+1/2$ does belong to $I(V_j)$, and the same for $I(W_j)$. Indeed we have

$$||y^{(i)}_{m,2} - y^{(j)}_{m,2}|| = ||\frac{\beta_i - \beta_j}{2}||$$

and

$$||y^{(i)}_{m,2} - (y^{(j)}_{m,2} + 1/2)|| = ||y^{(j)}_{m,2} - (y^{(i)}_{m,2} + 1/2)|| = ||\frac{\beta_i}{2} - (\frac{\beta_j}{2} + 1/2)||,$$

where the notation $||\ ||$ stands for the usual distance on the unit circle. But for m large enough, the lengths of the intervals of the partition by the points of $\mathcal{P}^{(i)}(m,2)$ and $\mathcal{P}^{(j)}(m,2)$ are small enough (see for instance [5, 2]) to ensure the result.

We thus get $\beta_i = \beta_j$, for every $(i,j) \in \mathbb{Z}^2$, which completes the proof. ■

5 Application to plane partitions

The aim of this section is first to introduce a notion of three-dimensional palindromes for plane partitions and second, to compute the number of these palindromes of given size in a discrete plane.

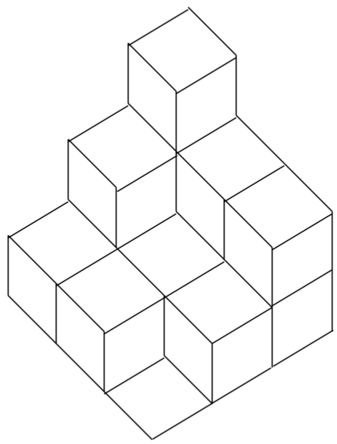
5.1 Definitions

Let us recall the definition of a discrete plane. Let $\mathcal{P}$ be the plane of equation $z = -\alpha x - \beta y + \gamma$, with $\alpha > 0$, $\beta > 0$, and $1, \alpha, \beta$ rationally independent. One can associate to such a plane a discrete plane by approximating $\mathcal{P}$ by unit square faces: consider the set of the integral translates of the fundamental cube that intersect the lower half-space $z < -\alpha x - \beta y + \gamma$; we call **discretization** of the plane $\mathcal{P}$ the **discrete plane** P defined as the boundary of this set. Note that the heights of the horizontal faces are of the form

$$H_{i,j} = \lfloor -i\alpha - j\beta + \gamma \rfloor, \text{ where}(i,j) \in \mathbb{Z}^2.$$

A **plane partition** PP on the discrete plane P is defined as

$$PP = PP(m,n;i,j) := \begin{matrix} H_{i-m+1,j} - H_{i,j} & \cdots & H_{i,j} - H_{i,j} \\ \vdots & & \vdots \\ H_{i-m+1,j-n+1} - H_{i,j} & \cdots & H_{i,j-n+1} - H_{i,j} \end{matrix}$$



2	1	0
2	1	1
3	2	1

Figure 1: An example of local configuration.

We say that (i,j) is an index of occurrence of the plane partition PP and that PP has size (m,n). As $\alpha > 0$ and $\beta > 0$, note that the heights in $PP(m,n;i,j)$ decrease with the indices in line and column.

A **local configuration** on the discrete plane P is defined as the **three-dimensional geometric realization** of a plane partition $PP(m,n;i,j)$, i.e., the translate by vector $(0,0,-H_{i,j})$ of the subset of the discrete plane P of those points of coordinates (x,y,z), with $i-m+1 \leq x \leq i$ and $j-n+1 \leq y \leq j$. Two plane partitions such that $PP(m,n;i,j) = PP(m,n;i',j')$ have the same geometric realization, up to a translation. Note that we represent in Figure 1 and 2 the local configurations up to a rotation, for visualization reasons.

We now define the **centrosymmetric complement** of the plane partition $PP(m,n;i,j)$ as the following plane partition:

$$\tilde{PP}(m,n;i,j) := \begin{array}{ccc} H_{i-m+1,j-n+1} - H_{i,j-n+1} & \cdots & H_{i-m+1,j-n+1} - H_{i-m+1,j-n+1} \\ \vdots & & \vdots \\ H_{i-m+1,j-n+1} - H_{i,j} & \cdots & H_{i-m+1,j-n+1} - H_{i-m+1,j} \end{array}$$

We define the **centrosymmetric complement** of a local configuration as the geometric realization of the centrosymmetric complement of the corresponding plane partition.

In geometrical terms, the three-dimensional realization of the centrosymmetric complement of the plane partition $PP = PP(m,n;i,j)$ is the image by the central symmetry with respect to the center of the box of size $(m, n, (H_{i-m+1,j-n+1} - H_{i,j}))$ containing the three-dimensional realization of the plane partition PP.

Definition 5 *A plane partition of size (m,n) is said to be* **a palindromic plane partition** *if*

$$PP(m,n;i,j) = \tilde{PP}(m,n;i,j).$$

In this case, the corresponding geometric realization is called **a palindromic local configuration**.

These plane partitions are called $m-n$ local configurations in the work of Reveillès (see [20]). Similarly Gérard investigates the arrangement of the different combinatorial structures contained in a digital hyperplane (see [16]). The notion of palindromic local configurations also appears in the work of Vittone and Chassery [22] under the name of neutral $m-n$ cubes.

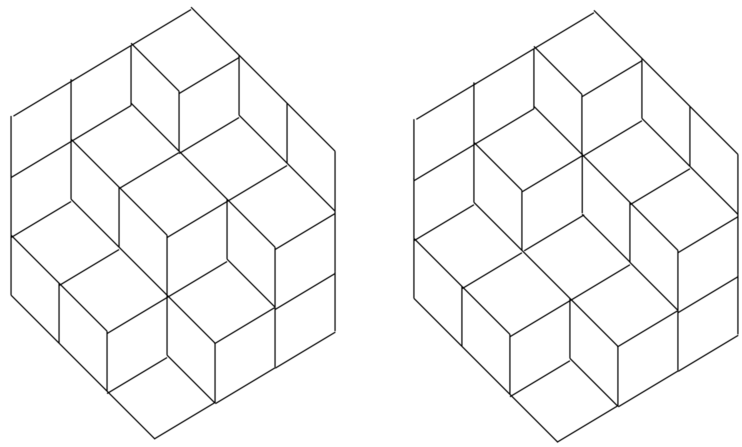


Figure 2: A local configuration and its complement.

5.2 Palindromic local configurations in discrete planes

Theorem 4 *Let $\mathcal{P}$ be the plane of equation $z = -\alpha x - \beta y + \gamma$, with $\alpha > 0$, $\beta > 0$, and $1, \alpha, \beta$ rationally independent. Let P be its discretization. The number of palindromic local configurations of size (m, n) in P equals two if m or n is even and one, otherwise.*

Proof Let us reduce this statement to a combinatorial problem. Consider the sequence of heights reduced modulo 2 in P, i.e., the double sequence $U = (U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ with values in the alphabet $\{0, 1\}$ defined by

$$\forall (i,j) \in \mathbb{Z}^2,\ U_{i,j} = H_{i,j} \text{ modulo } 2.$$

It is shown in [24] that this sequence satisfies

$$\forall (i,j) \in \mathbb{Z}^2,\ \forall l \in \{0,1\},\ (U_{i,j} = l \Longleftrightarrow -i\alpha/2 - j\beta/2 + \gamma/2 \in I_l \text{ modulo } 1),$$

where $I_1 = [0, 1/2[$ and $I_0 = [1/2, 1[$.

Let $\alpha' = \alpha/2$, $\beta' = \beta/2$. From Lemma 1, the intervals $I(W)$ associated to the rectangular factors of size (m, n) are in one-to-one correspondence with the intervals of endpoints in the set

$$\mathcal{P}_{m,n} = \{-i\alpha' - j\beta' + l/2,\ 0 \le i \le m-1,\ 0 \le j \le n-1,\ l = 0, 1\}.$$

Let us apply Theorem 2. The sequence U has exactly two centrosymmetric factors of size (m, n). Indeed, the points

$$y_{m,n} = \frac{-(m-1)\alpha' - (n-1)\beta' + 1/2}{2} \quad \text{and} \quad y_{m,n} + 1/2$$

are middle points of two distinct intervals (see Figure 3).

Consider now the symmetry $S'_{m,n}$ with respect to $y_{m,n} + 1/4$. We have for $l = 0, 1$:

$$S'_{m,n}(R_\alpha^{-i+1} R_\beta^{-j+1}\ \mathring{I}_l) = R_\alpha^{-m+i} R_\beta^{-n+j}\ \mathring{I}_{l'},$$

where $l' = 1 - l$. Hence this symmetry maps the interior of the interval $I(W)$ to the interior of the interval $I(\tilde{\overline{W}})$, where $\tilde{\overline{W}}$ denotes the image of W under centrosymmetric-complementation. The points $y_{m,n} + 1/4$ and $y_{m,n} + 3/4$ belong to the set $\mathcal{P}_{m,n}$ if and only if m and n are odd. In other words, there exist two factors of size (m,n) invariant under centrosymmetric complementation if and only if m or n is even, and otherwise, there is no such factor.

¿From [24], given any plane partition, there exist (at least) two indices of occurrence (i,j) and (i',j'), one with $H_{i,j}$ even and the other with $H_{i',j'}$ odd. This follows from the fact that the set of factors of size (m,n) of the sequence U is stable by complementation, which corresponds to the translation of $1/2$; for more details, see [24]. Let $PP = PP(m,n;i,j)$ be a plane partition. Let us associate to this plane partition the rectangular factor W of the sequence U which appears at index $(i-m+1, j-n+1)$. Note that if one considers another index of occurrence (i',j') of the plane partition PP, then the factor of size (m,n) of U which appears at index $(i'-m+1, j'-n+1)$ is either equal to W or to $\overline{W}$, since $U_{k,l} = H_{k,l}$ modulo 2, for every (k,l). The plane partition PP is palindromic if and only if W is centrosymmetric, if $H_{i-m+1,j-n+1} - H_{i,j}$ is even (respectively stable by centrosymmetric complementation, if $H_{i-m+1,j-n+1} - H_{i,j}$ is odd) (see Figure 3). Note that if W is a centrosymmetric factor (respectively stable by centrosymmetric complementation), then $\overline{W}$ is also centrosymmetric (respectively stable by centrosymmetric complementation). Hence, if m and n are odd, there exists at most one palindromic local configuration of size (m,n), and at most two, otherwise.

Conversely, let us exhibit palindromic local configurations. Let W be a factor of size (m,n) of the sequence U. Let (i,j) be one index of occurrence. One can "reconstruct" a plane partition from this binary word W by considering the plane partition $PP(m,n;i+m-1,j+n-1)$. As α and β are irrational numbers, the "steps" $H_{k,l} - H_{k+1,l}$ (respectively $H_{k,l} - H_{k,l+1}$) take only two values (one odd and one even value); for more details, see [24]. Hence if one considers two distinct indices of occurrence of W, one gets the same plane partition. Furthermore, W and $\overline{W}$ define the same plane partition. Hence, if W is centrosymmetric or stable by centrosymmetric complementation, then the plane partition $PP(W)$ is palindromic.

Suppose m and n are odd. Let W be one of the two palindromes of size (m,n) in U. The second palindrome factor of size (m,n) is thus its complement $\overline{W}$. Consider an index (i,j) of occurrence of W. The plane partition $PP(m,n;i+m-1,j+n-1) = PP(W)$ associated with W is palindromic.

Suppose that m or n is even. The two palindromes W_1 and $\overline{W_1}$ produce a palindromic plane partition $PP(W_1)$. There exist furthermore exactly two factors W_2 and $\overline{W_2}$ of size (m,n) invariant by centrosymmetric complementation. Hence, the plane partition $PP(W_2)$ is also palindromic. ■

References

[1] P. ALESSANDRI *Codages de rotations et basses complexités*, Université Aix-Marseille II, Thèse, 1996.

[2] P. ALESSANDRI, V. BERTHÉ *Three distance theorems and combinatorics on words*, l'Enseignement Mathématique **44** (1998), 103–132.

[3] J.-P. ALLOUCHE, *Sur la complexité des suites infinies*, Bull. Belg. Math. Soc. Simon Stevin **1** (1994), 133–143.

[4] P. ARNOUX, C. MAUDUIT, I. SHIOKAWA, J. TAMURA *Complexity of sequences defined by billiards in the cube*, Bull. Soc. math. France **122** (1994), 1–12.

[5] V. BERTHÉ *Fréquences des facteurs des suites sturmiennes*, Theoret. Comput. Sci. **165** (1996), 295–309.

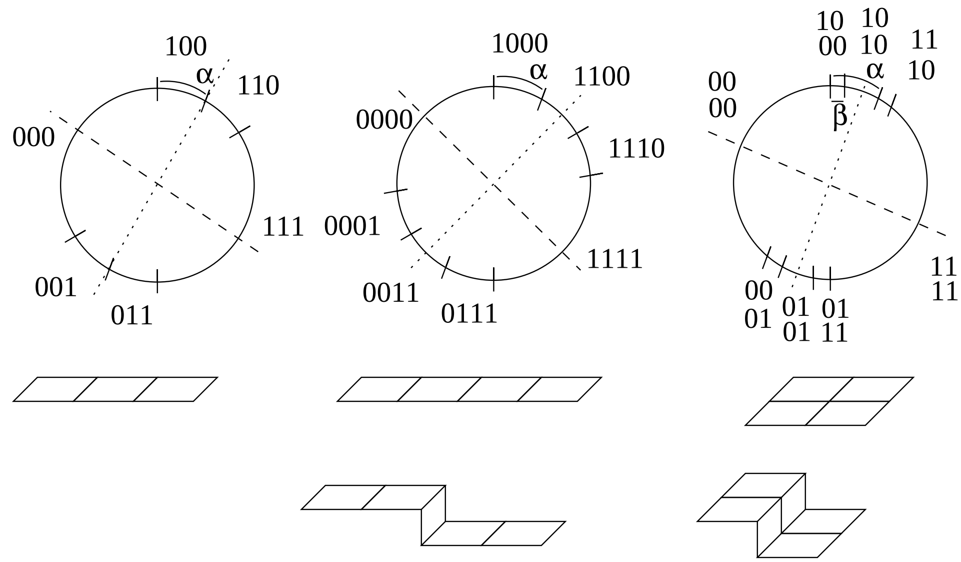


Figure 3: Examples of partitions, words and 3d-palindromes.

[6] V. BERTHÉ, L. VUILLON *Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences*, Discrete Math. **223** (2000), 27–53.

[7] V. BERTHÉ, L. VUILLON *Suites doubles de basse complexité*, J. Théor. Nombres Bordeaux **12** (2000), 179–208.

[8] J. BERSTEL *Tracé de droites, fractions continues et morphismes itérés*, in Mots, Lang. Raison. Calc., Éditions Hermès, Paris (1990), 298–309.

[9] J. BERSTEL *Recent results in Sturmian words*, Developments in Language Theory II (Dassow, Rozenberg, Salomaa eds) World Scientific 1996, 13–24.

[10] A. de LUCA *Sturmian words: structure, combinatorics and their arithmetics*, Theoret. Comput. Sci. **183** (1997), 45–82.

[11] A. de LUCA et F. MIGNOSI *Some combinatorial properties of Sturmian words*, Theoret. Comput. Sci. **136** (1994), 361–385.

[12] X. DROUBAY *Palindromes in the Fibonacci word*, Inform. Process. Lett. **55** (1995), 217–221.

[13] X. DROUBAY, G. PIRILLO *Palindromes and Sturmian words*, Theoret. Comput. Sci. **223** (1999), 73–85.

[14] X. DROUBAY, J. JUSTIN, G. PIRILLO *Episturmian words and some constructions of de Luca and Rauzy*, prepublication 98.

[15] S. FERENCZI *Complexity of sequences and dynamical systems*, Discrete Math. **206** (1999), 145–154.

[16] I. GÉRARD *Local configurations of digital hyperplanes*, (Bertrand, Couprie, and Poroton eds) Lecture Notes Comput. Sci. **1568** (1999), 65–75.

[17] M. LOTHAIRE *Algebraic Combinatorics on Words*, Chapter 2: Sturmian words, by J. Berstel and P. Séébold.

[18] M. MORSE, G. A. HEDLUND *Symbolic dynamics*, Amer. J. Math. **60** (1938), 815–866.

[19] M. MORSE, G. A. HEDLUND *Symbolic dynamics II: Sturmian trajectories*, Amer. J. Math. **62** (1940), 1–42.

[20] J.-P. REVEILLÈS *Géométrie discrète, calcul en nombres entiers et algorithmique*, Université Louis Pasteur, Strasbourg, Thesis, 1991.

[21] G. ROTE *Sequences with subword complexity* $2n$, J. Number Th. **46** (1996), 196–213.

[22] J. VITTONE,. CHASSERY $(n,m)-$*Cubes and Farey nets for naive planes understanding*, (Bertrand, Couprie, and Perroton eds) Lecture Notes Comput. Sci. **1568** (1999), 76–87.

[23] L. VUILLON *Combinatoire des motifs d'une suite sturmienne bidimensionnelle,* Theoret. Comput. Sci. **209** (1998), 261–285.

[24] L. VUILLON *Local configurations in a discrete plane*, Bull. Belgian Math. Soc. **6** (1999), 625–636.

[25] Z.-Y. WEN, Z.-X. WEN *Some properties of the singular words of the Fibonacci word*, Europ. J. Combinatorics **15** (1994), 587–598.