Mnëv's Universality Theorem revisited

Jürgen Richter-Gebert

July 10, 2002

Abstract

This article presents a complete proof of Mnëv's Universality Theorem and and a first complete proof of Mnëv's Universal Partition Theorem for oriented matroids. The Universality Theorem states that, for every primary semialgebraic set V there is an oriented matroid $\mathcal{M}$, whose realization space is stably equivalent to V. The Universal Partition Theorem states that, for every partition $\mathcal{V}$ of $\mathbb{R}^n$ induced by m polynomial functions $f_1, \ldots, f_m$ with integer coefficients there is a corresponding family of oriented matroids $(\mathcal{M}_\sigma)_{\sigma \in \{-1,0,+1\}^m}$ such that the collection of their realization spaces is stably equivalent to the family $\mathcal{V}$.

1 Introduction

Oriented matroids (also known as *combinatorial geometries*) form a combinatorial model for point configurations in linear vector spaces. The oriented matroid $\mathcal{M}(\boldsymbol{P})$ of a (linear) point configuration $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ in $\mathbb{R}^d$ is a list of all the partitions of points in $\boldsymbol{P}$ induced by linear hyperplanes in $\mathbb{R}^d$. The *realization space* of the oriented matroid $\mathcal{M}(\boldsymbol{P})$ is the space of all point configurations $\boldsymbol{P}'$ in $\mathbb{R}^d$ that generate the same partitions as $\boldsymbol{P}$ does. In particular, oriented matroids contain complete information about the incidence structure of $\boldsymbol{P}$ (i.e. information about which point sets in $\boldsymbol{P}$ are linearly dependent). One can as well describe oriented matroids on the level of (signed) *affine* point configurations and partitions by *affine* hyperplanes. For a broad introduction to the theory of oriented matroids see [1] and [2].

One of the most prominent and surprising facts of oriented matroid theory is Mnëv's *Universality Theorem* [4, 5]. It states that the realization spaces of oriented matroids can become arbitrarily complicated. More precisely:

> For every primary semialgebraic set V defined over $\mathbb{Z}$ there is an oriented matroid M, whose realization space is stably equivalent to V.

The key idea behind the proof is to encode an arbitrary system of polynomial equations and strict inequalities $\mathcal{E}$ into the geometry of a point configuration $\boldsymbol{P}(\mathcal{E})$ (the space of

Supported by the DFG Gerhard-Hess-Forschungsförderungspreis of G.M. Ziegler

Keywords: Oriented matroids, Realization spaces, universality, semialgebraic sets, stable equivalence, Partition theorem

1991 *Mathematics Subject Classification:* Primary 52B40; Secondary 14P10, 51A25, 52B30.

solutions $V(\mathcal{E})$ of $\mathcal{E}$ is a primary semialgebraic set). This can be done in principle by the use of the classical *von Staudt constructions* of projective geometry. The different solutions of $\mathcal{E}$ correspond then to (classes of) different realizations of $\mathcal{M}(\boldsymbol{P}(\mathcal{E}))$). The main problem that occurs in such a kind of construction is to arrange the configuration in a way such that the combinatorial structure is stable for all possible solutions of $\mathcal{E}$.

Mnëv's original proof is very technical and complicated. It consists of an algebraic part, that models the computation of the system $\mathcal{E}$ by elemtentrary operations. The "space of computations" of a given system of polynomials is subdivided into certain strata Ξ_i. Each stratum represents a set of computations in which the total orders of results of sub-computations are fixed. A tricky perturbation technique is used to show that one can arrange the computation in a way such that there is a stratum Ξ that is already equivalent to $V(\mathcal{E})$. After this, von Staudt constructions are used to encode this into a geometric configuration. The control in the total order of intermediate results translates to the control of the oriented matroid of the point configuration. The values of the variables are encoded by the positions of points on a line ℓ with respect to a *projective scale* given by the position of points $\mathbf{0}$, $\mathbf{1}$ and $\boldsymbol{\infty}$ on ℓ.

Much clarification was achieved by an alternative proof of Shor. A sketch of this proof is given in [7]. Shor replaced the algebraic part of Mnëv's proof by a normal form algorithm:

> Every primary semialgebraic set V over $\mathbb{Z}$ is stably equivalent to a semialgebraic set $V' \in \mathbb{R}^n$ whose variables $1 = x_1 < x_2 < \ldots < x_n$ are totally ordered and for which all defining equations have the form $x_i + x_j = x_k$ or $x_i \cdot x_j = x_k$ for certain $1 \leq i \leq j < k \leq n$.

This normal form is achieved by certain replacement rules. The problem there is to choose the replacement rules in a way that preserves the algebraic structure of the solution space and successively creates a total order on all variables that are involved.

It is one of the scopes of this paper is to give an even simpler proof of Mnëv's Universality Thoerem. In our proof, we also aim for a normal form in which the variables are strictly totally ordered. However, the elementary operations that occur are *quadrilateral set relations* rather than additions and multiplications. The values of the variables are retrieved by interpreting them with respect to *individual* projective scales, one scale for each variable and one scale for each elementary operation. By this the total ordering of the variables can be achieved by lining up the different projective scales one after the other as clusters of points on a line. The different scales are linked by quadrilateral set relations.

As a second scope of this paper, this construction provides a proof of an even stronger theorem: The *Universal Partition Theorem* as it was originally stated by Mnëv in [6]. While the Universality Theorem is concerned with a single primary semialgebraic set, the Universal Partition Theorem is concerned with a family of such sets that are nested in a complicated way. The main statement of the Universal Partition Theorem is that (up to stable equivalence) one can recover certain families of semialgebraic sets as a family of realization spaces of oriented matroids. These realization spaces are nested in a way that is topologically equivalent to the nesting of the original semialgebraic sets. For a long time no proof of this fact was available. A proof of a slightly weaker statement has recently been provided by Günzel [3]. The idea of using individual projective scales for each variable was already used there. However, in Günzel's approach stable equivalence is only obtained up to the product of the semialgebraic sets with a non-controllable smooth manifold. We here prove the original statement as it was claimed by Mnëv. The main difficulty in the proof of such a kind of statement is that one has to keep track of many semialgebraic sets at the same time, encoding them all into the same geometric situation.

2 Basic definitions and main result

2.1 Oriented matroids and chirotopes

Oriented matroids and their close relatives *chirotopes* encode the combinatorial structure of point configurations in $\mathbb{R}^n$ (compare [1]). We can restrict ourselves to the case of 2-dimensional affine point configurations, and start with the basic definitions on the level of chirotopes.

Definition 2.1. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in \mathbb{R}^{2 \cdot n}$ be a finite 2-dimensional point configuration on an index set X. We set $\boldsymbol{p}_i = (x_i, y_i)$, for $i = 1, \ldots, n$. The map

$$\begin{array}{rcl} \chi\colon X^3 & \longrightarrow & \{-1, 0, 1\} \\ (i, j, k) & \longmapsto & \det \begin{pmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{pmatrix} \end{array}$$

is called the *chirotope* of $\boldsymbol{P}$. A point configuration $\boldsymbol{P}$ is called a *realization* of a map $\chi\colon X^3 \longrightarrow \{-1, 0, 1\}$ if $\chi^{\boldsymbol{P}} = \chi$. The triple (i, j, k) is called a *basis* of χ if $\chi(i, j, k) \neq 0$. If $\chi(i, j, k) = +1$, then the *realization space* $\mathcal{R}(\chi, (i, j, k))$ is the set of all realizations $\boldsymbol{P}$ of χ with $\boldsymbol{p}_i = (0, 0)$, $\boldsymbol{p}_j = (1, 0)$, and $\boldsymbol{p}_k = (0, 1)$.

The map $\chi^{\boldsymbol{P}}$ indicates for any triple of points whether they are clockwise oriented, counterclockwise oriented, or collinear. An alternating map $\chi\colon X^3 \to \{-1, 0, 1\}$ is called *non-realizable* if there is no point configuration $\boldsymbol{P}$ with $\chi^{\boldsymbol{P}} = \chi$.

In general an alternating map $\chi\colon X^3 \to \{-1, 0, 1\}$ is a chirotope when additional conditions (known as Grassmann-Plücker relations) are satisfied. We will omit the detailed definition here. However, these relations are always fulfilled if χ comes from a point configuration. All sign maps, that play a rôle in this article are indeed chirotopes.

2.2 Semialgebraic sets and stable equivalence

Let $\Omega = (\{f_i\}_{0<i\leq r}, \{g_i\}_{0<i\leq s}, \{h_i\}_{0<i\leq t})$ be a finite collection of polynomials

$$f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t \in \mathbb{Z}[x_1, \ldots, x_n]$$

with integer coefficients. The *basic semialgebraic set* $V(\Omega) \in \mathbb{R}^n$ is the set

$$V = V(\Omega) := \Big\{\, \boldsymbol{x} \in \mathbb{R}^n \;\Big|\; \begin{array}{l} f_i(\boldsymbol{x}) = 0 \text{ for } i = 1, \ldots, r \\ g_i(\boldsymbol{x}) < 0 \text{ for } i = 1, \ldots, s \\ h_i(\boldsymbol{x}) \leq 0 \text{ for } i = 1, \ldots, t \end{array} \Big\}$$

defined as the solution of a finite number of polynomial equations and polynomial inequalities. A basic semialgebraic set is called *primary*, if the defining equations contain no non-strict inequalities (i.e. $t = 0$ in the above notion). Thus, for example, the set $\{0,1\}$ and the open interval $]0,1[\subset \mathbb{R}$ are primary semialgebraic sets, while the closed interval $[0,1]$ is a basic semialgebraic set in $\mathbb{R}$ that is not primary. Semialgebraic sets form a general setting to define subsets of $\mathbb{R}^n$ by polynomial equations and inequalities. To see that the realization space of a chirotope is a (primary) semialgebraic set one checks that the realization space is the set of all matrices $\boldsymbol{Q} \in \mathbb{R}^{2\cdot n}$ for which some entries are fixed, and the determinants have to be positive, negative or zero (compare [1]).

For an exact statement of a Universal Partition Theorem, we have to introduce the concept of simultaneous stable equivalence of a family of basic semialgebraic sets. We call a finite (ordered) collection $(V_1,\dots,V_m)$ of pairwise disjoint basic semialgebraic sets $V_i \subseteq \mathbb{R}^n$ a *semialgebraic family.* Let $\mathcal{V} = (V_1,\dots,V_m)$ with $V_i \subseteq \mathbb{R}^n$ and let $\mathcal{W} = (W_1,\dots,W_m)$ with $W_i \subseteq \mathbb{R}^{n+d}$ be semialgebraic families with $\pi(W_i) = V_i$ for $i = 1,\dots,n$, where π is the canonical projection $\pi : \mathbb{R}^{n+d} \to \mathbb{R}^d$ that deletes the last d coordinates. $\mathcal{V}$ is a *stable projection* of $\mathcal{W}$ if for $i = 1,\dots,m$ the W_i have the form

$$W_i = \Big\{(\boldsymbol{v},\boldsymbol{v}') \in \mathbb{R}^{n+d} \mid \boldsymbol{v} \in V_i \text{ and } \phi_j(\boldsymbol{v})\cdot\boldsymbol{v}' > 0;\ \psi_k(\boldsymbol{v})\cdot\boldsymbol{v}' = 0 \text{ for all } j \in X;\ k \in Y\Big\}.$$

Here X and Y denote finite (possibly empty) index sets. For $j \in X$ and $k \in Y$ the functions ϕ_j and ψ_k are polynomial functions

$$\begin{aligned} \phi_j &= (\phi_j^1,\dots,\phi_j^d) : \mathbb{R}^n \to (\mathbb{R}^d)^* \quad \text{with } \phi_j^l \in \mathbb{Z}[x_1,\dots,x_n] \quad \text{and} \\ \psi_k &= (\psi_k^1,\dots,\psi_k^d) : \mathbb{R}^n \to (\mathbb{R}^d)^* \quad \text{with } \psi_k^l \in \mathbb{Z}[x_1,\dots,x_n], \end{aligned}$$

that associate a linear functional on $\mathbb{R}^d$ to every element of $\mathbb{R}^n$.

Two semialgebraic families $\mathcal{V}$ and $\mathcal{W}$ are *rationally equivalent* if there exists a homeomorphism $f\colon \bigcup_{i=1}^m V_i \to \bigcup_{i=1}^m W_i$ such that both f and f^{-1} are rational functions and $f(V_i) = W_i$ for $i = 1,\dots,m$.

<u>Definition 2.2.</u> Two semialgebraic families $\mathcal{V}$ and $\mathcal{W}$ are *stably equivalent*, denoted $\mathcal{V} \approx \mathcal{W}$, if they are in the same equivalence class with respect to the equivalence relation generated by stable projections and rational equivalence.

<u>Definition 2.3.</u> If $\mathcal{V} = (V)$ and $\mathcal{W} = (W)$ are semialgebraic families consisting of a single semialgebraic set and $\mathcal{V} \approx \mathcal{W}$, then V is *stably equivalent* to W.

<u>Definition 2.4.</u> Let $V \in \mathbb{R}^n$ be a primary semialgebraic set and let $f_i,\dots,f_m \in \mathbb{Z}[x_1,\dots,x_n]$ be polynomial functions on $\mathbb{R}^n$. For $\sigma \in \{-1,0,+1\}^m$ we abbreviate

$$V_\sigma := \{\boldsymbol{v} \in V \mid \mathbf{sign}(f_i(\boldsymbol{v})) = \sigma_i \text{ for all } i = 1,\dots,m\}.$$

The collection of primary semialgebraic sets $(V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ is called a *partition* of V.

In particular, partitions are special semialgebraic families. Moreover, we can recover any primary semialgebraic set $W \in \mathbb{R}^n$ as a component of a partition of $\mathbb{R}^n$. To see this, we simply consider the partition that is induced by the polynomials of the defining equations $f_1(\boldsymbol{v}) = 0, \ldots, f_k(\boldsymbol{v}) = 0$ and defining strict inequalities $f_{k+1}(\boldsymbol{v}) > 0, \ldots, f_m(\boldsymbol{v}) > 0$ of W. We then have $W = V_\sigma$ with $\sigma = (\underbrace{0, \ldots, 0}_{k \text{ times}}, \underbrace{+1, \ldots, +1}_{m-k \text{ times}})$.

Figure 1 illustrates a partition $\mathcal{V}$ of $\mathbb{R}^2$ that is induced by two linear polynomials (the two lines) and two quadratic polynomials (the circle and the hyperbola). The elements of $\mathcal{V}$ that have maximal dimension are marked by the letters $a, \ldots, m$. In particular the sets $a, b, \ldots, e$ are disconnected.

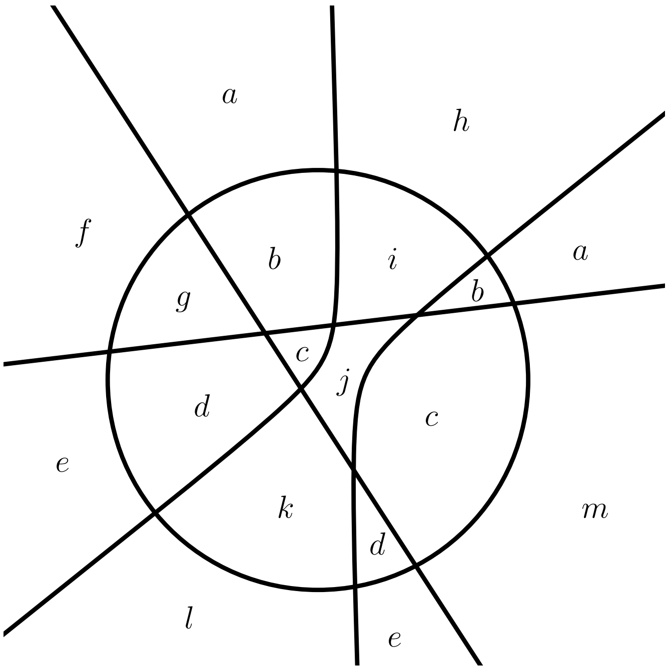

Figure 1

The Universal Partition Theorem for oriented matroids may now be stated as follows.

<u>THEOREM 2.5.</u> *For any partition $\mathcal{V} = (V_\sigma)_{\sigma \in \{-1,0,+1\}^m}$ of $\mathbb{R}^n$ there is an index set X and a collection of alternating sign maps $(\chi_\sigma\colon X^3 \to \{-1,0,1\})_{\sigma \in \{-1,0,+1\}^m}$ with common basis B such that*

$$\mathcal{V} \approx (\mathcal{R}(\chi_\sigma, B))_{\sigma \in \{-1,0,+1\}^m}.$$

In particular this theorem implies the Universality Theorem for oriented matroids. For of the sets V_σ in $\mathcal{V}$ the above statement ensures that there is a χ_σ such that $\mathcal{R}(\chi_\sigma, B)$ is stably equivalent to $V_s igma$. Since every semialgebraic set can occur as a component of a semialgebraic family, this implies the original Universality Theorem.

We will give the proof of the Universal Partition Theorem for oriented matroids in the next few sections. The proof given here does not rely on a Shor normal form.

3 The algebraic part

3.1 Projective scales and quadrilateral sets

The *cross ratio* $(\boldsymbol{p}_1, \boldsymbol{p}_2 | \boldsymbol{p}_3, \boldsymbol{p}_4)$ of four points on a line ℓ is defined by

$$(\boldsymbol{p}_1, \boldsymbol{p}_2 | \boldsymbol{p}_3, \boldsymbol{p}_4) \;=\; \frac{|\boldsymbol{p}_1, \boldsymbol{p}_3| \cdot |\boldsymbol{p}_2, \boldsymbol{p}_4|}{|\boldsymbol{p}_1, \boldsymbol{p}_4| \cdot |\boldsymbol{p}_2, \boldsymbol{p}_3|}.$$

Here $|\boldsymbol{p}_i, \boldsymbol{p}_j|$ denotes the (oriented) euclidean distance of $\boldsymbol{p}_i$, and $\boldsymbol{p}_j$ and we assume that none of the points lies at infinity. The cross ratio is invariant under projective transformations; therefore we can also extend the above definition to the case where one or more of the points lies at infinity. In particular, if ℓ is equipped with a euclidean scale then we have $(x, \mathbf{1} | \mathbf{0}, \boldsymbol{\infty}) = x$. In other words, after the choice of three distinct positions of $\mathbf{0}, \mathbf{1}$ and $\boldsymbol{\infty}$ on a line, the cross ratio exactly measures the euclidean scale. We say that $\mathbf{0}, \mathbf{1}$ and $\boldsymbol{\infty}$ define a *projective scale*. We will encode our variables x_i by points on a line with respect to individual projective scales $\mathbf{0}^i, \mathbf{1}^i$ and $\boldsymbol{\infty}^i$ for each variable. Our points on a line are related by *quadrilateral set relations*. These relations are our key to translate arithmetic relations into geometric conditions. Like cross ratios they form projective invariants.

<u>Definition 3.1.</u> A 6-tuple of numbers $(a, b, c, d, e, f) \in \mathbb{R}^6$ forms a *quadrilateral set* provided

$$q(a, b, c, d, e, f) := \frac{(a-d)(c-f)(e-b)}{(a-f)(c-b)(e-d)} = 1.$$

The number $q(a, b, c, d, e, f)$ is called the *quadrilateral ratio* and is a projective invariant for six points $a, \ldots, f$ on a line. In particular we get

$$\lim_{e \to \infty} q(a, b, c, d, e, f) = \frac{(a-d)(c-f)}{(a-f)(c-b)} \qquad \text{and} \qquad \lim_{e,f \to \infty} q(a, b, c, d, e, f) = \frac{a-d}{c-b}.$$

Five numbers in a quadrilateral set uniquely determine the sixth number. Since the formula q involves only differences between the indeterminants, we have

$$q(a, b, c, d, e, f) = q(a+t, b+t, c+t, d+t, e+t, f+t),$$

for any number $t \in \mathbb{R}$. This effect can be also considered as a consequence of the fact that translation by a scalar t is a projective transformation. We cover the limit case by setting $\infty + t = \infty$. In particular, addition and multiplication is modeled by the following quadrilateral set relations:

$$q(x, y, 0, x+y, \infty, \infty) = 1, \qquad q(x, y, 1, x \cdot y, \infty, 0) = 1.$$

For the Universal Partition Theorem, we make use of the following quadrilateral set relations and their translates:

$$\begin{aligned} q(0, 0, -x, x, \infty, \infty) &= 1 \\ q(0, y, -x, x+y, \infty, \infty) &= 1 \\ q(1, 1, 1/x, x, \infty, 0) &= 1 \\ q(1, y, 1/x, x \cdot y, \infty, 0) &= 1 \end{aligned}$$

We will use quadrilateral set relations as basic operations and obtain normal forms that are closely related to the original system of polynomials. In the next few sections we aim for a normal form that has the following properties.

- The variables that occur are strictly totally ordered.
- The only relations that occur are quadrilateral set relations and "perturbed" quadrilateral set relations.
- Additions, multiplications, and equalities are represented by quadrilateral set relations.
- Inequalities are represented by perturbed quadrilateral set relations.

The resulting normal form may be considered as a structure in which each variable, each elementary addition or multiplication, each equation, and each inequality is represented by a "cluster" of points that forms an individual projective scale and encodes the corresponding relation. Within each cluster the points are totally ordered by construction. We obtain an overall total ordering on the variables by simply lining up all the individual clusters one after the other. The elements of different clusters will be linked by quadrilateral set relations.

3.2 Computations of polynomials

The first steps of our approach to a normal form follow the approach of Günzel [3]. We first observe that it is sufficient to restrict our considerations to partitions of the set $(1,\infty)^n$ consisting of all vectors of $\mathbb{R}^n$ with all entries strictly greater than 1.

LEMMA 3.2. *For any partition $\mathcal{V} = (V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ of $\mathbb{R}^n$ there is a partition $\mathcal{W} = (W_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ of $(1,\infty)^{2n}$ such that $\mathcal{V} \approx \mathcal{W}$.*

PROOF. Let $f_1(\boldsymbol{x}),\dots,f_m(\boldsymbol{x}) \in \mathbb{Z}[x_1,\dots,x_n]$ be the defining equations of $\mathcal{V}$. Then the defining equations of $\mathcal{W}$ are $f_1(\boldsymbol{u}-\boldsymbol{v}),\dots,f_m(\boldsymbol{u}-\boldsymbol{v}) \in \mathbb{Z}[u_1,\dots,u_n,v_1,\dots,v_n]$ together with the inequalities $u_i > 1$ and $v_i > 1$ for all $i = 1,\dots,n$. We show this by proving $\mathcal{V} \approx_1 \mathcal{W}' \approx_2 \mathcal{W}$, where $\approx_1$ is a stable projection and $\approx_2$ is a rational equivalence. The partition $\mathcal{W}'$ is a partition of the semialgebraic set

$$\widetilde{\mathbb{R}}^{2n} := \Big\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{x} \in \mathbb{R}^n \text{ and } y_i > x_; \, y_i > -x_i \text{ for } i = 1,\dots,n\Big\}.$$

The defining equations for $\mathcal{W}'$ are given by the polynomials

$$f_1(\boldsymbol{x}),\dots,f_m(\boldsymbol{x}) \in \mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n].$$

By definition this gives a stable projection from $\mathcal{W}'$ to $\mathcal{V}$. The rational equivalence between $\mathcal{W}'$ and $\mathcal{W}$ is given by the affine transformation

$$\begin{aligned} T\colon \mathbb{R}^{2n} &\longrightarrow \mathbb{R}^{2n} \\ (\boldsymbol{x},\boldsymbol{y}) &\longmapsto (\frac{\boldsymbol{x}+\boldsymbol{y}}{2}+1, \frac{-\boldsymbol{x}+\boldsymbol{y}}{2}+1). \end{aligned}$$

We have $T(\widetilde{\mathbb{R}}^{2n}) = (1,\infty)^{2n}$. Furthermore, if $(\boldsymbol{u},\boldsymbol{v}) = T(\boldsymbol{x},\boldsymbol{y})$, we get $\boldsymbol{x} = \boldsymbol{u} - \boldsymbol{v}$. $\square$

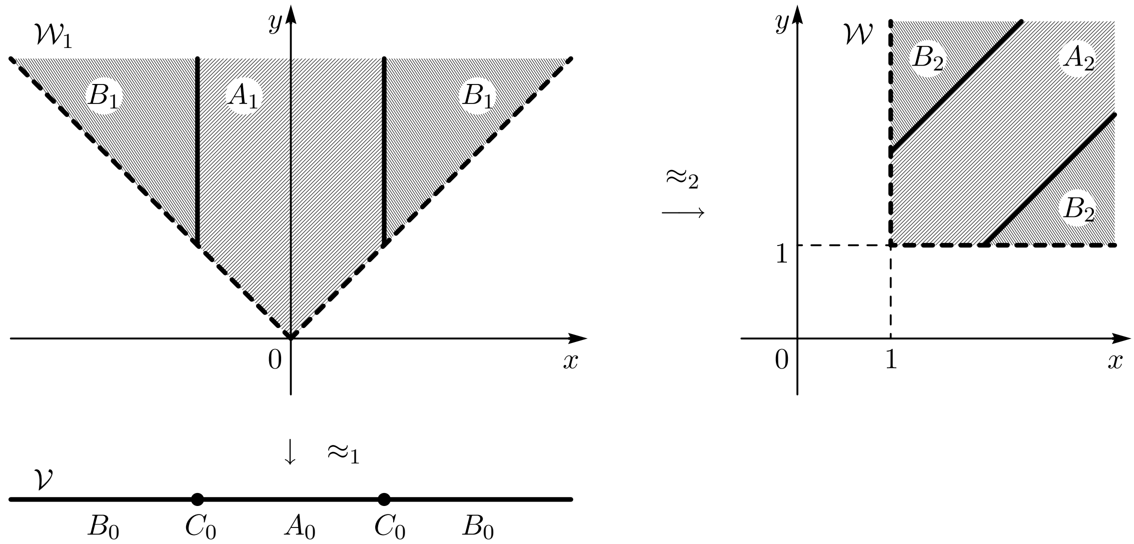

Figure 2

Figure 2 illustrates the equivalences of the last proof in a simple example. The original partition is 1-dimensional and is defined by one polynomial $f(x) = x^2 - 1$. The corresponding partition consists of three semialgebraic sets

$$A_0 = \{x \mid x^2 - 1 < 0\}, \quad B_0 = \{x \mid x^2 - 1 > 0\}, \quad C_0 = \{x \mid x^2 - 1 = 0\}.$$

A_0 is just an open line segment, B_0 consists of two open intervals and C_0 consists of two points. The stable projection $\approx_1$ increases the dimension of each of the sets by one. The semialgebraic sets that are stably equivalent to A_0 and B_0 are marked A_1 and B_1, respectively. By tha stable projection $\approx_1$ the wedge $\widetilde{\mathbb{R}}^2$ is mapped onto $\mathbb{R}$. Finally, the affine transformation T rotates and shifts $\widetilde{\mathbb{R}}^2$ and maps it to $(1,\infty)^2$. The corresponding cells of full dimensions are marked A_2 and B_2.

Now we consider a partition $\mathcal{V}$ of $(1,\infty)^n$ by polynomials $f_1,\ldots,f_m \in \mathbb{Z}[x_1,\ldots,x_n]$. Each such polynomial f_i can be written as $f_i^+ - f_i^-$ with $f_i^+, f_i^- \in \mathbb{N}[x_1,\ldots,x_n]$. The polynomial f_i^+ collects all terms of f_i with positive coefficients, the polynomial f_i^- collects all terms with negative coefficients.

The computation of a polynomial $f(x_1,\ldots,x_n) \in \mathbb{N}[x_1,\ldots,x_n]$ can be decomposed into a sequence of elementary additions and multiplications that start from the values $1, x_1, \ldots, x_n$ and compute f step by step. We consider f as bracketed in a way in which each bracket contains exactly one elementary addition or multiplication. The integral coefficients may be decomposed into a summation of ones. For instance, the polynomial $x^2 + 3y^3$ can be bracketed as

$$x^2 + 3y^3 \;=\; ((x \cdot x) + (((1+1)+1) \cdot ((y \cdot y) \cdot y))).$$

For each bracket $\alpha = (\ldots)$ that occurs we introduce an additional variable V_α. In our example we get

$$V_{x^2} = x \cdot x,\ V_{y^2} = y \cdot y,\ V_{y^3} = V_{y^2} \cdot y,$$

$$V_2 = 1 + 1,\ V_3 = V_2 + 1,\ V_{3y^3} = V_3 \cdot V_{y^3},\ V_{x^2+3y^3} = V_{x^2} \cdot V_{3y^3}.$$

We call such a decomposition of f a *computation of* f. If all variables $x_1, \ldots, x_n$ are greater than one, then (since the coefficients of f are also greater than one) the values of all intermediate variables V_α are greater than one, as well. Compared to the Shor normal form, a computation of a polynomial does not provide any control on the order of intermediate variables (except that they are all greater than one). For instance if we consider $x + y = z$ we do not know whether $x < y$ or $y < x$.

<u>LEMMA 3.3.</u> *For each polynomial $f \in \mathbb{N}[x_1, \ldots, x_n]$ there is*

- *an integer $k \geq n$,*
- *additional variables $x_{n+1}, \ldots, x_k$,*
- *a collection $A \subset \{$"$x + y = z$" $\mid x, y, z \in (0, 1, x_1, \ldots, x_k)\}$ of additions, and*
- *a collection $M \subset \{$"$x \cdot y = z$" $\mid x, y, z \in (0, 1, x_1, \ldots, x_k)\}$ of multiplications*

such that for every "input" $(x_1, \ldots, x_n) \in (1, \infty)^n$ a solution of the system $A \cup M$

- *determines all variables $x_1, \ldots, x_k$,*
- *satisfies $(x_1, \ldots, x_k) \in (1, \infty)^k$, and*
- *satisfies $x_k = f(x_1, \ldots, x_n)$.*

PROOF. The statement is a summary of the properties of the stepwise decomposition of f into elementary additions and multiplications we just presented. $\square$

<u>REMARK 3.4.</u> Under complexity theoretical aspects, the decomposition of integers into a summation of ones is by far not optimal. It creates a number of intermediate variables that is exponential in the bit coding length of the integers. If one is heading for good complexity bounds, one can bypass this problem by choosing a more efficient coding method that does use additions *and* multiplications. Using binary coding mechanisms one can in principle achieve that the number of intermediate variables is linear in the bit coding length of the integers.

We now model a computation of polynomials by introducing "clusters" of variables that are related by quadrilateral set operations.

DEFINITION 3.5. Let $Y = (0, 1, x_1, \dots, x_k)$ be a set of formal variables. A *cluster* $\mathcal{B} = (X, \mathcal{Q})$ is a pair consisting of an ordered collection $X = (x'_0, \dots, x'_l)$ of variables that satisfies $\{0, 1\} \subseteq \{x'_0, \dots, x'_l\} \subseteq Y$ and a (possibly empty) set $\mathcal{Q}$ of signed quadrilateral set relations

$$\mathcal{Q} \subset \Big\{ \text{“}\mathbf{sign}(q(a, b, c, d, e, f) - 1) = \sigma\text{”} \mid a, b, c, d, e, f \in X \text{ and } \sigma \in \{-1, 0, +1\} \Big\}.$$

We set $\mathcal{B}^{\downarrow} = x'_0$ and $\mathcal{B}^{\uparrow} = x'_l$. Concrete values $x'_0, \dots, x'_l \in \mathbb{R}$ *satisfy* a cluster $\mathcal{B} = (X, \mathcal{Q})$ if the are totally ordered by $x'_0 < x'_1 < \dots x'_l$, and they fulfill the requirements in $\mathcal{Q}$.

We now consider the original (partition defining) polynomial system $f_1, \dots, f_m \in \mathbb{Z}(x_1, \dots, x_n)$, where the input values of the x_i may be taken in $(1, \infty)$. The next lemma is proved by modeling a computation of all polynomials $f_1, \dots, f_m$ by a collection of clusters.

LEMMA 3.6. *For any partition $\mathcal{V} = (V_\sigma)_{\sigma \in \{-1,0,+1\}^m}$ of $(1, \infty)^n$ induced by polynomials $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ there exists integers K and N, such that the semialgebraic family*

$$\begin{aligned}\mathcal{W} = \Big(\Big\{\boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{R}^N \mid \quad & 1 < y_1 < y_2 < \dots < y_N \text{ and} \\ & q_i(\boldsymbol{y}) = 1 \text{ for } i = 1, \dots, K \text{ and} \\ & \mathbf{sign}(\overline{q}_i(\boldsymbol{y}) - 1) = \sigma_i \text{ for } i = 1, \dots, m\Big\}\Big)_{\sigma \in \{-1,0,+1\}^m}\end{aligned}$$

is stably equivalent to $\mathcal{V}$. Here q_i and $\overline{q}_i$ denote quadrilateral ratios on certain 6-tuples in $\{-1, 0, 1, y_1, \dots, y_N, \infty\}$.

PROOF. For each polynomial $f_1^+, f_1^-, \dots, f_m^+, f_m^-$ we consider the decomposition into elementary additions and multiplications given in Lemma 3.3. We collect all n_V intermediate variables that occur in all calculations into a set $Y = \{0, 1, x_1, \dots, x_n, x_{n+1}, \dots, x_{n_V}\}$. We assume that there are $n_A + n_M$ elementary operations altogether; n_A additions and n_M multiplications. Furthermore, we have to implement $n_S = m$ sign conditions. We fix a certain sign vector $\sigma \in \{-1, 0, +1\}^m$ and consider the following collection of clusters:

(0): We set $\mathcal{B}_0 = ((V^0_{-1}, 0^0, 1^0),\ \ \{q(0^0, 0^0, V^0_{-1}, 1^0, \infty, \infty) = 0\})$.

(V): For each variable x_i with $i \in \{x_1, \dots, x_{n_V}\}$ we introduce a cluster

$$\begin{aligned}\mathcal{B}_i = (\ (\ & V^i_{-x_i}, 0^i, V^i_{1/x_i}, 1^i, x^i_i\ ), \\ & \{\ q(0^i, 0^i, V^i_{-x_i}, x^i_i, \infty, \infty) = 1, \\ & \ \ q(1^i, 1^i, V^i_{1/x_i}, x^i_i, \infty, 0^i) = 1\ \}\ ).\end{aligned}$$

(A): For the i-th ($i \in \{1, \dots, n_A\}$) elementary addition $x_a + x_b = x_c$ we set $j = n_V + i$ and introduce a cluster

$$\begin{aligned}\mathcal{B}_j = (\ (\ & V^j_{-x_a}, 0^j, x^j_b, x^j_c\ ), \\ & \{\ q(x^j_b, 0^j, V^j_{-x_a}, x^j_c, \infty, \infty) = 1\ \}\ ).\end{aligned}$$

(M): For the i-th ($i \in \{1, \ldots, n_M\}$) elementary multiplication $x_a \cdot x_b = x_c$ we set $j = n_V + n_A + i$ and introduce a cluster

$$\begin{aligned} \mathcal{B}_j = (\; (\; 0^j, V^j_{1/x_a}, 1^j, x^j_b, x^j_c \;), \\ \{ \; q(1^j, x^j_b, V^j_{1/x_a}, x^j_c, \infty, 0^j) = 1 \; \} \;). \end{aligned}$$

(S): For the i-th ($i \in \{1, \ldots, n_S\}$) polynomial $f_i = f_i^+ - f_i^-$ with $x_a = f_i^+$ and $x_b = f_i^-$ we set $j = n_V + n_A + n_M + i$ and introduce a cluster

$$\begin{aligned} \mathcal{B}_j = (\; (\; V^j_{-x_a}, 0^j, x^j_b \;), \\ \{ \; \mathbf{sign}(q(0^j, 0^j, V^j_{-x_a}, x^j_b, \infty, \infty) - 1) = \sigma_i \; \} \;). \end{aligned}$$

In the above description 0^i and 1^i represent formal variables that (together with ∞) form a projective scale for the cluster $\mathcal{B}_i$. We set $M = n_V + n_A + n_M + n_S$. For each pair of cluster variables W^i, W^j with $W \in \{0, 1, V_{-1}, x_1, V_{-x_1}, V_{1/x_1}, \ldots, , x_k, V_{-x_k}, V_{1/x_k}\}$ and $i, j \in \{0, \ldots, N\}$ with $i \neq j$, we also introduce a quadrilateral relation

$$q(0^i, W^j, 0^j, W^i, \infty, \infty) = 1. \tag{$*$}$$

These last *linking relations* force that $W^i - 0^i = W^j - 0^j$, i.e., the variable W has identical values with respect to the projective scales of the clusters $\mathcal{B}_i$ and $\mathcal{B}_j$. Obviously, some of these linking relations are redundant. However, since we do not aim for complexity theoretic results, we may neglect this redundancy. Finally, we identify $\mathcal{B}^{\uparrow}_{i-1} = \mathcal{B}^{\downarrow}_i$ for $i = 1, \ldots, N$, i.e., the "last" point of the cluster $\mathcal{B}_i$ is the "first" point of the cluster $\mathcal{B}_{i+1}$. We set $0^0 = 0$ and $1^0 = 1$ and $x^i_i - 0^i = x_i$.

For a given "input" $\boldsymbol{x} = (x_1, \ldots, x_n) \in (1, \infty)^n$ the conditions of the clusters **(0)** – **(M)** together with the linking relations ($*$) uniquely determine all variables that occur in the clusters. Moreover, within each cluster $\mathcal{B}_i = ((x'_0, x'_1, \ldots, x'_l), \mathcal{Q}_i)$ the variables are consistently ordered: $x'_0 < x'_1 < \ldots < x'_l$. Thus all cluster variables taken together form a strictly ordered chain $V_{-1} < 0 < 1 < y_1 < y_2 < \ldots < y_N$. Finally, the requirements given by the clusters in **(S)** encode the sign conditions σ on the polynomials $f_1, \ldots, f_m$. Thus all quadrilateral set conditions are satisfied simultaneously if and only if $\boldsymbol{x} \in V_\sigma$. We denote the set of all points $(y_1, \ldots, y_n)$ in $\mathbb{R}^N$ satisfying all of the above requirements by W_σ. The values of $(y_1, \ldots, y_n) \in W_\sigma$ are given by a rational function $f : V_\sigma \to W_\sigma$ in the values $(x_1, \ldots, x_n) \in V_\sigma$. The inverse function f^{-1} is given by $x_i = x^i_i - 0^i$. Thus V_σ and W_σ are rationally equivalent. Since f and f^{-1} are the same for all $\sigma \in \{-1, 0, +1\}^m$ we have $\mathcal{V} \approx \mathcal{W}$. □

We close this section by exemplifying the concept of clusters in the easiest possible example. We consider the partition of $\mathbb{R}$ given by the polynomial $g(x) = x + 1$. Applying the technique of Lemma 3.2, this translates to a partition of $(1, \infty)^2$ defined by the polynomial $f(x_1, x_2) = x_1 - x_2 + 1$. A decomposition of this polynomial into a positive and a negative part consists only of elementary expressions already. We set

$$f^+(x_1, x_2) = x_1 + 1 = x_3 \quad \text{and} \quad f^-(x_1, x_2) = x_2.$$

So, we have to encode three variables, one addition and one comparison. The next figure shows the corresponding collection of cluster variables. The variables are illustrated as points in their final total ordering. Observe that the last variable of the cluster $\mathcal{B}_i$ and the first variable of the cluster $\mathcal{B}_{i+1}$ are identified.

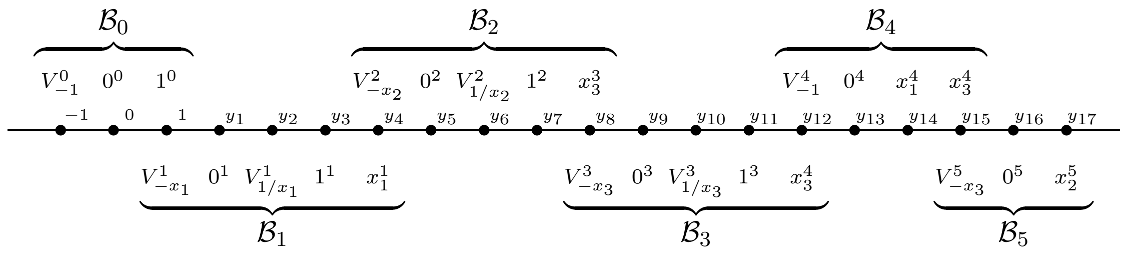

Figure 3

After relabelling, this translates into 17 variables $y_1, \ldots, y_{17}$ altogether (plus additional points for -1, 0, 1, and ∞). The clusters **(0)** – **(M)** taken together force 8 quadrilateral set relations on these variables. After deleting redundancies, the linking relations force 8 additional quadrilateral set relations. Finally, the sign condition is expressed as one perturbed quadrilateral set relation $\mathbf{sign}(q(y_{16}, y_{16}, y_{15}, y_{17}, \infty, \infty) - 1) = \sigma$.

4 The geometric part

We now conclude the proof of the Universal Partition Theorem for oriented matroids, by encoding the quadrilateral sets into a point configuration.

Proof of Theorem 2.5. With the method of encoding a partition into a collection of clusters as given by Lemma 3.6, our proof is almost finished. It remains to apply a standard construction that encodes quadrilateral sets into point configurations and thereby fixes the orientations. This process was already described by Mnëv [4, 5] and by Shor [7]. In principle this can be done by a slightly refined "von Staudt construction". We consider our variables together with $\mathbf{-1}$, $\mathbf{0}$, $\mathbf{1}$, and $\boldsymbol{\infty}$ as points on a line and consider the projective scale defined by $\mathbf{0}$, $\mathbf{1}$, and $\boldsymbol{\infty}$. We have to implement the quadrilateral set relations of Lemma 3.6 by suitable point configurations. This can be done by intersecting the sides of a complete quadrilateral with a line (that is where the name "quadrilateral set" comes from). Up to translation, the only cases that occur are

$$\begin{array}{rclcrcl} q(0,0,-x,x,\infty,\infty) & = & 1, & \qquad & q(0,y,-x,x+y,\infty,\infty) & = & 1, \\ q(1,1,1/x,x,\infty,0) & = & 1, & \qquad & q(1,y,1/x,x\cdot y,\infty,0) & = & 1, \end{array}$$

and the sign conditions $\mathbf{sign}(q(0,0,-x,y,\infty,\infty) - 1) = \sigma$.

The corresponding point configurations are shown in Figure 4 In each of the cases four new points labeled $a, \ldots, d$ are introduced. Each of these configurations contains an "information line" ℓ, on which the values of the variables are represented by points. Points a and b lie on a line ℓ'. The position of the points c and d as well as all orientations are fixed by the incidence structure of the configuration and the ordering of the points on ℓ and on ℓ'. Observe that, by choosing b very close to a one can achieve that the points b, c, and d are in an ε-neighborhood of a, for arbitrary small $\varepsilon > 0$.

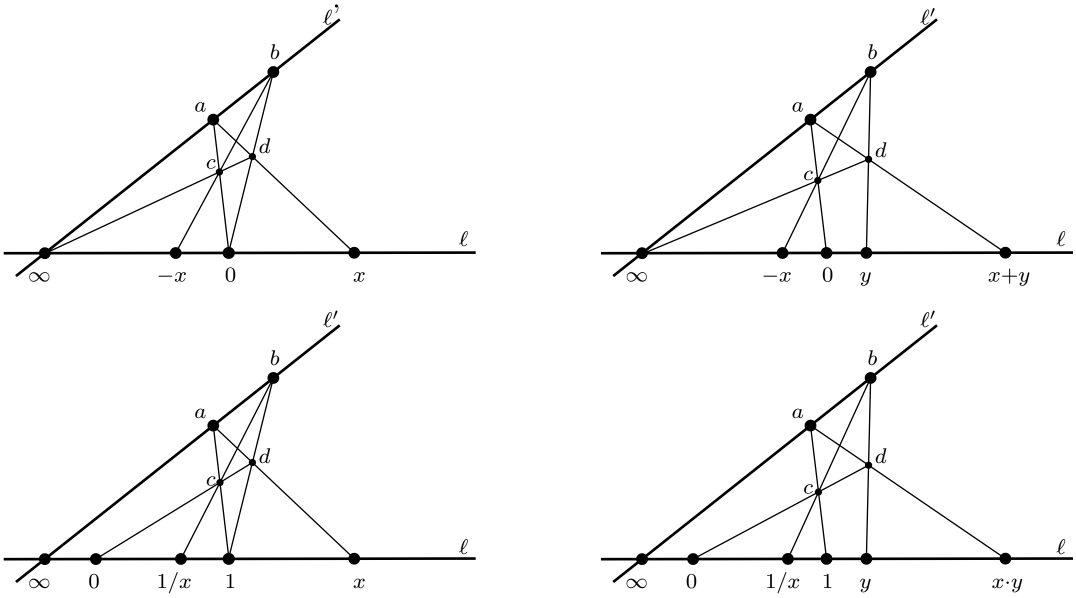

Figure 4

The sign conditions can be encoded into perturbed versions of the right upper configuration of Figure 4. For this the points c, d, and $\boldsymbol{\infty}$ are no longer assumed to be collinear. Depending on the orientation of the triple $(c, d, \boldsymbol{\infty})$ we get either $x < y$, $x > y$, or $x = y$. The two perturbed situations are shown in Figure 5.

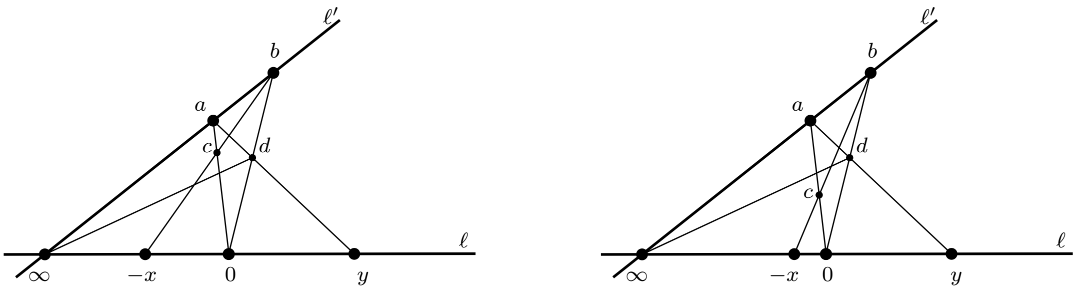

Figure 5

Finally, all the necessary quadrilateral set relations have to be encoded into one point configuration, thereby fixing all the orientations. We start with the line ℓ, which we identify with the x-axis, and a line ℓ', which we identify with the y-axis. The origin is labeled $\boldsymbol{\infty}$. All points in the final configuration will have non-negative x and y coordinates. First the quadrilateral set relations $q_1, \ldots, q_K$ corresponding to our classes **(0)** – **(M)** together with the linking relations $(*)$ are encoded. We take concrete values $\boldsymbol{y} = (y_1, y_2, \ldots, y_N)$ that satisfy $q_1, \ldots, q_K$, and are totally ordered $y_1 < y_2 < \ldots < y_N$. By a projective transformation we map the points $-1, 0, 1, y_1, \ldots, y_N, \infty$ onto the line ℓ

such that ∞ becomes the origin and and the sequence $-1, 0, 1, y_2, \ldots, y_N$ starts to the right from the origin. We assume that the point labeled -1 has coordinates $(1, 0)$. Now we iteratively add the point configurations that encode the quadrilateral set relations. For each q_i we introduce four new points $a_i, \ldots, d_i$ in the following way. We set $a_1 = (0, 1)$ and $b_1 = (0, 2)$. The points c_1 and d_1 are chosen according to the point configuration of Figure 4 that encodes q_1. For $i = 2, \ldots, K$ we set $a_i = (0, \alpha_i)$ where $\alpha_i > 0$ is chosen large enough that it is above all the lines spanned by points that are already constructed (except for line ℓ' itself on which a_i lies). Now let $b_i = (0, \alpha_i + \varepsilon_i)$ where $\varepsilon_i > 0$ is a very small number. We construct c_i and d_i according to the configuration that encodes q_i. Choosing ε_i small enough we can achieve that all lines that are spanned by points that are so far constructed (except of those passing through the origin) have negative slope (i.e., they intersect ℓ' above the origin). For small ε_i it happens as well that the signs of all orientations involving one of the points b_i, c_i, or d_i are completely determined by the type of the quadrilateral set relation q_i and do not depend on the actual choice of $\boldsymbol{y}$. This can be shown by a simple case analysis. The obstructions to the choices of the α_i and the ε_i can be expressed as stable projections. Finally, we have to add our m sign conditions. For each of the relations $\overline{q}_i$ in Lemma 3.6, with $i = 1, \ldots, m$, we add a configuration of the incidence type as given in Figure 5. We do this by adding points $a_{K+i}, \ldots, d_{K+i}$ in the same way as described above. We call the oriented matroid of the resulting point configuration $\chi[\boldsymbol{y}]$. The construction fixes all orientation except of $\chi[\boldsymbol{y}](\boldsymbol{\infty}, c_{K+i}, d_{K+i})$, for $1 = 1, \ldots, m$. These signs are dependent on the choice of our input parameters $\boldsymbol{y}$. The construction forces

$$\chi[\boldsymbol{y}](\boldsymbol{\infty}, c_{K+i}, d_{K+i}) = \mathbf{sign}(\overline{q}_i(\boldsymbol{y}) - 1),$$

for $i = 1, \ldots, m$. Hence the orientations of $\chi[\boldsymbol{y}]$ depend just on the choice of the input parameters $\boldsymbol{y}$. We now define χ_σ as an alternating sign function on

$$X^3 = \{\mathbf{-1}, \mathbf{0}, \mathbf{1}, y_1, \ldots, y_N, \boldsymbol{\infty}, a_1, b_1, \ldots, c_{K+m}, d_{K+m}\}^3.$$

The labels of X are equipped with a total order "$\prec$". The map χ_σ is the determined by its values $\chi_\sigma(i, j, k)$ with $i \prec j \prec k$. For this we define

$$\chi_\sigma(i, j, k) = \begin{cases} \chi[\boldsymbol{y}](i, j, k), & \text{if } (i, j, k) \neq (\boldsymbol{\infty}, c_l, d_l),\ K < l \leq K + m, \\ \sigma_i, & \text{if } (i, j, k) = (\boldsymbol{\infty}, c_l, d_l),\ K < l \leq K + m. \end{cases}$$

By construction this choice of χ_σ has the desired properties: if $\boldsymbol{y} \in W_\sigma$ (with the W_σ of Lemma 3.5), then we have $\chi_\sigma = \chi[\boldsymbol{y}]$; if W_σ is empty, then χ_σ is non-realizable. As common basis of all χ_σ we choose $B = (\boldsymbol{\infty}, \mathbf{-1}, a_0)$. The desired stable equivalence between the realization spaces $\mathcal{R}(\chi_\sigma, B)$ and the sets W_σ are given by the stable projections that determine the values α_i and ε_i and the rational equivalence that describes the actual construction of the points. □

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White & G.M. Ziegler, *Oriented Matroids, Encyclopedia of Mathematics*, Vol. 46, Cambridge University Press 1993.

[2] J. Bokowski & B. Sturmfels, *Computational Synthetic Geometry,* Lecture Notes in Mathematics, **1355**, Springer-Verlag, Berlin Heidelberg 1989.

[3] H. Günzel, *The universal partition theorem for oriented matroids,* Preprint TH-Aachen (1994), 29p.; *Discrete Comput. Geometry*, to appear.

[4] N.E Mnëv, *On manifolds of combinatorial types of projective configurations and convex polyhedra, Soviet Math. Doklady*, **32** (1985), 335–337.

[5] N.E Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties,* in: Viro, O.Ya. (ed.): *Topology and Geometry — Rohlin Seminar*, Lecture Notes in Mathematics **1346**, Springer, Heidelberg 1988, 527–544.

[6] N.E Mnëv, *The universality theorems on the oriented matroid stratification of the space of real matrices,* in: *Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift* (P. Gritzmann, B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc., Providence, RI, **6** (1991), 237–243.

[7] P. Shor, *Stretchability of pseudolines is NP–hard,* in: *Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift* (P. Gritzmann, B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc., Providence, RI, **4** (1991), 531–554.

Technische Universität Berlin,
FB Mathematik, Sekr. 6-1,
Straße des 17. Juni 136,
D-10623 Berlin, Germany
e-mail: `richter@math.tu-berlin.de`