A variational approach to one dimensional Phase Unwrapping

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Abstract

Over the past ten years, many phase unwrapping algorithms have been developed and formulated in a discrete setting. Here we propose a variational formulation to solve the problem. This continuous framework will allow us to impose some constraints on the smoothness of the solution and to implement them efficiently. This method is presented in the one dimensional case, and will serve as a basis for future developments in the real 2D case.

1. Introduction

Interferometric radar techniques have been widely used to produce high-resolution ground digital elevation models (DEM's). In space-borne SAR (Synthetic Aperture Radar) interferometry, two images of the same scene are acquired using two different geometries. The phase difference between the registered images (the so-called *interferogam*) is related to a desired physical quantity of interest such as the surface topography. The phase difference can be registered only modulo 2π and interferometric techniques consist mainly of recovering the absolute phase (the unwrapped phase) from the registered one (the wrapped phase).
Over the past ten years, many phase unwrapping algorithms have been developed. Commonly they first differentiate the phase field and subsequently reintegrate it, adding the missing integral cycles to obtain a more continuous result. Three basic classes are representative of these algorithms:

• Residue-cut "tree" algorithms: Branch cut methods (Goldstein et al.,1988 [4]) unwrap by integrating the estimated neighboring pixel differences of the unwrapped phase along paths that avoid the regions where these estimated differences are inconsistent.

• Least-square algorithms were adapted to SAR interferometry by Guiglia and Romero [5]. They applied a mathematical formalism to determine the vector gradient of the phase field and then integrate it subject to regularity constraints.

• Franceschetti and Fornaro [3], and recently Maitre et Lyuboshenko [6] proposed a phase unwrapping algorithms based on the Green function. This kind of methods have been shown to be mathematically equivalent to least-squares solution, but differ in computational efficiency.
It is interesting to note that most of these algorithms are formulated in a discrete setting. Instead, we investigate a continuous formulation of the problem with a variational approach. Traditionally developed in physics and mechanics, this framework has been intensively applied in image analysis since the 1990s. The reasons for this are that the models can be justified theoretically, and that suitable numerical schemes exist for computing the solution.

Section 2 of this paper is dedicated to the mathematical statement of the unwrapping problem. The two following sections deal with the proposed variational approach depending on the regularity of the wrapped phase. Section 3 deals with the case of regular unwrapped phase corresponding to a smooth ground surface. The unwrapped phase is obtained by including in the energy regularity constraints at phase jumps in the wrapped signal. To achieve these constraints, we propose to minimize a set of functionals. In section 4, we allow the absolute phase to have discontinuities apart from the phase jumps. In order to preserve these ground discontinuities, the minimization is now performed in BV (space of functions of bounded variation). Numerical schemes and results on a synthetic noisy wrapped signal are given.

2. 1D phase unwrapping

Let $\varphi_m : I \subset \mathbb{R} \longrightarrow [-\pi, \pi[$ be the given phase difference. The unwrapping problem consists in finding a function $\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$W(\varphi) = \varphi_m \tag{1}$$

where W is the wrapping operator defined by:

$$\begin{aligned} W : \mathbb{R} &\longrightarrow [-\pi,\ \pi[\\ u &\longmapsto W(u) \\ \forall u \in \mathbb{R},\ \exists k_u &\in \mathbb{Z} : W(u) = u - 2k_u\pi. \end{aligned}$$

k_u is an integer such that $u - 2k_u\pi \in [-\pi, \pi[$. Let φ_m be the observation obeying $\forall x \in I =]a, b[\subset \mathbb{R}$, $\varphi_m(x) \in [-\pi, \pi[$. The observation φ_m is a function defined modulo 2π on a bounded interval I of $\mathbb{R}$. By definition φ and

φ_m share the same regularity properties except at points x where the value of k_u changes. We denote by

$$P_{\varphi_m} = \{d_0, ..., d_D\} \quad (2)$$

these points, referred to as phase jumps. In this work, we assume that the phase jumps d_k are known. For real applications, they will be estimated from the bi-dimensional interferogram.

In order to retrieve the unwrapped phase φ from the observation φ_m from equation (1), it is necessary to add the following conditions, for every $d_k \in P_{\varphi_m}$:

$$\varphi_{|]d_k, d_{k+1}[} \text{ and } \varphi_{m|]d_k, d_{k+1}[} \text{ share the same regularity} \quad (3)$$

$$\begin{array}{l} \varphi \text{ is continuous at } d_k \\ \varphi'^{+}(d_k) = \varphi_m'^{+}(d_k) \text{ and } \varphi'^{-}(d_k) = \varphi_m'^{-}(d_k) \end{array} \quad (4)$$

Let us comment on these conditions:

- We assume in condition (4) that the limits $\lim\limits_{x \to d_k^+} \varphi'_m$ and $\lim\limits_{x \to d_k^-} \varphi'_m$ exist.
- In (3), the regularity of φ_m depends on the observed areas. Two cases may arise. The first is when the only discontinuities of φ_m are due to phase jumps and so we suppose that φ_m is piecewise differentiable on I. The second is when the wrapped function φ_m admits a finite number of ground discontinuities $\{t_j\}_j$. We assume that they are located between two phase jumps. The $\{t_j\}_j$ model ground discontinuities with a jump small enough not to generate points d_k or unrecoverable areas. Note that the phase discontinuities $\{d_k\}_k$ are assumed to be known, while the ground discontinuities $\{t_j\}_j$ are unknown.

Now the goal is to describe a functional to be minimized with respect to φ such that φ satisfies conditions (1), (3) and (4).

3. Variational approach

3.1. Description of the functional

In this section we assume that:

- φ_m has no ground discontinuities.
- $\varphi_m \in \mathcal{C}^p(I \backslash P_{\varphi_m})$ for some $p > 1$.

In this case the distributional derivative $D\varphi_m$ can be decomposed as the sum of a regular measure (absolutely continuous with respect to the Lebesgue measure) and a singular measure: $D\varphi_m = \underbrace{\varphi'_m dx}_{\text{regular}} + \underbrace{D_s\varphi_m}_{\text{singular}}$.

We consider a quadratic functional $E(\varphi)$ defined by:

$$E(\varphi) = E_1(\varphi) + \lambda E_2(\varphi) \quad (5)$$

where:

$$E_1(\varphi) = \int_I |\varphi' - \varphi'_m|^2 dx \quad (6)$$

$$E_2(\varphi) = \int_I (\varphi - \varphi_m)^2 \chi_{I_0} dx \quad (7)$$

and $I_0 =]a_0, b_0[\subset I$ is an open interval of reference that does not contain any phase jumps of φ_m. χ_{I_0} is the characteristic function of I_0.
The problem is to minimize $E(\varphi)$ on the Sobolev space $H^1(I) = \{f \in L^2(I) \text{ such that } Df \in L^2(I)\}$.
Let us comment on the energy terms:

- $E_1(\varphi)$ is the data term which contains requirements (1) and (3). Indeed $W(\varphi) = \varphi_m$ implies that $\varphi' = \varphi'_m$ on $I \backslash P_{\varphi_m}$.
- $E_2(\varphi)$ is a reference term. It will play an important role in the uniqueness of the phase φ we want to retrieve. We can prove the existence and uniqueness of a minimizer $\varphi \in H^1(I)$ of (5). Moreover in one dimension we know that functions in $H^1(I)$ are equal to a continuous function almost everywhere. (4).

Therefore we introduce a third term that will enforce the regularity of the unwrapped function φ at every phase jump point d_k. The idea is to consider:

$$E_3(\varphi) = \sum_{d_k \in P_{\varphi_m}} e_{3,k}(\varphi) \quad \text{with:}$$

$$e_{3,k}(\varphi) = \left(\varphi'^{+}(d_k) - \varphi_m'^{+}(d_k)\right)^2 + \left(\varphi'^{-}(d_k) - \varphi_m'^{-}(d_k)\right)^2 + \left(\varphi^{+}(d_k) - \varphi^{-}(d_k)\right)^2.$$

However, since we are looking for a function φ in $H^1(I)$, we cannot give a meaning to the derivatives φ'^{+} and φ'^{-}. So we introduce $\alpha \in \mathbb{R}^+$ to approximate the backward and forward derivatives of φ at points d_k. We define:

$$E_{3,\alpha}(\varphi) = \sum_{d_k \in P_{\varphi_m}} e_{3,k}^{\alpha}(\varphi) \quad \text{with:} \quad (8)$$

$$\begin{aligned} e_{3,k}^{\alpha}(\varphi) = & \left(\frac{\varphi(d_k + \alpha) - \varphi(d_k)}{\alpha} - \varphi'_m(d_k + \alpha)\right)^2 \\ & + \left(\frac{\varphi(d_k) - \varphi(d_k - \alpha)}{\alpha} - \varphi'_m(d_k - \alpha)\right)^2 \\ & + \left(\varphi(d_k + \alpha) - \varphi(d_k - \alpha) - \alpha\left(\varphi'_m(d_k + \alpha) + \varphi'_m(d_k - \alpha)\right)\right)^2. \end{aligned}$$

The term $\alpha\left(\varphi'_m(d_k + \alpha) + \varphi'_m(d_k - \alpha)\right)$ is introduced for numerical reasons. Notice that it tends to 0 when α tends to 0. Hence the functional (5) is approximated by the functional E_α defined by:

$$E_\alpha(\varphi) = E_1(\varphi) + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi).$$

To find the unwrapped phase, we minimize the functional E_α over $H^1(I)$, with no constraints.

In the next section, we establish the optimality conditions satisfied by the solution φ_α.

3.2. Optimality conditions for φ_α

The Euler-Lagrange equations satisfied by φ_α are:

$$-\left(\varphi'_\alpha - \varphi'_m\right)' + \lambda(\varphi_\alpha - \varphi_m)\,\chi_{I_0} = 0, \text{ over } I \tag{9}$$

$$\varphi'_\alpha(a) = \varphi'_m(a) \quad \text{and} \quad \varphi'_\alpha(b) = \varphi'_m(b) \tag{10}$$

And for all $d_k \in P_{\varphi_m}$: (11)

$$\varphi_\alpha(d_k{-}\alpha) - \varphi_\alpha(d_k) = \alpha\,\varphi'_m(d_k{-}\alpha)$$
$$\varphi_\alpha(d_k{+}\alpha) - \varphi_\alpha(d_k) = \alpha\,\varphi'_m(d_k{+}\alpha)$$
$$\varphi_\alpha(d_k{+}\alpha) - \varphi_\alpha(d_k{-}\alpha) = \alpha\left(\varphi'_m(d_k{+}\alpha) + \varphi'_m(d_k{-}\alpha)\right)$$

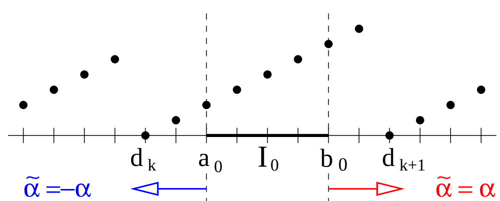
We can prove the existence and uniqueness in $H^1(I)$ of the solution φ_α of equation (9) with the boundary conditions (10) and the limit conditions (11). The behavior of φ_α as α tends to 0, will be studied in a forthcoming paper.

3.3. Discretization and numerical result

Equation (9) is a linear parabolic equation and thus there is no real difficulty in implementing it. After standard finite difference discretization, an iterative method such as Jacobi or Gauss-Seidel can be used to solve the linear sustem. The difficulty comes from the approximation of condition (11), where we need to take into account a direction of propagation with respect to the reference interval I_0. This is intuitive since the values on I_0 are constrained and influence both sides in a certain direction.
Thus we propose the following discretization. For φ^n_α given, we have:

$$\varphi^{n+1}_\alpha(d_k{+}\widetilde{\alpha}) = \varphi^n_\alpha(d_k{+}2\widetilde{\alpha}) - \varphi_m(d_k{+}2\widetilde{\alpha}) + \varphi_m(d_k{+}\widetilde{\alpha})$$
$$\varphi^{n+1}_\alpha(d_k) = \varphi^n_\alpha(d_k{-}\widetilde{\alpha}) - \varphi_m(d_k{-}2\widetilde{\alpha}) + \varphi_m(d_k{-}\widetilde{\alpha})$$
$$\varphi^{n+1}_\alpha(d_k{-}\widetilde{\alpha}) = \varphi^n_\alpha(d_k{+}\widetilde{\alpha}) + \varphi'_m(d_k{+}\widetilde{\alpha}) + \varphi'_m(d_k{-}\widetilde{\alpha}) \tag{12}$$

$$\text{where} \quad \widetilde{\alpha} = \begin{cases} \alpha & \text{if } d_k > b_0 \\ -\alpha & \text{if } d_k < a_0. \end{cases}$$



Notice that the discretization of the derivatives $\varphi'_m(d_k + \widetilde{\alpha})$ and $\varphi'_m(d_k - \widetilde{\alpha})$ depends on which side of I_0 they are estimated (jumps have to be avoided). As suggested by a discrete viewpoint, and shown in the previous figure, α is chosen to be equal to the grid spacing.

An example of a result is shown in Figure 1. Several iterations are displayed. Note that the noise is not removed but does not disturb the unwrapping process. Because of the convexity of the problem, changes in the values of the parameters have little influence on the solution.

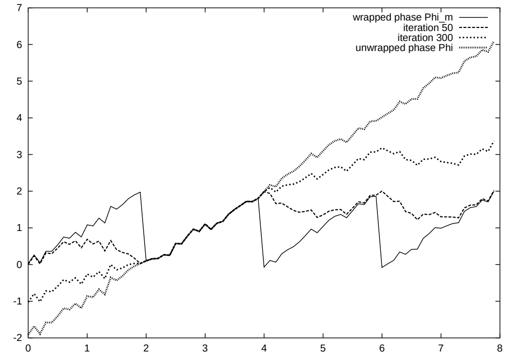


Figure 1. Phase unwrapping without terrain discontinuities. Intermediate stages are displayed. φ_m was created with additive Gaussian noise ($\sigma = 0.08$). $I_0 =]2.5, 3.5[$

4. Case of φ_m with ground discontinuities

In this section we assume that φ and φ_m have ground discontinuities (denoted by $T_{\varphi_m} = \{t_j\}_j$) due to geological faults. We recall that $t_j \neq d_k\ \forall j,\ \forall k$ (see section 2). Therefore we need to recover the unknown discontinuities T_{φ_m}. Unfortunately, this is not possible with Sobolev spaces. When a function is discontinuous, its gradient has to be understood as a measure, and the space of functions of bounded variations, $BV(I)$ is then suitable. We recall that $BV(I)$ is the set (see [1]):

$$\left\{ f \in L^1(I) : \sup_{\substack{g \in L^\infty(I) \\ |g|_{L^\infty} \le 1}} \int_I f(x) g'(x) dx < \infty \right\}.$$

The most important property is that the distributional derivative (which is a measure) can be decomposed into three terms. The total variation is:

$$\int_I |Df| = \underbrace{\int_I |f'(x)| dx}_{\text{regular part}} + \underbrace{\sum_{x \in S_f} |f(x^+) - f(x^-)|}_{\text{jump part}} + \underbrace{|C_f|}_{\text{Cantor part}}.$$

where $f(x^\pm) = \lim_{\rho \to 0} \dfrac{1}{\rho} \displaystyle\int_x^{x \pm \rho} f(\tau) d\tau$ and S_f is the set of discontinuities of f. In this section $S_{\varphi_m} = P_{\varphi_m} \cup T_{\varphi_m}$. So we search for a function φ_α in $BV(I)$ which minimizes the functional

$$\widetilde{E}_\alpha(\varphi) = \int_I |D(\varphi - \varphi_m)| + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi).$$

where E_2 and $E_{3,\alpha}$ are defined by (7) and (8). We can prove the existence and uniqueness of a minimizer in $BV(I)$ for E_α. But we do not pursue further this theoretical BV-based approach. The main reason is that it is very difficult to obtain efficient optimality conditions in BV. So in the sequel we adopt the same formalism used for optical images, that is, instead of $\tilde{E}_\alpha$ we consider the simpler functional:

$$\widetilde{J}_\alpha(\varphi) = \int_I |\varphi' - \varphi'_m| dx + \lambda E_2(\varphi) + \mu E_{3,\alpha}(\varphi).$$

4.1. Half-quadratic minimization

In order to avoid the non-differentiability of the absolute value function, we approximate it (see [2]) by an edge preserving Φ-function: $|u| \approx \Phi_\varepsilon(u)$ with $\Phi_\varepsilon(u) = \sqrt{\varepsilon^2 + u^2} - \varepsilon$, for small values of ε. The functional $\widetilde{J}_\alpha(\varphi)$ is then approximated by:

$$J_{\alpha,\varepsilon}(\varphi) = \int_I \Phi_\varepsilon(\varphi' - \varphi'_m) + \lambda E_2(\varphi) + E_{3,\alpha}(\varphi).$$

The term with the Φ_ε-function is non-quadratic, which implies a non-linear diffusion operator in the Euler-Lagrange equations. A way to overcome this difficulty is to propose an half quadratic algorithm based on duality results. Because Φ_ε is edge-preserving it is shown in [1, 2] that it is always possible to find a function Φ_ε^* of the form:

$$\Phi_\varepsilon^*(t, b) = bt^2 + G_\varepsilon(b)$$

such that $\Phi_\varepsilon(t) = \inf_{b\in[0,1]} \Phi_\varepsilon^*(t, b)$. Applying this transformation, we can rewrite the problem as:

$$\inf_{\varphi_\alpha} J_{\alpha,\varepsilon}(\varphi_\alpha) = \inf_{\varphi_\alpha, b} J^*_{\alpha,\varepsilon}(\varphi_\alpha, b)$$

There are two main advantages: $J^*_{\alpha,\varepsilon}$ is quadratic in φ_α when b is fixed; for φ_α fixed, the minimizer in b can be found explicitly [1, 2]. So a convergent algorithm [2] is to minimize alternately with respect to each variable. Starting from $\varphi^0_{\alpha,\varepsilon} \equiv 0$,

repeat

$$b^{n+1} = \frac{\Phi'(|(\varphi^n_{\alpha,\varepsilon})' - \varphi'_m|)}{2|(\varphi^n_{\alpha,\varepsilon})' - \varphi'_m|} \tag{13}$$

$$-\left(b^{n+1}(\varphi^{n+1}_{\alpha,\varepsilon} - \varphi_m)'\right)' + \lambda(\varphi^{n+1}_{\alpha,\varepsilon} - \varphi_m)\chi_{I_0} = 0 \tag{14}$$

until convergence,

with the boundary conditions (10) and limit conditions (11). (14) is the Euler-Lagrange equation satisfied by $\varphi_{\alpha,\varepsilon}$, a minimizer of $J_{\alpha,\varepsilon}$.

4.2. Numerical result

Let us comment on the discretization of the previous equations:

- (13) is an explicit formula.
- (14) is a linear equation which can be solved with an iterative method. The term $(bf')'$ where $f = (\varphi_{\alpha,\varepsilon} - \varphi_m)$ is approximated by:

  $$(bf')'\Big|_i \approx \frac{1}{h^2}\Big[b_{i-1}f_{i-1} + b_i f_{i+1} - (b_{i-1} + b_i)f_i\Big].$$

  with h the grid spacing.
- conditions (10) and (11) are discretized as in section 3.3.

An example is shown in Figure 2.

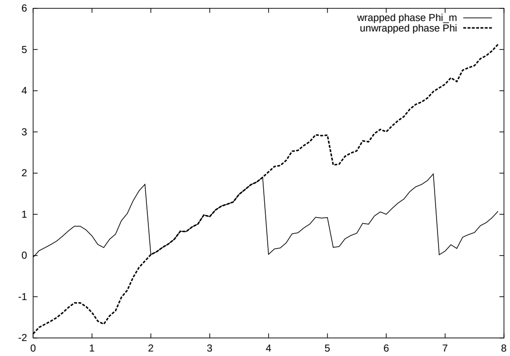


Figure 2. Phase unwrapping with terrain discontinuities. φ_m was created with two ground discontinuities $x = 1.1$ and $x = 5.1$ and with additive Gaussian noise ($\sigma = 0.08$). φ is computed with small values of ε. $I_0 =]2.5, 3.5[$.

5. Conclusion

This paper establishes the mathematical foundations of the 1D phase unwrapping problem in a continuous setting. A variational approach preserving terrain discontinuities was presented. Of course our 1D numerical results are similar to any standard unwrapping algorithm. Our approach is rigorous and promising. Future work will focus on the development of variational approaches to 2D-interferograms. We plan to estimate the phase jumps (curves of discontinuities) from the bidimensional interferogram using a level set approach, and then to apply the method from this paper in the 2D case.

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