Modelling default correlation with multivariate intensity processes

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Abstract

In this paper we present a review of current intensity-based approaches to modelling the joint distribution of default times in a pool of obligors. First of all Cox processes are introduced for a single obligor, then the probabilistic framework is extended to the multivariate case. i^{th}-to-default claims and percentile basket derivatives are described as the financial claims that generate our pricing problem. The definition of the linear correlation coefficient is recalled and some of its limitations as a measure of association are discussed. Three classes of model from the existing literature are then analysed: models relying on the assumption of conditional independence of default times, contagion models and models based on a copula representation of dependent defaults. In particular we show that copula functions give a complete representation of the dependence structure of continuous multivariate distributions and separate it from the univariate margins. Among scalar measures of association, nonparametric correlation coefficients are shown to be more robust global estimators of dependence than the Pearson coefficient, especially outside of the the class of elliptical distributions. Finally, we briefly discuss upper and lower tail dependence as local dependence measures of bivariate extremes.

1 Introduction

The interest in the financial industry for the modelling of correlated default events arises mainly in two respects: in the management of credit risk at a portfolio level and in the valuation of basket credit derivatives. These two complementary perspectives have added to the richness of the debate and to the considerable modelling effort related to this subject in the recent literature.

In particular the strong growth of a market for credit risk transfer in synthetic form [3] has motivated the financial community to evolve in terms of sophistication as well. This evolution from simple single-name Credit Default Swap transactions, a market rapidly coming to maturity, to more advanced products where correlation among reference entities in the underlying basket plays an important role, finds an interesting parallel in the technology shift required to move from simple interest rate swaps to volatility products such as swaptions.

Although intensity-based models are now widespread and commonly used for the modelling of the default process in the case of a single firm, literature on the pricing and risk management of default correlation products is still evolving. In this paper we aim to review the solutions proposed in the financial literature to the problem of extending intensity-based models to the multivariate case by introducing a dependence structure among the defaults of different obligors.

Also, the modelling of default dependence has a natural application in the assessment of counterparty risk, a major source of risk that, although outside the scope of this paper, should deserve more attention on the part of the financial industry.

The paper is organised as follows: in section 2 well known results about Cox processes and their use in intensity models are recalled. In particular we stress the importance of properly defining the information framework in terms of the relevant filtrations available to an agent within the considered time horizon. In section 3 we define the payoff associated with the claims that generate our pricing problem, namely i^{th}-to-default products and percentile basket derivatives. Sections 4 and 7 discuss scalar measures of association, their robustness and invariance properties with respect to the multivariate distribution they describe. In the rest of the paper three classes of model are presented, following the evolution path in the existing literature: first we review the ideas contained in two seminal works [10] [8] co-authored by Duffie, dated 1998 and 1999 respectively. Then we discuss the contagion model proposed in 2000 by Jarrow and Yu [20]. Finally in section 8 we move on to more recent multivariate default models based on copula functions. This class of model seems poised to prevail on other approaches for the great flexibility that it provides by completely characterising the dependence structure for a wide choice of functional specifications and by separating this from the problem of calibrating to term structures of protection premia the default intensity process of single obligors in the underlying pool.

2 Probabilistic Framework

We start from the univariate case by describing the intensity-based framework now broadly accepted by the market as a standard for the pricing of single name credit derivative transactions such as Credit Default Swaps (CDS). In particular we utilise Cox processes [23], also known as *doubly stochastic Poisson processes*. As Schönbucher and Schubert [27] point out, “almost all practical implementations of intensity-based models can be translated into a Cox-process framework.” This class of model has the advantage that it can be easily fitted to term structures of protection premia, which currently represent the price of synthetic credit risk in its most liquid form.

2.1 From Poisson to Cox processes

For the following we assume that we are given a reference complete filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$. Following [21] we recall the definition of a standard Poisson process.

Definition 2.1 *After drawing a sequence (θ_k) of independent exponential random variables of parameter 1, we let T_m be the partial sum of the first m terms of the sequence*

$$T_m = \sum_{k=1}^{m} \theta_k \tag{1}$$

and define the stochastic process

$$N_t := \sum_{m=1}^{\infty} \mathbf{1}_{\{T_m \le t\}} \ . \tag{2}$$

This process is a standard Poisson process *of parameter 1.*

We then pick a right-continuous with left limits (RCLL) non-decreasing function Λ such that $\Lambda_0 = 0$, $\Lambda_t < \infty \ \forall\, t$ and $\Lambda_\infty = \infty$, and consider the time-changed Poisson process

$$\bar{N}_t = N_{\Lambda_t} \ . \tag{3}$$

This new process is called *inhomogeneous Poisson process* with intensity Λ. Finally we let the intensity Λ be stochastic and we thus obtain the so-called *Cox process*, also known as *doubly stochastic Poisson process*. Conditionally on the knowledge of its intensity – that is the σ-field $\mathcal{F}^{\Lambda}_{\infty} = \sigma(\Lambda_t,\ t \ge 0)$ – a Cox process $\tilde{N}$ is an inhomogeneous Poisson process of intensity Λ.

In the following we assume that Λ admits a density λ, so that $\Lambda_t = \int_0^t \lambda_s \, \mathrm{d}s$, and what we call intensity is simply the density λ.

2.2 Default time and information

From now on we assume that we have a pool of n obligors, so that univariate quantities relating to the default processes of single obligors will be indexed with $i \in \{1, \ldots, n\}$; in particular, for bivariate examples we will assume $n = 2$. To model the occurrence of credit events, we associate with each obligor i a $(\mathcal{F}_t)$-adapted, non-negative and continuous process λ_i and set $\Lambda_{it} := \int_0^t \lambda_{is} \, \mathrm{d}s$ with a.s. $\int_0^t \lambda_{is} \, \mathrm{d}s < \infty$ and $\Lambda_{i\infty} = \infty$. We can now introduce the *default time* τ_i as the time of the first jump of a Cox process associated with firm i. More precisely, gven a Cox process $\tilde{N}_i$ with intensity λ_i, we set

$$\tau_i := \inf \{ t \ge 0 \mid \tilde{N}_{it} > 0 \} \ . \tag{4}$$

In general the τ_i are not stopping times with respect to $(\mathcal{F}_t)$, which can be denoted as a "background filtration" [27] generated by all state variables (economic variables, interest rates, currencies, etc.) including the intensity processes (λ_{it}). For this reason we consider the filtration generated by the survival process

$$\mathcal{H}_{it} = \sigma(\tau_i > s,\, s \leq t)\ , \tag{5}$$

which contains information on the occurrence or non-occurrence of a credit event relative to firm i. It is therefore convenient to introduce the enlarged σ-field

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_{1t} \vee \ldots \vee \mathcal{H}_{nt}\ , \tag{6}$$

which represents the 'true' public information available to an agent at time t. A contingent claim of maturity T is thus a random variable X (positive or bounded, so that all expectations will exist) measurable with respect to $\mathcal{G}_t$. In particular, results for the pricing of a derivative security when there is only one defaultable firm are obtained by conditioning on (6) with $n = 1$ (see Lando [23]).

3 Correlation products

The modelling approaches reviewed in this paper represent a first step towards the pricing and hedging of basket products which can be grouped in two main categories: i^{th}-to-default contingent claims and percentile basket derivatives. As we will see in this section, from the definition of the payoff structures and in order to evaluate the impact of diversification on the credit risk generated by the underlying basket of exposures, multivariate survival modelling is required and independence between the default processes cannot be assumed. It will also appear that analytical solutions for evaluation and risk management purposes is in most cases very difficult, if not impossible to obtain. For this reason we will pay particular attention to the simulation methods available within each modelling framework.

3.1 i^{th}-to-default contingent claims

This class of claim is characterised by a payout depending on the temporal ranking of the credit events. We associate with the collection $\tau_1, \ldots, \tau_n$ of default times the ordered sequence of random times $\tau_{(1)} \leq \tau_{(2)} \leq \ldots \leq \tau_{(n)}$. By definition, $\tau_{(1)} = \min(\tau_1, \ldots, \tau_n)$, also written as $\bigwedge_{i=1}^{n} \tau_i$, is the first-to-default time (FTD) and $\tau_{(n)} = \max(\tau_1, \ldots, \tau_n)$.

We will consider i^{th}-to-default contracts in the form of a credit default swap [7] maturing at time T with an underlying basket of n reference entities. In this perspective, we assume that at $\tau_{(i)}$ the buyer of protection can deliver to the seller of protection a nominal amount equal to the nominal of the contract, of obligations issued by obligor k with $\tau_k = \tau_{(i)}$ in exchange for par. If we indicate with φ_k a recovery process for obligor k, representing as a fraction of face value the market value at τ_k of obligations issued by obligor k, the payoff structure of the protection leg of a i^{th}-to-default contract with a unit nominal can be written from the point of view of the protection buyer as

$$\sum_{i=1}^{n} (1 - \varphi_k)\, \mathbf{1}_{\{\tau_k = \tau_{(i)}\}}\ . \tag{7}$$

In exchange for this claim, the protection buyer pays to the seller an amount κ periodically until $\tau_{(i)}$.

3.2 Percentile basket derivatives

Percentile basket derivatives are characterised by a payout dependent on percentiles of the underlying portfolio's loss induced by credit events and typically take the form of a Collateralised Debt Obligation (CDO). CDOs are usually much more complex transactions than i^{th}-to-default swaps. They involve the tranching into different classes of security of the cash flows generated by a pool of collateral, which is most commonly composed of obligations such as bonds or receivables, but can also contain credit default swaps as in more recent transactions. Moreover, if a typical FTD structure has an underlying basket of up to ten names, the collateral pool of a CDO will more likely contemplate more than a hundred different exposures, making simulation more computationally intensive.

Following [12], in order to represent very schematically the payoff profiles generated by a CDO transaction let us define a random variable $L : \Omega \to [0,1]$ representing the total percentage loss at maturity T due to credit events experienced by the collateral pool during the life of the CDO, and let L_x denote the x-percentile of the distribution of L. Now we pick $\alpha, \beta \in (0,1)$ and assume that the portfolio flows are divided into three zero coupon tranches maturing at time T: *junior* (J) tranche, covering the first loss in the collateral up to L_α; *mezzanine* (M) tranche, covering loss realisations between L_α and L_β; *senior* (S) tranche, taking the residual loss for $L > L_\beta$. The respective payoffs at time T of the three tranches can then be written

$$\Pi_J = \max(L_\alpha - L, 0) \tag{8}$$
$$\Pi_M = L_\beta - L_\alpha - \max(L - L_\alpha, 0) + \max(L - L_\beta, 0) \tag{9}$$
$$\Pi_S = 1 - L_\beta - \max(L - L_\beta, 0) \tag{10}$$

This is of course an oversimplified representation; in the reality of things, for the pricing and hedging of a CDO one has to consider the following elements (see also [8] for a detailed example of CDO design):

- The composition of the collateral pool, which represents the source of multivariate credit risk to be modelled.
- The cash flow structure of the tranches.
- The prioritisation scheme, i.e. the collateral cash flow waterfall which determines the priority (and thus the likelihood) of payment of the CDO tranches.

We mention that Moody's developed a method to determine the loss distribution in the collateral pool of a CDO. This approach, called *binomial expansion technique*, is based on the assumption that the distribution of the number of defaults among I risky obligors of the same sector can be summarised by using only D indipendent issuers, i.e.

$$\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \le t\}} = \frac{I}{D} \sum_{d=1}^{D} \mathbf{1}_{\{\tilde{\tau}_d \le t\}} \ , \tag{11}$$

where $\mathbf{1}_{\{\tilde{\tau}_1 \le t\}}, \ldots, \mathbf{1}_{\{\tilde{\tau}_D \le t\}}$ are *i.i.d.* Bernoulli random variables with parameter p. The probability of k defaults in the collateral pool can then be approximated with the binomial expansion formula

$$q(k, D) = \frac{D!}{(D-k)!\, k!} p^k \, (1-p)^{(D-k)} \ . \tag{12}$$

According to Moody's, the *diversity score* D is computed in order to match the two first moments on both sides of (11) on empirical observations (see [1] and [21] for details).

4 Introducing dependence

In the existing literature the traditional way of evaluating default dependence in a bivariate distribution via the linear correlation coefficient is broadly adopted (see for instance [22] [18]).

Definition 4.1 *Given a pair of random variables* X_1 *and* X_2*, the* linear correlation coefficient*, also called the* Pearson correlation coefficient*, is defined by*

$$\varrho(X_1, X_2) = \frac{\mathbb{E}[X_1 \, X_2] - \mathbb{E}[X_1] \, \mathbb{E}[X_2]}{\sqrt{\mathbb{E}[X_1^2] \, \mathbb{E}[X_2^2]}} \ . \tag{13}$$

In particular, two correlation measures between pairs of obligors are often introduced [24] [21]: *discrete default correlation* which corresponds to $\varrho(\mathbf{1}_{\{\tau_1 > t\}}, \mathbf{1}_{\{\tau_2 > t\}})$ and the correlation between the random survival times $\varrho(\tau_1, \tau_2)$, called the *survival time correlation.*

We notice that the linear correlation coefficient always lies in the range $[-1, 1]$, and that its definition (13) is based on the assumption of the finiteness of second moments, i.e. $\mathbb{E}[X_1^1]$, $\mathbb{E}[X_2^2] \in (0, \infty)$. This assumption will be recalled when we will look for more robust measures of association in Section 7. In particular, as Hult and Lindskog [19] point out, linear correlation is not always a meaningful measure of dependence for non-elliptical distributions. Elliptical distributions have density of the form $f(\mathbf{x}^{\mathrm{T}}\mathbf{x})$, so that density contours are ellipsoids. Classical examples of elliptical distributions are the multivariate normal and the multivariate t-distribution. Furthermore, $\varrho(X_1, X_2) = 0$ does not imply independence between X_1 and X_2.

5 Conditional independence models

Models reviewed in this section are implicitly based on the assumption of *conditional independence of default times* [2].

Definition 5.1 *The random times* $\tau_i\,,\ i=1,\ldots,n$ *are said to be* conditionally independent *with respect to the filtration* $(\mathcal{F}_t)$ *under* $\mathbb{P}$ *if for any* $T>0$ *and arbitrary* $t_1,\ldots,t_n\in[0,T]$

$$\mathbb{P}\{\tau_1>t_1,\ldots,\tau_n>t_n \mid \mathcal{F}_T\}=\prod_{i=1}^{n}\mathbb{P}\{\tau_i>t_i \mid \mathcal{F}_T\}\,. \tag{14}$$

At the intuitive level, the assumption of conditional independence means that once the common risk factors are fixed, the idiosyncratic risk factors become independent of each other.

5.1 Correlated intensity processes

A first approach to modelling default dependence consists in introducing correlation in the dynamics of the default intensities of the obligors. In Duffie and Gârleanu [8] for instance, correlation in default timing is induced in an affine jump-diffusion setting by writing the intensity process for obligor i as the sum of a process X_c governing common aspects of economic performance and a specific process X_i driving idiosyncratic default risk.

First of all, as in [9] a *basic affine process* with parameters $(\kappa,\theta,\sigma,\mu,\xi)$ is introduced of the form

$$\mathrm{d}X_t=\kappa(\theta-X_t)\,\mathrm{d}t+\sigma\sqrt{X_t}\,\mathrm{d}W_t+\mathrm{d}J_t, \tag{15}$$

where W is a standard Brownian motion and J is an independent pure-jump process the jump sizes of which are independent and exponentially distributed with mean μ and the jump times of which are given by an independent Poisson process with mean jump arrival rate ξ. After noting that the sum of two independent basic affine processes with respective parameters $(\kappa,\theta_1,\sigma,\mu,\xi_1)$ and $(\kappa,\theta_2,\sigma,\mu,\xi_2)$ is a basic affine process of parameters $(\kappa,\theta_1+\theta_2,\sigma,\mu,\xi_1+\xi_2)$, the intensity process for obligor i is written as

$$\lambda_i=X_c+X_i\,, \tag{16}$$

where X_c and $X_i\,,\ i\in\{1,\ldots,n\}$ are independent basic affine processes. In particular, X_c can be viewed as a process governing common aspects of economic performance, while X_i represents the idiosyncratic default risk specific to obligor i.

All correlation between the default processes of the obligors is driven by the parameterisation of the common state process. Although this setup yields analytic solutions for default probabilities, the dependence structure that it produces is difficult to analyse as it connot be translated into intuitive measures of default correlation. Moreover, the simulation of the default processes, as the authors themselves admit, is not easy to implement, so that a simplified simulation methodology is proposed [8]. Finally, the default correlations that can be attained in this setup are typically too low when compared with empirical default correlations (see for instance [21] for a demonstration in a copula framework).

5.2 Multivariate exponential default times

In the model proposed by Duffie and Singleton [10], certain credit events are triggered by systemic shocks and may be common to a number of obligors, so that

$$\mathbb{P}\{\tau_i = \tau_j, i \neq j\} > 0 \ . \tag{17}$$

In a multivariate exponential setting (see also [12]), m independent time-homogeneous (i.e. with constant intensity) Poisson processes $N_1, \ldots, N_m$ are introduced. Whenever, for any $j \in \{1, \ldots, m\}$, there is a jump in the process N_j, any entity $i \in \{1, \ldots, n\}$ might default with given probability p_{ij}. With this model, the arrival of a credit event for entity i is given by

$$g_i = \sum_{j=1}^{m} p_{ij} \lambda_j \ , \tag{18}$$

where λ_j is the (constant) intensity of process N_j. The intensity of arrival of simultaneous credit events for entities i and k can be written

$$g_{ik} = \sum_{j=1}^{m} p_{ij} \lambda_j p_{jk} \ . \tag{19}$$

Finally, the survival time correlation between entities i and k is equal to

$$\rho(\tau_i, \tau_k) = \frac{g_{ik}}{g_i + g_k - g_{ik}} \ . \tag{20}$$

In this setup a more convenient measure of association between the default processes is thus obtained. Also, stronger correlations can be reached, though at the expense of the ease of calibration: an intensity must be specified for each possible joint default event. Moreover, it is unrealistic to suppose that two or more firms would default at exactly the same time, unless there is a parent-subsidiary or similar contractual relationship.

6 Contagion models

A more reasonable distribution of the incidence of defaults over time is produced by contagion models such as the static *infectious defaults* model by Davis and Lo [6], or the *looping defaults* introduced by Jarrow and Yu [20]. In Jarrow and Yu [20], the occurrence of a credit event relative to obligor j can produce a jump in the default intensity of obligor i and viceversa, so that the default intensity process for obligor $i \in \{1, \ldots, n\}$ can be written

$$\lambda_{it} = \alpha_{it} + \sum_{j=1}^{n} \beta_{ijt} \, \mathbf{1}_{\{\tau_j \leq t\}} \ , \tag{21}$$

where (α_{it}) and (β_{ijt}) are $(\mathcal{F}_t)$-adapted processes. This behaviour is associated with the possibility of cross-holding of assets among the obligors in the pool, so that "contractual relationship" among reference entities in the sense of (17) is here explicitly modelled.

The fact that the distributions of default times are defined recursively is termed "looping defaults" by the authors. To avoid the complexity of analysing the multivariate distributions that arise in this setting, in [20] a simplified framework is proposed, where the pool of obligors being modelled is split into two categories: *primary firms* with independent default intensity processes and *secondary firms* the intensity of which is made to jump of a given amount at the time of default of any primary firm.

Although this model requires heroic assumptions on the holding structure in the pool of obligors without any particular advantage in terms of tractability of the dependence structure and of ease of calibration, it is nevertheless interesting to note that the intensity processes in (21) are not $(\mathcal{F}_t)$-adapted, and that conditional independence of default times cannot be assumed in this setting.

7 Nonparametric measures of association

A further step in modelling default dependence is obtained by moving from the linear correlation coefficient to *nonparametric* or *rank correlation measures.*

As we mentioned in Section 4, outside of the class of elliptical distributions the use of the Pearson coefficient may be problematic. The need and the possibility (made available with ease through copula functions described below) to model a wide range of functional forms for the multivariate distribution of default processes, motivated the financial community to look for more robust measures of association with respect to the Pearson coeeficient.

Two popular nonparametric measures of association are Spearman's rho and Kendall's tau [19].

Definition 7.1 Kendall's tau *for a random vector* (X_1, X_2) *is defined as*

$$\tau(X_1, X_2) := \mathbb{P}\{(X_1 - X_1')\,(X_2 - X_2') > 0\} - \mathbb{P}\{(X_1 - X_1')\,(X_2 - X_2') < 0\} \ , \qquad (22)$$

where (X_1', X_2') *is an independent realisation of the joint distribution of* (X_1, X_2)*.*

Definition 7.2 Spearman's rho *for a random vector* (X_1, X_2) *is defined as*

$$\varrho_S(X_1, X_2) := 3\left(\mathbb{P}\{(X_1 - X_1')\,(X_2 - X_2'') > 0\} - \mathbb{P}\{(X_1 - X_1')\,(X_2 - X_2'') < 0\}\right) \ , \qquad (23)$$

where (X_1', X_2') *and* (X_1'', X_2'') *are independent realisations of the joint distribution of* (X_1, X_2)*.*

An important property of Kendall's tau and Spearman's rho is that they are invariant under strictly increasing transformations of the underlying random variables. Also, it can be seen from the definitions that they do not rely on the existence of certain moments.

In fact, the key concept of nonparametric correlation consists in replacing the variates (whatever their distributions) with their rank in the sample. This is more evident in the sample version of Kendall's tau and Spearman's rho (for which we refer to [26]), where the rank sample is drawn from a known distribution function, namely uniformly from the set of integers $\{1, \ldots, N\}$, where N is the number of pairs in the sample. Nonparametric correlation is more robust than linear correlation in the same sort of sense that the median is more robust than the mean. In particular, if both nonparametric correlation coefficients remain a meaningful measure of dependence for non-elliptical distributions, we mention that Kendall's tau presents stronger invariance properties than Spearman's rho [19].

Kendall's tau and Spearman's rho also find convenient treatment whithin the theory of copula functions as shown in Section 8.1.

8 Modelling dependent defaults with copulas

8.1 Copula functions

Copulas were first introduced in 1959 in surveys on random metric spaces, but they were not applied to finance until 1999. A copula function is essentially a multivariate uniform distribution, i.e. a multivariate distribution with uniform margins. Due to the fact that the c.d.f. of a continuous random variable is uniformly distributed and that copulas are invariant under strictly increasing transformations, it is clear that copula functions may be used to capture those dependence properties of a joint distribution that are invariant with respect to its (continuous) margins.

Definition 8.1 *A* n-dimensional copula *is a function* $\mathbf{C} : \mathbf{I}^n = [0,1]^n \to [0,1]$ *with the following properties:*

1. (grounded) $\forall\, u \in \mathbf{I}^n$, $\mathbf{C}(u) = 0$ *if at least one coordinate* $u_j = 0$ *with* $j \in \{1, \dots, n\}$.
2. (reflective) *if all coordinates of* u *are* 1 *except* u_j *then* $\mathbf{C}(u) = u_j$ *with* $j \in \{1, \dots, n\}$.
3. (n-increasing) *the* $\mathbf{C}$*-volume of all hypercubes with vertices in* $\mathbf{I}^n$ *is positive, or equivalently*
$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-)^{i_1+\ldots+i_n}\, \mathbf{C}(u_{1i_1}, \dots, u_{ni_n}) \geq 0 \tag{24}$$
for all $(u_{11}, \dots, u_{n1})$ *and* $(u_{12}, \dots, u_{n2})$ *in* $\mathbf{I}^n$ *with* $u_{j1} \leq u_{j2}\ \ \forall\, j = 1, \dots, n$.

It is useful to consider the following definition of **concordance ordering**.

Definition 8.2 *We say that the copula* $\mathbf{C}_1$ *is smaller than the copula* $\mathbf{C}_2$ *(or* $\mathbf{C}_2$ *is larger than* $\mathbf{C}_1$*), and write* $\mathbf{C}_1 \prec \mathbf{C}_2$ *(or* $\mathbf{C}_2 \succ \mathbf{C}_1$*) if*

$$\forall\, (u_1, \dots, u_n) \in \mathbf{I}^n\,, \quad \mathbf{C}_1(u_1, \dots, u_n) \leq \mathbf{C}_2(u_1, \dots, u_n)\,. \tag{25}$$

We then define three specific copulas that play an important role: the lower and upper Fréchet bounds $\mathbf{C}^-$ and $\mathbf{C}^+$ and the product copula $\mathbf{C}^\perp$:

$$
\begin{aligned}
\mathbf{C}^-(u_1,\ldots,u_n) &= \max\left(\sum_{i=1}^{n} u_i - n + 1,\, 0\right) && (26)\\
\mathbf{C}^\perp(u_1,\ldots,u_n) &= \prod_{i=1}^{n} u_i && (27)\\
\mathbf{C}^+(u_1,\ldots,u_n) &= \min(u_1,\ldots,u_n) && (28)
\end{aligned}
$$

For any copula $\mathbf{C}$ the following order holds:

$$\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+ \, . \tag{29}$$

We now state an important representation theorem [28], which provides a way to decouple the analysis of the dependence structure of multivariate distributions from the problem of identifying the marginal distributions.

Theorem 8.1 (Sklar) *Let* $\mathbf{F}$ *be an* n*-dimensional distribution function with continuous margins* $\mathbf{F}_1,\ldots,\mathbf{F}_n$*. Then* $\mathbf{F}$ *has a unique copula representation:*

$$\mathbf{F}(x_1,\ldots,x_n) = \mathbf{C}(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n)) \, . \tag{30}$$

Thus, for continuous multivariate distributions the univariate margins and the dependence structure can be separated. The dependence structure is completely characterised by the copula $\mathbf{C}$.

Now the need for robust scalar measures of dependence discussed in Section 7 can be better understood. Both Kendall's tau and Spearman's rho can be written in terms of copula functions as measures of concordance [4]:

$$
\begin{aligned}
\tau &= 4 \iint_{\mathbf{I}^2} \mathbf{C}(u_1,u_2)\, \mathrm{d}\mathbf{C}(u_1,u_2) - 1 && (31)\\
\rho_S &= 12 \iint_{\mathbf{I}^2} u_1\, u_2\, \mathrm{d}\mathbf{C}(u_1,u_2) - 3 && (32)
\end{aligned}
$$

We notice that it is interesting to use scalar dependence measures because the direct comparison between survival copulas may not be obvious. Moreover, concordance ordering as in definition 8.2 above is only a *partial* ordering of the set of copulas [14].

8.2 The survival copula approach

Li [24] demostrates that the CreditMetrics approach [17] of modelling default correlation through a multivariate unobserved gaussian asset process is equivalent to using a gaussian copula to describe the joint distribution of default times (defined as in (4)). The CreditMetrics approach is also known in the econometrics literature as a *hidden variables probit regression* [16].

Let us quickly describe the CreditMetrics approach to modelling the joint default of a pair of obligors. Given a fixed time horizon T (CreditMetrics works with T equal to one year) we can map the marginal distribution of default times to a standard normal distribution as follows: if $\mathbf{F}_i$ is the distribution of default time τ_i with $i \in \{1, 2\}$, for each entity CreditMetrics would calculate the realisation z_i of a standard normal variable Z_i describing a "hidden asset process" such that $\mathbf{\Phi}(z_i) = \mathbf{F}_i(T)$, where $\mathbf{\Phi}$ is the standard normal distribution. An interpretation is given for z_i as thresholds for the hidden asset process in a Merton-like setting. This setup can also be extended to model joint rating migration [17]. The probability of a joint default is given by

$$\mathbf{F}(T, T) = \mathbb{P}\{\tau_1 \leq T, \tau_2 \leq T\} = \mathbf{\Phi}_\rho(z_1, z_2) \;, \tag{33}$$

where we indicate with $\mathbf{\Phi}_\rho$ the standard multivariate (bivariate in this case) normal distribution with correlation matrix ρ.

Definition 8.3 (multivariate normal copula) *Let ρ be a symmetric, positive definite matrix with* $\operatorname{diag} \rho = \mathbf{1}$. *The multivariate gaussian copula is defined as follows:*

$$\mathbf{C}(u_1, \ldots, u_n; \rho) := \mathbf{\Phi}_\rho(\mathbf{\Phi}^{-1}(u_1), \ldots, \mathbf{\Phi}^{-1}(u_n)) \;. \tag{34}$$

From this definition it is clear that the CreditMetrics approach applies a bivariate gaussian copula to the distributions of default time for a pair of obligors, in the sense that

$$\mathbf{F}(T, T) = \mathbf{C}(\mathbf{F}_1(T), \mathbf{F}_2(T); \rho) \;, \tag{35}$$

where ρ is equal to the CreditMetrics correlation coefficient between the hidden asset processes of the two obligors.

Li [24] also proposes a Monte Carlo technique for pricing first-to-default claims in a simple intensity-based setting.

Georges *et al.* [14] show that applying a copula to the marginal distributions $\mathbf{F}_i$ of default time is equivalent to applying a *survival copula* $\breve{\mathbf{C}}$ to the survival functions $\mathbf{S}_i(t) = \mathbb{P}\{\tau_i > t\} = 1 - \mathbf{F}_i(t)$ of the single obligors. The relationship between the copula $\mathbf{C}$ of default times and the survival copula is given by

$$\breve{\mathbf{C}}(u_1, \ldots, u_n) = \bar{\mathbf{C}}(1 - u_1, \ldots, 1 - u_n) \;, \tag{36}$$

with

$$\bar{\mathbf{C}}(u) = \sum_{j=0}^{n} \left[(-)^j \sum_{v \in \mathcal{Z}(u, n-j)} \mathbf{C}(v) \right] \tag{37}$$

where

$$\mathcal{Z}(u, m) = \{v \in \mathbf{I}^n \mid v_i \in \{u_i, 1\} \;\forall\, i = 1, \ldots, n \;,\; \sum_{j=0}^{n} \mathbf{1}_{\{v_j = 1\}} = m\} \;. \tag{38}$$

It is interesting to note that the Kendall's tau and the Spearman's rho of the survival copula $\breve{\mathbf{C}}$ are equal to the Kendall's tau and the Spearman's rho of the associated copula $\mathbf{C}$. For a study of the properties of survival copulas we refer to [14].

8.3 The threshold approach

Schönbucher and Schubert [27] propose the use of copulas to correlate the threshold exponential random variables for a default countdown process, rather than modelling the joint survival function directly with survival copulas of default times. More precisely, as a default occurs when the intensity process of a firm reaches a pre-specified trigger which is an exponential random variable of parameter 1 independent of the intensity process, the trick is to link the different thresholds with a copula.

Let (λ_{it}) be an $(\mathcal{F}_t)$-adapted RCLL stochastic process and define the *default countdown process* for obligor i

$$\gamma_i(t) := \exp\left(-\int_0^t \lambda_{is}\,\mathrm{d}s\right) . \tag{39}$$

In this setting we then define the default time

$$\tau_i := \inf\{t : \gamma_i(t) \le U_i\} , \tag{40}$$

with the *default threshold* random variable $\mathbf{U} = (U_1, \ldots, U_n)$ distributed according to the twice continuously differentiable copula $\mathbf{C}^{\mathbf{u}}$. We assume that the U_i are uniformly distributed on $[0,1]$ under $\mathcal{F}_0 \vee \sigma(\tau_i > 0)$; under the same filtration, if we take $\theta_i = -\ln U_i$ then θ_i is a unit exponential random variable and it follows that (40) is equivalent to the following definition of default time, which is usual in the existing literature (see for instance [23])

$$\tau_i := \inf\left\{t : \int_0^t \lambda_{is}\,\mathrm{d}s \ge \theta_i\right\} . \tag{41}$$

Equation (41) is in turn equivalent to (4) when λ_i is chosen to be the intensity of the Cox process $\tilde{N}$. However, the process λ_i, which controls the speed of the countdown process in (39), coincides with the default intensity of obligor i only in the independence case ($\mathbf{C}^{\mathbf{u}} = \mathbf{C}^{\perp}$) or when information is restricted to $\mathcal{F}_t \vee \sigma(\tau_i > s, s \le t)$ representing information about the general state of the economy and the default of obligor i alone. For this reason, in [27] λ_i is termed the *pseudo-intensity* of obligor i. We notice that the independence case represents the only dependence specification in this framework for which default times are conditionally independent in the sense of (14).

It is convenient to denote

$$\mathcal{H}_t = \bigvee_{i=1}^{n} \mathcal{H}_{it}, \tag{42}$$

where $\mathcal{H}_{it}$ is defined by (5). We call h_t^i the intensity of obligor i for an agent to whom complete information $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ is available at time t.

The regularity properties of $\mathbf{C}^{\mathbf{u}}$ imply that $\mathbb{P}\{\tau_i = \tau_j\} = 0$ for every $i, j = 1, \ldots, n$ such that $i \neq j$, i.e. simultaneous defaults occur with probability zero.

This approach has the main advantage of splitting the distribution of the single intensity processes and the joint law of the default triggers, in such a way that the individual intensities can be easily calibrated to term structures of protection premia, while the dependence structure can be specified separately. In addition, this setup provides a simple framework for the simulation of random default times using the following straightforward algorithm:

1. Simulate $(u_1, \ldots, u_n)$ from the copula $\mathbf{C}^{\mathbf{u}}$;

2. For each firm i, simulate the intensity process (λ_{it}) to compute $\gamma_i(t)$. Stop when $\gamma_i(t) \geq u_i$ and take $\tau_i = t$.

Furthermore, Schönbucher and Schubert [27] show that in the case of positive dependence, the default of one obligor causes the intensities of the other obligors to jump upwards, thus producing credit contagion features in an endogenous way. Namely, if no default has happened until time t we can write the intensity process for obligor i under $\mathcal{G}_t$ as follows:

$$\frac{\mathrm{d}h_i}{h_i} = \frac{\mathrm{d}\lambda_i}{\lambda_i} + \left(h_i\left(1 - \frac{\mathbf{C}_{u_i u_i}\,\mathbf{C}}{\mathbf{C}^2_{u_i}}\right) - \lambda_i\right)\mathrm{d}t - \mathrm{d}N_i + \sum_{j=1, j\neq i}^{n}\left(\frac{\mathbf{C}_{u_i u_j}\,\mathbf{C}}{\mathbf{C}_{u_i}\mathbf{C}_{u_j}} - 1\right)(\mathrm{d}N_j - h_j\,\mathrm{d}t)\,, \tag{43}$$

where $N_i(t) = \mathbf{1}_{\{\tau_i \leq t\}}$ is the default indicator process and the time argument has been suppressed. We notice that the summation term in this equation contains the potential influence of a credit event occurring to obligor j on the intensity of the default process for obligor i induced by the dependence structure $\mathbf{C}$ via the jump component $\mathrm{d}N_j$. See [27] for a full interpretation of the terms in (43).

8.4 Characterisation of copula functions

Indeed, for many financial applications, the problem does not lie in the use of a given multivariate distribution, but consists in finding a convenient distribution to describe some stylised facts about the underlying process while preserving tractability. Although copulas provide a wide range of well known functional specifications to choose from, it is not always obvious which copula to select. In the literature some papers propose the utilisation of certain classes of copula, like the Frank copula [24] or Archimedean copulas [27], other tend to privilege other classes depending on the ease of parameterisation or on the tractability with regard to the model they propose. A way of avoiding arbitrary distributional assumptions is to develop characterisation theorems from some basic assumptions of the real life situation we face. In this respect, rather than looking at global measures of association like Kendall's tau and Spearman's rho, further insight could be gained by considering more local characterisations. Upper and lower tail dependence, also termed *quantile measures of dependence*, are local dependence measures of bivariate extremes, which are important in survival modelling, because they indicate the behaviour of the joint survival times in the limit cases respectively of "immediate joint death" and "long term joint survival".

Definition 8.4 *If a bivariate copula* $\mathbf{C}$ *is such that*

$$\lim_{u \to 1} \frac{1 - 2u + \mathbf{C}(u,u)}{1-u} = \lambda_U \tag{44}$$

exists, then $\mathbf{C}$ *has upper tail dependence if* $\lambda_U \in (0,1]$ *and no upper tail dependence if* $\lambda_U = 0$*. Similarly, if*

$$\lim_{u \to 0} \frac{\mathbf{C}(u,u)}{u} = \lambda_L \tag{45}$$

exists, $\mathbf{C}$ *has lower tail dependence if* $\lambda_L \in (0,1]$ *and no lower tail dependence if* $\lambda_L = 0$*.*

We notice that λ_U and λ_L depend only on the copula functions, not on the margins. Upper (lower) tail dependence is a measure of *joint* extremes, that is it measures the probability that one component is extremely large (small) given that the other is extremely large (small). This appears more readily by looking at the definition of tail dependence in probabilistic terms, i.e. outside of the copula framework, for which we refer to [19]. In particular, if we indicate with $\mathbf{F}_1$ and $\mathbf{F}_2$ the marginal distribution functions of default times in the bivariate copula $\mathbf{C}$, Georges *et al.* [14] show that the arguments of the limits in (44) and (45) can be written

$$\frac{1 - 2u + \mathbf{C}(u,u)}{1-u} = \mathbb{P}\{\tau_2 > \mathbf{F}_2^{-1}(u) \mid \tau_1 > \mathbf{F}_1^{-1}(u)\} \tag{46}$$

$$\frac{\mathbf{C}(u,u)}{u} = \mathbb{P}\{\tau_2 < \mathbf{F}_2^{-1}(u) \mid \tau_1 < \mathbf{F}_1^{-1}(u)\} \; . \tag{47}$$

It can be easily verified [4] that the gaussian copula has no upper or lower tail dependence, meaning that extremes are asymptotically independent under the multivariate normal dependence specification (34). We refer to [4] and [14] for a summary of different copula specifications and for a broader range of characterisations of the dependence structure of multivariate distributions.

9 Conclusion

Much progress has been made in the field of credit derivatives modelling since the working paper by Duffie and Singleton appeared which first proposed intensity-based models for the pricing of defaultable securities (now published in [11]). However, a lot remains to be done with regard to multivariate extensions of the univariate Cox process framework. In particular, thanks to increased transparency and availability of time series of protection levels and of market prices of structured credit products, the need is much felt for applied studies that evaluate with empirical tests the assumptions being made and the models proposed in the literature.

It must be said that such applications of theory to practical cases are often only possible in cooperation with research departments which operate inside financial institutions that are at the forefront of credit trading and correlation product design. Our work intends to stimulate such empirical studies by providing a quick reference to available models.

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