# Auction-Based Spectrum Sharing

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**Abstract**

We study auction-based mechanisms for sharing spectrum among a group of users, subject to a constraint on the interference temperature at a measurement point. The users access the channel using spread spectrum signaling and so interfere with each other. Each user receives a utility that is a function of the received signal-to-interference plus noise ratio. We propose two auction mechanisms for allocating the received power. The first is an SINR-based auction, which, when combined with logarithmic utilities, leads to a weighted max-min fair SINR allocation. The second is a power-based auction that maximizes the total utility when the bandwidth is large enough and the receivers are co-located. Both auction mechanisms are shown to be socially optimal for a limiting "large system" with co-located receivers, where bandwidth, power and the number of users are increased in a fixed proportion. We also formulate an iterative and distributed bid updating algorithm, and specify conditions under which this algorithm converges globally to the NE of the auction.

## I. Introduction

There has been growing interest in making more efficient use of spectrum by shifting from the conventional "command-and-control" spectrum usage models to more flexible "Exclusive Use" and "Commons" models (e.g., see [1]). In the Exclusive Use model, the licensee has exclusive rights to the spectrum, but could allow other users to purchase access rights to the spectrum when it is underutilized. In the Commons model, spectrum is unlicensed and an unlimited number of users can share spectrum with usage rights governed by technical standards. In either model, a basic question is how to share the available spectrum efficiently and fairly. One metric that has

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A preliminary version of this paper was presented at the 2nd Workshop on Modelling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt '04), Cambridge, UK, March 24-26, 2004.

been suggested for this purpose is to require that the *interference temperature* in the spectrum band be kept under some threshold, where interference temperature is defined to be the RF power measured at a receiving antenna per unit bandwidth.

In this paper, we study a spectrum allocation problem under such an interference temperature constraint. This model is motivated by the scenario in which users wish to purchase a local, relatively short-term data service. The spectrum to be used may be licensed to an independent entity (e.g., private company) or controlled by a government agency, either of which we refer to as a *manager*. Users may transmit to receivers at different locations, or to co-located receivers at a single Access Point. In both cases, the manager controls the amount of bandwidth and power assigned to each user in order to keep the interference temperature at a given measurement point below a certain threshold. We assume that all users adopt a spread spectrum signaling format, in which the transmitted power is evenly spread across the entire available band controlled by the manager. This allows efficient multiplexing of data streams from different sources corresponding to different applications, and reduces the combined power-bandwidth allocation problem to a received power allocation problem. Each user has a utility, which is a function of the received Signal-to-Interference plus Noise Ratio (SINR), reflecting his desired Quality of Service (QoS). The interference a user receives depends on the other users' transmission powers and the cross-channel gains, as well as the bandwidth.

In this setting, an interference temperature constraint is equivalent to a constraint on the received power at the measurement point. This allows us to view the received power as a divisible good; we study auction mechanisms for allocating this good. It is well known that a Vickrey-Clarke-Groves (VCG) auction can be used to achieve a socially optimal allocation, i.e., maximize the total utility [2]. However, as discussed in Sect. II-B, this may not be suitable here due to the required information from the users and the computational burden on the manager. Instead, we propose two auctions mechanisms that allocate the received power as a function of bids submitted by the users and the price announced by the manager. We model the resulting problem as a noncooperative game [2], and characterize the Nash equilibria and related properties of the two auctions. We first analyze these auctions as a simultaneous move game, assuming all information (i.e., utility functions and link channel gains) is available to the users (but not to the manager). We subsequently formulate an iterative and fully distributed algorithm, which only requires the users to obtain limited local information in order to converge globally to the Nash Equilibrium (NE). This makes the auction mechanisms easily implementable and scalable with the population size.

Our approach is similar to a *share auction* (see [3]–[7] and the references therein), or *divisible auction*, where a perfectly divisible good is split among bidders whose payments depend solely on the bids. A common form of bids in a share auction is for each user to submit his demand curve (e.g., [3]–[5]), i.e., the amount of goods a user desires as a function of the price. The auctioneer can then compute a market clearing price based on the set of demand curves. However, in our problem, a user's demand curve for received power also depends on the demands of other users due to interference. On the other hand, if the demand curves are viewed in terms of SINR so that they are mutually independent, the market clearing price of SINR is not easy to find since the constraint is on the received power. To overcome these difficulties, we adopt a signaling system similar to [6], [7], where users submit one dimensional bids for the resource.

We assume a weighted proportional allocation rule in which a user's power allocation is proportional to his bid. This type of allocation rule has been studied in a wide range of applications (e.g., see [8], [9]) including network resource allocation (e.g., [6], [7]). Given this allocation, the users participate in a game with the objective of maximizing their own benefit. It is well known that the NE of a game typically does not maximize the total system utility [10]. This has been referred to as the *price of anarchy* (e.g., [6]). In order to achieve a more desirable NE, we allow the manager to announce a unit price (e.g., [11], [12]) either for received SINR or received power. An SINR-based auction with logarithmic utilities leads to a weighted max-min fair SINR allocation. A power-based auction maximizes the total utility for a large enough bandwidth with co-located receivers. Both pricing schemes maximize the total utility in a large enough system with co-located receivers if the total power and bandwidth are increased in fixed proportion to the number of users. Related work on uplink power control for CDMA has appeared in [12]–[15]. A key difference here is that there is a constraint on the total received power at all times[1]. Because of this, a user's interference depends on his own power allocation, which can make the problem non-convex.

[1]We assume that any transmission power constraint for each user is large enough so that it can be ignored.

We assume the user population is stationary, i.e., the users and their corresponding utilities stay unchanged during the time period of interest. On a larger time-scale one can view time divided into periods, during which the number of users and each user's utility are fixed and the proposed auction algorithm is used. When a new period begins, users may join or leave the system. Remaining users may update their utility functions to reflect changes in their QoS requirements. For example, a user with data that must be delivered by a deadline might increase his utility (as a function of SINR) as the deadline approaches. Here we do not consider the mechanisms and associated dynamics over multiple periods.

The remainder of the paper is organized as follows. After introducing the auction mechanisms in Sect. II, we analyze the performance for a finite system and for a limiting "large system" in Sect. III and IV, respectively. In Sect. V we give an iterative and distributed bid updating algorithm, and show that it converges globally to the unique NE of the auction when one exists. Numerical results are given in Sect. VI and conclusions in Sect. VII. Several of the main proofs are given in the Appendix.

## II. Auction Mechanisms

### A. System Model

Spectrum with bandwidth $B$ is to be shared among $M$ spread spectrum users, where a user refers to a transmitter and an intended receiver pair. User $i's$ valuation of the spectrum is characterized by a utility function $U_i(\gamma_i)$, where $\gamma_i$ is the received SINR at user $i$'s receiver. We primarily consider the case where each user's utility function is given by $U_i(\gamma_i) = U(\theta_i, \gamma_i)$, where $\theta_i$ is a user-dependent parameter. As a particular example, we consider the *logarithmic utility* $U(\theta_i, \gamma_i) = \theta_i \ln(\gamma_i)$. [2] For notation we use $U'(\theta_i, \gamma_i) = \partial U(\theta_i, \gamma_i)/\partial \gamma_i$ and $U''(\theta_i, \gamma_i) = \partial^2 U(\theta_i, \gamma_i)/\partial \gamma_i^2$, and at times we omit the user index $i$.

*Assumption 1:* For each user $i$, $U(\theta_i, \gamma_i)$ is increasing, strictly concave, and twice continuously differentiable in $\gamma_i$.

[2] This approximates the weighted rate of user $i$ in the high SINR regime.

Utilities that satisfy this assumption are commonly used to model "elastic" data applications [16]. For each $i$, the received SINR is given by

$$\gamma_i = \frac{p_i h_{ii}}{n_0 + \frac{1}{B}\left(\sum_{j \neq i} p_j h_{ji}\right)}, \tag{1}$$

where $p_i$ is user $i$'s transmission power, $h_{ij}$ is the channel gain from user $i$'s transmitter to user $j$'s receiver, and $n_0$ is the background noise power that is assumed to be the same for all users. To satisfy an interference temperature constraint, the total received power constraint at a specified measurement point must satisfy

$$\sum_{i=1}^{M} p_i h_{i0} \leq P, \tag{2}$$

where $h_{i0}$ is the channel gain from user $i$'s transmitter to the measurement point. The system model is shown in Fig. 1. A power allocation is *Pareto optimal* if no user's utility can be increased without decreasing another user's utility.

*Lemma 1:* A power allocation scheme is Pareto optimal if and only if the total received power constraint is tight, i.e., $\sum_{i=1}^{M} p_i h_{i0} = P$.

This follows because if the power constraint is not tight, then each user can increase their power by $P / \sum_{i=1}^{M} p_i h_{i0}$, which increases the SINR for every user. Lemma $1$ does not require Assumption 1; in particular, $U(\theta_i, \gamma_i)$ does not have to be concave in $\gamma_i$, although it must be strictly increasing. Note that Pareto optimality does not indicate how to split resources among users, only that the resource should be fully utilized. A stronger condition is *social optimality*, where the total utility $\sum_{i=1}^{M} U(\theta_i, \gamma_i)$ is maximized. Social optimality implies Pareto optimality, but the reverse is not true. Therefore, to achieve social optimality, the manager should always ensure that the received power constraint is tight.

A special case, on which we will focus, is when the receivers are all co-located with the measurement point. This could model a situation where a service provider purchases the spectrum usage rights from the manager and provides service from a single Access Point. In this case, $h_{ij} = h_{i0}$ for all $i, j \in \{1, ..., M\}$, and we denote user $i$'s received power as $p_i^r = p_i h_{i0}$. In a Pareto optimal allocation for this co-located receiver case, we have for each user $i$,

$$\gamma_i \equiv \gamma_i(p_i^r) = \frac{p_i^r}{n_0 + \frac{1}{B}(P - p_i^r)}, \tag{3}$$

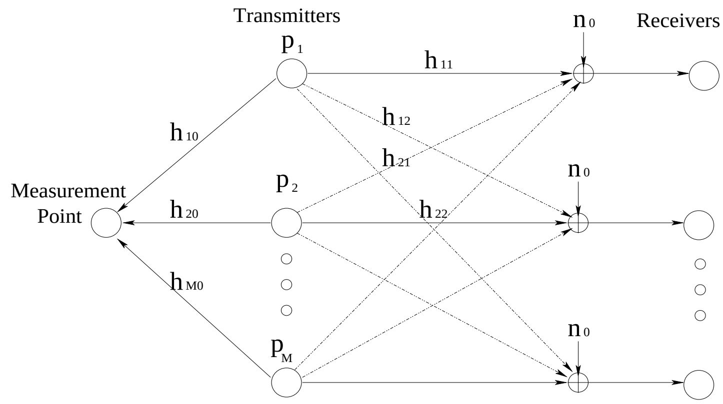


Fig. 1. System model for $M$ transmitter-receiver pairs

so that the users' utilities $U\left(\theta_i, \gamma_i\left(p_i^r\right)\right)$ under a Pareto optimal allocation are separable in assigned powers.

We assume that each user's utility function is private information, i.e., only known to the user himself. The manager must then devise a mechanism for allocating power without having this knowledge *a priori*. Also the manager may not have *a priori* knowledge of the channel gains, $h_{ij}$'s. One such mechanism is the generalized VCG auction.

### *B. VCG Auction for Spectrum Sharing*

A VCG auction results in a socially optimal outcome, and it is a (weakly) dominant strategy for users to bid truthfully (i.e., state their true utility functions). In our context, a VCG auction can be described as follows: First, users are asked to submit their utility functions $\{U\left(\theta_i, \gamma_i\right)\}$. The manager then computes the maximum total utility $U_{\mathsf{max}} = \max_{\{p_j\}} \sum_{j=1}^{M} U\left(\theta_j, \gamma_j\right)$ given the received power constraint, and allocates power to the users accordingly. Furthermore, the manager computes the maximum total utility if user $i$ is excluded from the auction, i.e., $U_{\mathsf{max/i}} = \max_{\{p_j\}/p_i} \sum_{j \neq i} U\left(\theta_j, \gamma_j\right)$ for each $i \in \{1, ..., M\}$. In total, the manager must solve $M+1$ optimization problems. The manager then charges user $i$ the amount $U_{\mathsf{max}} - U_{\mathsf{max/i}}$, which is the incremental social benefit derived from including user $i$ in the auction.

The VCG auction may not be suitable in this context for several reasons: $(i)$ In order to completely specify the users' utility functions, in particular, the SINR in $(1)$, for each user $i$, the channel gains $h_{ij}$ for all $i, j \in \{1, ..., M\}$ must be measured by the users and reported to the manager. This might be a heavy burden for the users in a large network. $(ii)$ The manager must solve $M+1$ optimization problems, which are typically non-convex due to the interference. This becomes computationally expensive for large $M$, and may not be suitable for online allocations. For these reasons, we examine mechanisms that require less information exchange and less computation for the manager.

### *C. One-Dimensional Auctions with Pricing*

We now describe two auctions (SINR- and power-based) in which users submit one-dimensional bids representing their willingness to pay, and the manager simply allocates the received power in proportion to the bids. The users then pay an amount proportional to their SINR (or power). The manager announces a nonnegative reserve bid $\beta$, and uses a corresponding reserve power that interferes with the other users. In contrast with the situation where the manager submits a reserve bid to extract more revenue from the other bidders [17], here the main purpose of the reserve bid is to guarantee a unique desirable outcome of the auction. We will show that the interference generated by the manager can be made arbitrarily small. Although the two auctions are relatively simple, we show that under some mild conditions they give power allocations that are arbitrarily close to the allocation from a VCG auction.

Regarding the information structure of the auction, we first assume that it is a complete information game, i.e., all users' utility functions and all channel gains are known to all users. In Sect. V, we present a distributed algorithm that can achieve the NE of the auction with limited information, where each user $i$ only needs to measure the background noise density $n_0$, the channel gain $\hat{h}_{ii} = h_{ii}/h_{i0}$ and the SINR at his own receiver.

**Simultaneous Auction Algorithm:**

1) The manager announces a reserve bid $\beta \geq 0$, and a price $\pi^s > 0$ (in an SINR-based auction) or $\pi^p > 0$ (in a power-based auction).
2) After observing $\beta$, $\pi^s$ (or $\pi^p$), user $i \in \{1, ..., M\}$ submits a bid $b_i \geq 0$.
3) The manager keeps reserve power $p_0$, and allocates to each user $i$ a transmission power

$p_i$ so that the received power at the measurement point is proportional to the bids, i.e.,

$$h_{i0}p_i = \frac{b_i}{\sum_{j=1}^{M} b_i + \beta}P, \text{ and } p_0 = \frac{\beta}{\sum_{j=1}^{M} b_i + \beta}P. \tag{4}$$

The resulting SINR for user $i$ is

$$\gamma_i = \frac{p_i h_{ii}}{n_0 + \frac{1}{B}\left(\sum_{j\neq i} p_j h_{ji} + p_0 h_{0i}\right)}, \tag{5}$$

where $h_{0i}$ is the channel gain from the manager (measurement point) to user $i$'s receiver[3]. If $\sum_{i=1}^{M} b_i + \beta = 0$, then $p_i = 0$.

4) In an SINR-based (power-based) auction, user $i$ pays $C_i = \pi^s \gamma_i$ $(C_i = \pi^p p_i)$

A *bidding profile* is the vector containing the users' bids $\boldsymbol{b} = (b_1, ..., b_M)$. The *bidding profile of user $i$'s opponents* is defined as $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_M)$, so that $\boldsymbol{b} = (b_i; b_{-i})$. In the preceding auctions, each user $i$ submits a bid $b_i$ to maximize his *surplus function*

$$S_i(b_i; b_{-i}) = U(\theta_i, \gamma_i(b_i; b_{-i})) - C_i(b_i; b_{-i}). \tag{6}$$

Here we omit the dependence on $\beta$ and $\pi$.

An NE of the auction is associated with a bidding profile $\boldsymbol{b}^*$ such that $S_i(b_i^*; b_{-i}^*) \geq S_i(b_i'; b_{-i}^*)$ for any $b_i' \in [0, \infty)$ and any user $i$. Define user $i$'s *best response* given $b_{-i}$ as the set

$$\mathcal{B}_i(b_{-i}) = \left\{ \hat{b}_i \mid \hat{b}_i = \arg\max_{b_i \in [0,\infty)} S_i(b_i; b_{-i}) \right\}, \tag{7}$$

i.e., the set of $b_i$'s that maximize $S_i(b_i; b_{-i})$ given a fixed $b_{-i}$.[4] The NE bidding profile $\boldsymbol{b}^*$ is a fixed point, i.e., no user has the incentive to deviate unilaterally. The existence and uniqueness of an NE are shown in the following to depend on $\beta$ and $\pi$.

These auction mechanisms differ from some previously proposed auction-based network resource allocation schemes (e.g., [6], [7]) in that the bids here are not the same as the payments. Instead, the bids are signals of willingness to pay. The manager can therefore influence the NE by choosing $\beta$ and $\pi$. This alleviates the typical inefficiency of the NE, and allows us to reach Pareto optimal or even socially optimal solutions.

[3]If $h_{0i} = 0$ for all $i \in \{1, ..., M\}$, then the manager does not interfere with the users and many of the results in the following section still hold. However, in the co-located case, we have $h_{0i} = 1$ for all $i$.

[4]In general the best response set may contain more than one element.

## III. Finite System Analysis

### A. SINR-based Auction

In this case, $C_i(\gamma_i) = \pi^s \gamma_i = \pi^s \frac{p_i h_{ii}}{n_0 + \frac{1}{B}\left(\sum_{j \neq i}^{M} p_j h_{ji} + p_0 h_{0i}\right)}$, so that each user's payment depends on both the transmission power, and the interference. We focus on logarithmic utility functions, although the analysis applies to some other utility functions as well (e.g., $\theta_i \log(1+\gamma_i)$).

*Theorem 1:* In an SINR-based auction with logarithmic utility:

$(i)$ For $\beta > 0$, there exists a threshold price $\pi_{th}^s > 0$ such that a unique NE exists if $\pi^s > \pi_{th}^s$; otherwise, no NE exists.

$(ii)$ For $\beta = 0$, there are either an infinite number of Nash Equilibria, or no NE.

The proof is given in the appendix; as shown there, when $\beta > 0$ and $\pi_s > \pi_{th}^s$, the best response for each user is unique, and the vector of best responses across users is given by

$$\underline{\boldsymbol{b}}(\boldsymbol{b}) = \mathbf{K}\boldsymbol{b} + \boldsymbol{k}_0 \beta, \tag{8}$$

where the eigenvalue of the nonnegative matrix $\mathbf{K} = [k_{ij}]_{i,j \in \{1,\ldots M\}}$ with largest modulus, $\rho_K$, is real, and satisfies $0 < \rho_K < 1$, and the vector $\boldsymbol{k}_0 = (k_{10}, \ldots, k_{M0})$ has positive elements. The unique NE is

$$\boldsymbol{b}^* = (\mathbf{I} - \mathbf{K})^{-1} \boldsymbol{k}_0 \beta = \sum_{n=0}^{\infty} \mathbf{K}^n \boldsymbol{k}_0 \beta. \tag{9}$$

Notice that the value of $\beta$ does not affect the power allocation at the NE, since all equilibrium bids are proportional to $\beta$. Thus the manager only needs to announce an arbitrary $\beta > 0$. This observation can also be directly obtained from $(4)$, and so is applicable to any utility function. On the other hand, $\pi_{th}^s$ in Theorem 1 is typically difficult to find analytically. In the co-located receiver case, however, we have a closed-form relation between $\pi_{th}^s$ and the users' utility parameters. For $i \in \{1, \ldots, M\}$, define

$$k_i(\pi^s) = \frac{\theta_i (P + B n_0)}{B(\pi^s P - \theta_i n_0)}. \tag{10}$$

*Corollary 1:* In an SINR-based auction with logarithmic utilities and co-located receivers, $k_i(\pi_{th}^s) > 0$ for each user $i$ and $\sum_{i=1}^{M} k_i(\pi_{th}^s) / (1 + k_i(\pi_{th}^s)) = 1$. For $\beta = 0$, an infinite number of Nash Equilibria exist if $\pi^s = \pi_{th}^s$, otherwise no NE exists.

The proof of Theorem 1 can be modified for this corollary by using the fact that $k_{il}(\pi^s) = k_i(\pi^s)$ for all $l \in \{0, ..., M\}$, and then explicitly solving for the NE. The bidding and power profiles at the NE are:

$$b_i^* = \frac{\frac{k_i(\pi^s)}{1+k_i(\pi^s)}}{1 - \sum_{j=1}^{M} \frac{k_j(\pi^s)}{1+k_j(\pi^s)}}\beta \;\text{ and }\; p_i^* = \frac{k_i(\pi^s)}{1+k_i(\pi^s)}P \quad \text{for } i \in \{1, ..., M\}. \tag{11}$$

Given the existence of a unique NE, we next characterize the resulting resource allocation. We say an allocation $\{x_i\}_{i\in\{1,...,M\}}$ is *weighted max-min fair* with weights $\{w_i\}_{i\in\{1,...,M\}}$ if for each $i \in \{1, ..., M\}$, $x_i$ can not be increased without decreasing some $x_j$, $j \in \{1, ..., M\}$, for which $x_j/w_j \leq x_i/w_i$.

*Proposition 1:* If a unique NE exists in an SINR-based auction with logarithmic utilities, the SINR allocation $\{\gamma_i^*\}_{i\in\{1,...,M\}}$ and the payments $\{C_i^*\}_{i\in\{1,...,M\}}$ are weighted max-min fair with the weights $\{\theta_i\}_{i\in\{1,...,M\}}$ given a fixed reserve power $p_0^*$.

The proof is given in Appendix A. In [18], Kelly et al. consider an algorithm for rate allocation in a wire-line network with logarithmic utility functions $w_i \log(x_i)$ for all users $i \in \{1, ..., M\}$. In that case, the socially optimal rate allocation $\{x_i\}_{i\in\{1,...,M\}}$ is *weighted proportional fair* with weights $\{w_i\}_{i\in\{1,...,M\}}$, i.e., for any other feasible rate allocation $\{x_i'\}_{i\in\{1,...,M\}}$, $\sum_{i=1}^{M} w_i (x_i' - x_i)/x_i \leq 0$. Their utility maximization problem is convex and separable since there is no externality (i.e., interference) among different users. Here, due to the interference among users, the problem is generally not separable (except in the co-located receiver case) and is typically not convex; thus the allocation achieved by the SINR-based auction with logarithmic utilities typically is not socially optimal or proportional fair.[5]

We call a system *stable* if there exists a unique NE. In a stable system, define the *system usage efficiency* by $\eta = \sum_{i=1}^{M} p_i^* h_{i0}/P = \sum_{i=1}^{M} b_i^* / \left(\sum_{i=1}^{M} b_i^* + \beta\right)$. For Pareto optimality $\eta = 1$, but the necessary condition for stability is $\eta < 1$ due to the required positive reserve bid $\beta$, i.e., Pareto optimality and stability are conflicting objectives[6].

We define an $\varepsilon$-*system* as one with parameters $(P^\varepsilon, B^\varepsilon, M^\varepsilon, n_0^\varepsilon) = (P(1-\varepsilon), B, M, n_0 + \varepsilon P/B)$, where $\varepsilon \in (0,1)$. An $\varepsilon$-*Pareto optimal* allocation is defined as a Pareto optimal solution for the $\varepsilon$-system.

[5]Moreover, in this setting the socially optimal allocation with logarithmic utilities is not proportional fair.

[6]Here we are not including power used by the manager in our definition of Pareto optimality.

*Proposition 2:* In an SINR-based auction with logarithmic utility, there exists a unique price $\pi^{s\varepsilon}$ for any $\varepsilon \in (0,1)$, such that the system is stable and achieves an $\varepsilon$-Pareto optimal solution (i.e., $\eta = 1-\varepsilon$ in the original system).

*Proof:* From the proof of Theorem 1, it can be seen that as $\pi^s$ increases from $\pi^s_{th}$ to $\infty$, $\boldsymbol{k}_0$ and $\rho_K(\pi^s)$ continuously and monotonically decreases; in particular, $\rho_K(\pi^s)$ decreases from 1 to 0. Also, the NE bidding profile $\boldsymbol{b}^* = \left(\sum_{n=0}^{\infty} \mathbf{K}^n\right) \boldsymbol{k}_0 \beta$ continuously and monotonically decreases from $\infty$ (for at least one user's bid) to 0 (for all users' bids), and so does the summation $\sum_{i=1}^{M} b_i^*$, which means $\eta = \sum_{i=1}^{M} b_i^* / \left(\sum_{i=1}^{M} b_i^* + \beta\right)$ decreases from 1 to 0. So there must exist a unique price $\pi^{s\varepsilon} \in (\pi^s_{th}, \infty)$ that achieves any $\eta = 1-\varepsilon \in (0,1)$. ■

In practice, the manager can achieve a target $\eta^*$ by adjusting $\pi^s$ after observing the usage efficiency at the current NE: if it is too low, the price should be decreased. Note that if the price is decreased too much, the stability conditions in Theorem 1 may be violated.

### *B. Power-based Auction*

In this case $C_i(p_i) = \pi^p p_i$. For the co-located receiver case with logarithmic utility functions, Corollary 1 still holds with a more complicated expression for $k_i(\pi^s)$. The bidding and power profiles at the NE are again given by (11), but it may be impossible to find a price $\pi^{p\varepsilon}$ that gives an arbitrary $\eta = 1-\varepsilon$. This is because $U(\theta_i, \gamma_i)$ is not always concave in the received power $p_i^r$, and so the $p_i^r$ that maximizes user $i$'s surplus may not be continuous with price $\pi^p$, i.e., it may jump from one local optimum to the other. As a result, $\eta = \sum_{i=1}^{M} p_i^r / P$ may be discontinuous at some values.

We say that a power allocation is $\varepsilon$-*socially optimal* if it maximizes the total utility of the $\varepsilon$-system. In the case of co-located receivers, the power-based auction can achieve an $\varepsilon$-*socially optimal* allocation for a more general class of utility functions.

*Assumption 2:* For each $i \in \{1, ..., M\}$, $U(\theta_i, \gamma_i)$ satisfies Assumption 1 and

$$\frac{|U''(\theta_i, \gamma_i)|}{U'(\theta_i, \gamma_i)} (\gamma_i + B) > 2 \tag{12}$$

for any $\gamma_i \in [0, P/n_0]$.

Inequality (12) follows from letting $\partial^2 U(\theta_i, \gamma_i(p_i^r)) / \partial^2 p_i^r < 0$ for any $p_i^r \in [0, P/h_{i0}]$, i.e., the utility is strictly concave in the received power. In the case of logarithmic utility functions, Assumption $2$ is satisfied if $P/\left(Bn_0\right) < 0$ dB. For many utility functions (e.g., $\theta_i \log\left(1+\gamma_i\right)$, $1-e^{-\theta_i \gamma_i}$, and $\theta_i \gamma_i^{\alpha}$ $(\alpha \in (0,1))$), Assumption 2 is satisfied when the bandwidth is large enough, which implies that the interference among users is relatively small.

*Theorem 2:* In a power-based auction with co-located receivers and Assumption 2, there exists a price $\pi^{p\varepsilon}$ such that the system is stable and the NE achieves $\varepsilon$-social optimality for any $\varepsilon \in (0,1)$.

The proof of Theorem 2 involves writing the Kuhn-Tucker conditions [2] for the $\varepsilon$-socially optimal solution (which are necessary and sufficient under Assumption 2), as well as the best response of the users in the power-based auction. It can be shown that if the manager sets $\pi^p$ equal to the Lagrange multiplier for the received power constraint, the best responses of the users are equivalent to the Kuhn-Tucker conditions.

Theorem 2 shows that with large enough bandwidth, so that the externality effects among users are relatively small, the power-based auction with co-located receivers can achieve an allocation that is arbitrarily close to that produced by a VCG auction, and so is preferable to the SINR-based auction in terms of social optimality. When Assumption 2 is not satisfied, the power-based auction may not be able to achieve an $\eta$ close to $1$ (e.g., with logarithmic utilities); this can result in a lower total utility compared to the SINR-based auction, which can achieve any $\eta$.

### C. Revenue Comparison between SINR- and Power-Based Auctions

From the manager's point-of-view, revenue maximization might be another important objective. Here we restrict our discussion to the two auctions previously described for co-located receivers.[7] Let $R^p$ and $R^s$ be the revenue derived from the power- and SINR-based auctions, respectively. We first consider the case where users are symmetric (i.e., have the same utility functions) and the utilities are concave in power.

*Theorem 3:* In the co-located receiver case with symmetric utilities and Assumption 2, assume further that both auctions achieve the same system usage efficiency $\eta$. Then $R^p > R^s$, and $R^p/R^s \to 1$ as $M \to \infty$.

[7] We note that other auction mechanisms may extract more revenue.

*Proof:* For symmetric users, everyone is allocated the same received power $p^{r*}$ in both auctions with the same efficiency $\eta$. Let $U(\gamma(p^r)) = U(\gamma_i(p_i^r))$ for $1 \le i \le M$. From the first-order conditions for surplus maximization,

$$\pi^p = U'(\gamma(p^r))\,\gamma'(p^r)\,|_{p^r=p^{r*}} \text{ and } \pi^s = U'(\gamma(p^r))\,|_{p^r=p^{r*}} \tag{13}$$

so that

$$\frac{R^p}{R^s} = \frac{M\pi^p p^{r*}}{M\pi^s \gamma(p^{r*})} = \frac{U'(\gamma(p^r))\,\gamma'(p^r)\,|_{p^r=p^{r*}} p^{r*}}{U'(\gamma(p^r))\,|_{p^r=p^{r*}}\gamma(p^{r*})} = \frac{\frac{B(n_0B+P)}{(n_0B+P-p^{r*})^2}p^{r*}}{\frac{p^{r*}B}{n_0B+P-p^{r*}}} = \frac{n_0B+P}{n_0B+P-p^{r*}} > 1.$$

As $M \to \infty$, $p^{r*} \to 0$, and so $R^p/R^s \to 1$. ■

When Assumption 2 is not satisfied, the power-based auction may collect lower revenue than the SINR-based auction, since the former might not be able to achieve $\eta$ close to 1. However, for logarithmic utility functions the relation between the revenues remains the same.

*Proposition 3:* In the co-located receiver case with logarithmic utilities, assume there exists a $\bar{\theta}$ such that $\theta_i \le \bar{\theta}$ for $1 \le i \le M$. Then $R^p > R^s$ and $R^p/R^s \to 1$ as $M \to \infty$.

The proof is given in the appendix. Notice that in Proposition 3 we do not require symmetric users or the same $\eta$ in both auctions. Hence with logarithmic utilities the power-based auction always generates more revenue.

## IV. Large System Analysis

In this section we consider the asymptotic behavior as $P$, $B$, $M$ and $\beta$ go to infinity, while keeping $P/M$, $P/B$, $M/B$ and $\beta/M$ fixed. We focus on the co-located receiver case and assume that each user $i$'s utility parameter $\theta_i$ is independently chosen according to a continuous probability density $f(\theta)$ over $[\underline{\theta}, \bar{\theta}]$, where $0 \le \underline{\theta} < \bar{\theta} < \infty$. The expected value of $\theta$ is denoted as $E[\theta]$.

*Proposition 4:* In an SINR-based auction with logarithmic utility and co-located receivers, a unique NE exists in the limiting system if and only if

$$\pi^s > \pi^s_{th} = E[\theta](n_0 + P/B)\frac{M}{P}. \tag{14}$$

In this case, the power and SINR allocations at the NE are weighted max-min fair with weights $\{\theta_i\}_{1 \le i \le M}$, and user $i$ pays $\theta_i$. Otherwise, no NE exists.

The proof is given in the Appendix. The system usage efficiency at the NE is $\eta = \frac{E[\theta](n_0+P/B)}{\pi^s P/M}$. As $\eta \to 1$, the price $\pi^s$ converges to $\pi^s_{th}$, which is proportional to the system load $M/P$. This coincides with the *congestion pricing scheme* proposed in [15], where the equilibrium price reflects the system congestion.

In the limiting system with co-located receivers, all users receive the same fixed noise plus interference level $(n_0 + P/B)$ at the NE, because each user only gets a negligible proportion of the total power. This makes the SINR- and power-based auctions equivalent if $\pi^s = (n_0 + P/B)\,\pi^p$. The socially optimal allocation maximizes the average utility per user. (Note that the total utility is infinite.)

*Assumption 3:* The utility function $U(\theta, \gamma)$ is asymptotically sublinear with respect to $\gamma$, i.e.,

$$\lim_{\gamma \to \infty} \frac{1}{\gamma} U(\theta, \gamma) = 0, \ \forall \theta.$$

*Theorem 4:* In the limiting system with co-located receivers, if $U(\theta, \gamma)$ satisfies Assumptions 1 and 3, then both the SINR- and power-based auctions can achieve $\varepsilon$-social optimality for any $\varepsilon \in (0, 1)$.

A sketch of the proof is given in the Appendix D. Assumption 3 is valid for common utility functions, e.g., $\theta \ln(\gamma)$, $\theta \ln(1 + \gamma)$, and $\theta\gamma^\alpha$ for any $\alpha \in (0, 1)$, and any upper-bounded utility. Under this assumption, even if a finite number of users are allocated non-negligible proportions of the total power, their contributions to the average utility become negligible as the number of users increases. Because of this, the socially optimal allocation gives each user finite power, and so each user sees the same interference level $(n_0 + P/B)$. In this case, both auctions can achieve results that are arbitrarily close to that of a VCG auction.

## V. ITERATIVE AND DISTRIBUTED BID UPDATING ALGORITHM

In Sect. II, we assumed that the users' utility functions and all the channel gains are public information, so that the auction can be analyzed as a simultaneous-move game with complete information. In practice, the users' utility functions are likely to be private information, and it is difficult for user $i$ to measure the channel gains associated with other users, i.e., $h_{jk}$ for $j, k \neq i$. In that case, users cannot find the NE of the auction in one iteration. Next, we present an iterative and fully distributed algorithm that converges to the NE of these auctions[8].

[8]Note that here we are still referring to the Nash equilibrium of the simultaneous move game as in Sect. II-C.

As an example, consider the SINR-based auction with logarithmic utilities (not necessarily with co-located receivers). Suppose users update their bids according to the best response $(8)$ simultaneously in iterations $t = 1, 2, \cdots$, i.e.,

$$\boldsymbol{b}^{(t)} = \mathbf{K}\boldsymbol{b}^{(t-1)} + \boldsymbol{k}_0\beta, \tag{15}$$

where $\boldsymbol{b}^{(0)}$ is an arbitrarily chosen feasible initial bidding profile.

*Proposition 5:* If there exists a unique NE in the SINR-based auction with logarithmic utility functions, then the update algorithm $(15)$ globally converges to the NE from any $\boldsymbol{b}^{(0)}$.

*Proof:* For a unique NE we must have $\mathbf{K} \geq \mathbf{0}$ (componentwise), $\boldsymbol{k}_0 > 0$ and $\rho_K < 1$. Under these conditions iterating $(15)$ gives

$$\lim_{t\to\infty} \boldsymbol{b}^{(t)} = \lim_{t\to\infty} \left[\mathbf{K}^t\right] \boldsymbol{b}^{(0)} + \lim_{t\to\infty} \left[\sum_{n=0}^{t-1} \mathbf{K}^n\right] (\boldsymbol{k}_0\beta) = (\mathbf{I} - \mathbf{K})^{-1} \boldsymbol{k}_0\beta,$$

which is the unique NE. ■

Next, we show that $(15)$ can be equivalently written in a distributed fashion, where each user only needs to measure the channel gain $\hat{h}_{ii} = h_{ii}/h_{i0}$, the background noise density $n_0$, and his received SINR $\gamma_i^{(t)}$ in each iteration $t$.

*Proposition 6:* In the SINR-based auction with logarithmic utility functions, $(15)$ is equivalent to the following distributed updating algorithm for each user $i$ in iteration $t = 1, 2, ...$

$$b_i^{(t)} = \frac{\theta_i/\pi^s - \gamma_i^{(t-1)}\varphi_i}{\gamma_i^{(t-1)} - \gamma_i^{(t-1)}\varphi_i} b_i^{(t-1)}, \tag{16}$$

where $\varphi_i = n_0\theta_i/\left(\hat{h}_{ii}P\pi^s\right)$.

*Proof:* From the proof of Theorem 1, we know that by following the best response $(15)$ in iteration $t$, each user $i$ submits a bid $b_i^{(t)}$ in an attempt to achieve $\gamma_i\left(b_i^{(t)}; b_{-i}^{(t-1)}\right) = \theta_i/\pi^s$, which maximizes his surplus during iteration $t$ assuming the other bids are fixed at $b_{-i}^{(t-1)}$. Using $(4)$ and $(5)$, we have

$$b_i^{(t)} = \frac{\left(n_0\left(\sum_{j\neq i} b_j^{(t-1)} + \beta\right) + (P/B)\left(\sum_{j\neq i} b_j^{(t-1)}\hat{h}_{ji} + \beta h_{0i}\right)\right)\theta_i}{P\hat{h}_{ii}\pi^s - \theta_i n_0}. \tag{17}$$

Again using $(4)$ and $(5)$ for the SINR at iteration $t-1$, we have

$$n_0\left(\sum_{j\neq i}^{M} b_j^{(t-1)}+\beta\right)+(P/B)\left(\sum_{j\neq i}^{M} b_j^{(t-1)}\hat{h}_{ji}+\beta h_{0i}\right)=b_i^{(t-1)}\left(P\hat{h}_{ii}-\gamma_i^{(t-1)}n_0\right)/\gamma_i^{(t-1)}. \quad (18)$$

The desired result follows by substituting this into $(17)$. ■

The update $(16)$ requires only that user $i$ measure $\hat{h}_{ii}$. There is no need to know the other users' bids. This makes the algorithm completely distributed and easily scalable.

The update $(15)$ is similar to the Parallel Update Algorithm in [19], where users update their bids via a myopic strategy. Unlike the example in [19], here the sequence of bids does not oscillate if each user $i$ chooses an initial bid $b_i^{(0)}=0$. This is because the users' best responses have "strategic complementarity"– roughly, this means when one user submits a higher bid, the others want to do the same. Thus if all users start with the lowest bids, they converge monotonically to the unique NE in a stable system. Therefore incremental or random asynchronous updates do not improve convergence.

The update $(15)$ is mathematically similar to the power control algorithm proposed in [20] (see also [21], [22]) for a cellular network, where users adjust their powers (without any power constraints) to meet some preset target SINRs. In those papers, the matrix $\mathbf{K}$ depends only on the channel gains and the target SINRs, and so may not satisfy $\rho_K < 1$ (in which case there would not be a feasible allocation). There are several key differences between $(15)$ and the algorithm in [20]: (1) We consider elastic data traffic without a preset target SINR; (2) We have a total received power constraint; (3) We use the algorithm to adjust bids instead of the power itself; and (4) We can adjust the price so that a unique NE always exists. The mathematical similarity arises from the fact that by designing appropriate auction mechanisms, we convert the constrained power allocation problem into an unconstrained noncooperative game, in which each user updates his bid in an attempt to reach the desired equilibrium SINR level.

In practice, we would like to guarantee a unique NE, which requires $\pi^s > \pi^s_{th}$, and to achieve high efficiency $\eta$, which requires that $\pi^s$ be close to $\pi^s_{th}$, without knowing the exact value of $\pi^s_{th}$. The manager must adaptively search for a suitable price. In our simulations, we use the following search method:

1) Initialization: Set $(\underline{\pi},\overline{\pi})=(0,\infty)$; choose an arbitrary initial price $\pi^{(0)}>0$, and a maximum number of iterations $T$. Set $n=0$.

2) Start the auction at price $\pi^{(n)}$, set $n = n + 1$.
   a) If the auction does not converge within $T$ iterations, then stop. Let $\underline{\pi} = \pi^{(n-1)}$. If $\overline{\pi} = \infty$, set $\pi^{(n)} = 2\pi^{(n-1)}$; otherwise, set $\pi^{(n)} = \left(\underline{\pi} + \overline{\pi}\right)/2$. Go to 2.
   b) If the auction converges within $T$ iterations with $\eta < \eta^*$, then set $\overline{\pi} = \pi^{(n-1)}$ and $\pi^{(n)} = \left(\underline{\pi} + \overline{\pi}\right)/2$. Go to 2.
   c) If the auction converges within $T$ iterations with $\eta \geq \eta^*$, then stop.

Although we only discussed SINR-based auctions with logarithmic utilities, the bid updating algorithm also works for a power-based auction with co-located receivers and logarithmic utilities, as well as some other utility functions such as $U(\theta, \gamma) = \theta \log(1 + \gamma)$. [9]

## VI. Numerical Results

We next present some numerical results with logarithmic utility functions and co-located receivers. In these simulations, $\{\theta_i\}_{i \in \{1,\ldots,M\}}$ are independently and uniformly distributed in $[1, 100]$. Each graph represents the average result from $100$ independent realizations.

Fig. 2 shows average utility per user for the two auctions along with an upper bound obtained from solving the dual formulation for the utility maximization problem. In both auctions, we set the prices so that $\eta$ is close to $1$. From Theorem 2, the power-based auction achieves social optimality for $P/(Bn_0) < 0$ dB. Fig. $2(a)$ shows that the difference in utilities achieved by the two auctions is negligible in this regime. For $P/(Bn_0) > 0$ dB, the utility is not concave with power, hence the utility maximization problem may exhibit a duality gap. The two auctions achieve a utility close to the bound in this regime. In Fig. $2(b)$, we scale the system as in Sect. IV, and choose $P/(Bn_0) = 20$ dB so that the utility is not concave in power. When $M \leq 14$, the auctions do not achieve the upper bound on the maximum average utility. For large $M$, the utilities for both auctions and the socially optimal solution converge to a constant. For this example, the asymptotic behavior is accurate for $M \geq 14$.

Fig. 3 shows the performance of the distributed bid updating algorithm. Fig. $3(a)$ shows the users' bids starting from the origin and monotonically converging to the unique NE bids. Fig. $3(b)$ shows the performance of the updating algorithm as the system is scaled. The target system

[9] Again, we note that in some cases a target $\eta^*$ may not be achievable in the power-based auctions.

usage efficiency $\eta^*$ is chosen to be $0.90$, $0.95$ and $0.98$, respectively. We can see that the number of iterations needed for convergence increases with $M$ and approaches a constant when $M$ is large (i.e., $M > 20$). This shows that the algorithm scales well with the system size. The figure also shows that the number of iterations needed for convergence increases with $\eta^*$, implying that fast convergence and high system usage efficiency are generally conflicting objectives.

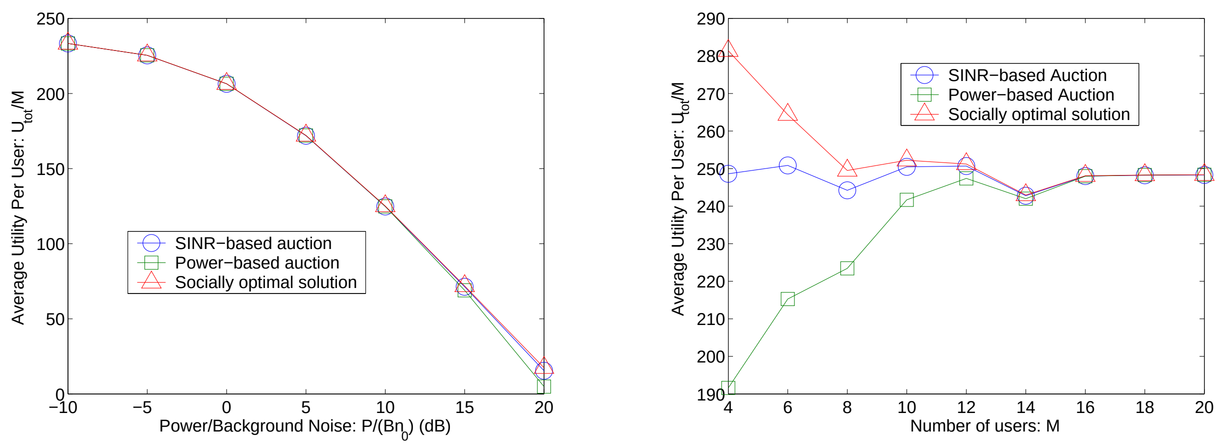


Fig. 2. The average utility for the two auctions and the maximum achievable utility with logarithmic utility functions and co-located receivers: $(a)$ finite system with $(P/n_0, M,) = (10^3, 10)$; $(b)$ system with $(P/n_0, B) = (10^4 M, 10^2 M)$.

## VII. Conclusions

We have considered spectrum sharing among a group of spread spectrum users with a constraint on the total interference temperature at a particular measurement point. We proposed two auction mechanisms, SINR- and power-based, that allocate power using a simple proportional bidding rule. When combined with logarithmic utilities, the SINR-based auction leads to a weighted max-min fair SINR allocation. The following results were obtained for the special case in which the receivers are co-located with the measurement point. Namely, the power-based auction maximizes the total utility with large enough bandwidth. Also, subject to certain assumptions on the utility functions, the power-based auction generates more revenue than the SINR-based auction, although the difference in revenue collected by the two auctions vanishes as the number of users increases. Both auction mechanisms achieve social optimality (i.e., maximize

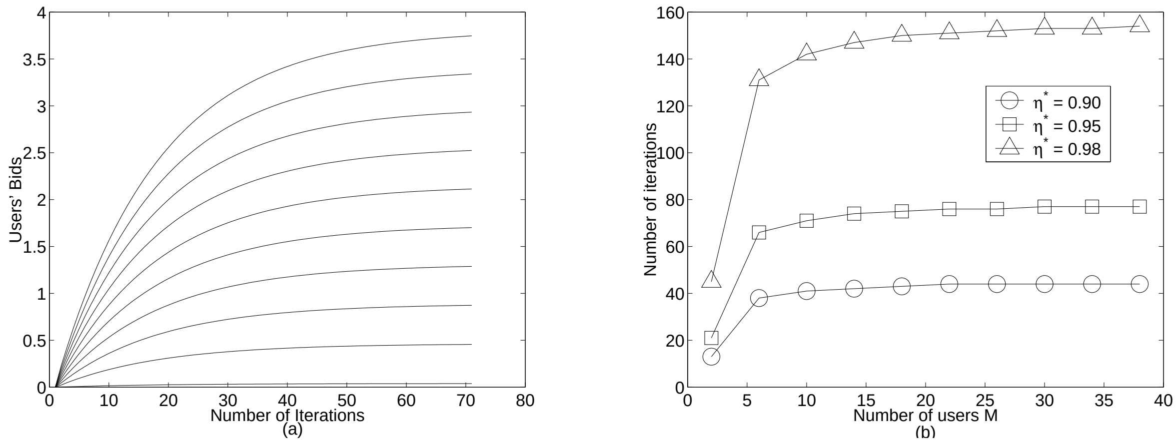


Fig. 3. Performance of the myopic bid updating algorithm with logarithmic utility functions and co-located receivers: $(a)$ bids for each user vs. iterations for a finite system with $(P/n_0, B, M, \beta) = \left(10^2, 10^3, 10, 1\right)$ and $\eta^* = 0.95$; $(b)$ Number of iterations required for a system with $(P/n_0, B) = \left(10^4 M, 10^2 M\right)$ and target $\eta^*$

utility per user) in the large system limit where bandwidth and power are increased in fixed proportion. We also presented an iterative, distributed bid updating algorithm, which for both auctions converges globally to the NE.

In this work we have assumed that the users and channels are static, and that the interference temperature is measured at a single location. Relaxing these assumptions leads to directions for future research. A related topic is how to assign bandwidth and power in the context of the Commons spectrum usage model, where there is no spectrum manager to preside over the resource allocation. In that situation, a primary goal is to avoid the "tragedy of commons".

## APPENDIX

### A. Proof of Theorem 1 and Proposition 1

We first specify the best response $\mathcal{B}\left(b_{-i}\right)$ for user $i \in \{1, ..., M\}$ with surplus

$$S_i\left(b_i; b_{-i}\right) = \theta_i \log\left(\gamma_i\left(b_i; b_{-i}\right)\right) - \pi^s \gamma_i\left(b_i; b_{-i}\right). \tag{19}$$

Define the normalized channel gain $\hat{h}_{ji} = h_{ji}/h_{j0}$ for all $j, i \geq 1$ so that

$$\gamma_i\left(b_i; b_{-i}\right) = \frac{b_i \hat{h}_{ii} P B}{n_0 B\left(\sum_{j=1}^{M} b_i + \beta\right) + P\left(\sum_{j \neq i} b_j \hat{h}_{ji} + \beta h_{0i}\right)}. \tag{20}$$

Notice that $\underline{b}_i > 0$, where $\underline{b}_i \in \mathcal{B}(b_{-i})$, since $S_i(0; b_{-i}) = -\infty$ for all $b_{-i}$. Since at the NE every user bids according to his best response, we can assume that $b_i > 0$ for all $i$. Differentiating $(19)$ with respect to $b_i$ yields

$$\frac{\partial S_i(b_i; b_{-i})}{\partial b_i} = \left(\frac{\theta_i}{\gamma_i(b_i; b_{-i})} - \pi^s\right) \frac{\partial \gamma_i(b_i; b_{-i})}{\partial b_i} \tag{21}$$

where

$$\frac{\partial \gamma_i(b_i; b_{-i})}{\partial b_i} = \frac{\left(n_0 B\left(\sum_{j\neq i} b_i + \beta\right) + P\left(\sum_{j\neq i} b_j \hat{h}_{ji} + \beta h_{0i}\right)\right)\hat{h}_{ii} P B}{\left(n_0 B\left(\sum_{j=1}^{M} b_i + \beta\right) + P\left(\sum_{j\neq i} b_j \hat{h}_{ji} + \beta h_{0i}\right)\right)^2} > 0. \tag{22}$$

Hence, the sign of $(21)$ only depends on the sign of

$$g(b_i) = \frac{\theta_i}{\gamma_i(b_i; b_{-i})} - \pi^s, \tag{23}$$

which is monotonically decreasing in $b_i$. Setting $g(b_i) = 0$ yields the unique solution

$$b_i = \sum_{j\neq i} k_{ij} b_j + k_{i0}\beta, \tag{24}$$

where

$$k_{ij} = \tfrac{\theta_i n_0}{P\pi^s \hat{h}_{ii} - \theta_i n_0} + \tfrac{\theta_i P}{B\left(P\pi^s \hat{h}_{ii} - \theta_i n_0\right)}\hat{h}_{ji} \text{ and } k_{i0} = \tfrac{\theta_i n_0}{P\pi^s \hat{h}_{ii} - \theta_i n_0} + \tfrac{\theta_i P}{B\left(P\pi^s \hat{h}_{ii} - \theta_i n_0\right)} h_{0i}. \tag{25}$$

If $\pi^s > \theta_i n_0 / \left(\hat{h}_{ii} P\right)$, then $k_{il} > 0$ for all $l \in \{0, ..., M\}$. In this case, $(24)$ is the unique maximizer of (19), and user $i$'s unique best response, given $b_{-i}$, is

$$\mathcal{B}_i(b_{-i}) = \sum_{j\neq i} k_{ij} b_j + k_{i0}\beta. \tag{26}$$

Therefore, if the auction has a unique NE $\boldsymbol{b}^*$, then it is the unique componentwise positive solution of the following matrix equation:

$$(\mathbf{I} - \mathbf{K})\,\boldsymbol{b} = \boldsymbol{k}_0\beta, \tag{27}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{K} = [k_{ij}]_{i,j\in\{1,...M\}}$ with $k_{ii} = 0$ for all $i$, and $\boldsymbol{k}_0 = (k_{10}, ..., k_{M0})$.[10] We consider two cases separately: $\beta > 0$ and $\beta = 0$.

**Case I ($\beta > 0$):** Define $\underline{\pi} = \max_{i\in\{1,...,M\}} \theta_i n_0 / \left(\hat{h}_{ii} P\right)$ and assume $\pi^s > \underline{\pi}$. Then $\mathbf{K}$ is a nonnegative matrix (i.e., all entries are nonnegative) and $\boldsymbol{k}_0$ is positive componentwise. We

[10] We denote all vectors as row vectors. The need for transposition should be clear from the context.

further assume that $\mathbf{K}$ is irreducible, which is true unless some users are completely isolated from each other.[11]. From the Perron-Frobenius theorem [23], the eigenvalue of $\mathbf{K}$ with maximum modulus is real, positive, simple, and the corresponding eigenvector is positive componentwise. Denote this eigenvalue as $\rho_K$. If $\rho_K < 1$, then $\lim_{n\to\infty} \mathbf{K}^n = 0$, and $(\mathbf{I} - \mathbf{K})^{-1} = \sum_{n=0}^{\infty} \mathbf{K}^n$ exists and is positive. In which case, there is a unique componentwise positive solution to $(27)$ given by

$$\boldsymbol{b}^* = \left(\sum_{n=0}^{\infty} \mathbf{K}^n\right) \boldsymbol{k}_0 \beta, \tag{28}$$

which represents the unique NE of the auction. On the other hand, if $\rho_K \geq 1$, then $\sum_{n=0}^{\infty} \mathbf{K}^n = \infty$, and the auction has no NE.

Next we show that $\rho_K$ is strictly decreasing with $\pi^s$. Let $\rho_K(\pi^s)$ and $k_{ij}(\pi^s)$ denote these parameters as functions of $\pi^s$. Let $\boldsymbol{x} = (x_1, ..., x_M)$ be a nonnegative vector. From Corollary $8.3.3$ of [23] and the fact that a square matrix has the same eigenvalues as its transpose, we have

$$\rho_K(\pi^s) = \max_{\substack{\boldsymbol{x}\geq 0 \\ \boldsymbol{x}\neq 0}} \min_{\substack{j\in\{1,...,M\} \\ x_j \neq 0}} \frac{1}{x_j} \sum_{i=1}^{M} k_{ij}(\pi^s) x_i. \tag{29}$$

Let $\boldsymbol{x}^*(\pi^s)$ be a vector that achieves $\rho_K(\pi^s)$ in $(29)$. Note that $\boldsymbol{x}^*(\pi^s)$ will have more than one positive entry, otherwise $\rho_K(\pi^s) = 0$. Assume that $\underline{\pi} < \pi^s < \pi^{s\prime}$. From $(25)$, $k_{ij}(\pi^s)$ is positive, continuous and strictly decreasing in $\pi^s > \underline{\pi}$. Hence,

$$\frac{1}{x_j} \sum_{i=1}^{M} k_{ij}(\pi^s) x_i > \frac{1}{x_j} \sum_{i=1}^{M} k_{ij}(\pi^{s\prime}) x_i, \tag{30}$$

for any nonnegative $\boldsymbol{x}$ that has more than one positive entry and $x_j \neq 0$. This implies that

$$\max_{\substack{\boldsymbol{x}\geq 0 \\ \boldsymbol{x}\neq 0}} \min_{\substack{j\in\{1,...,M\} \\ x_j \neq 0}} \frac{1}{x_j} \sum_{i=1}^{M} k_{ij}(\pi^s) x_i > \max_{\substack{\boldsymbol{x}\geq 0 \\ \boldsymbol{x}\neq 0}} \min_{\substack{j\in\{1,...,M\} \\ x_j \neq 0}} \frac{1}{x_j} \sum_{i=1}^{M} k_{ij}(\pi^{s\prime}) x_i, \tag{31}$$

i.e., $\rho_K(\pi^s) > \rho_K(\pi^{s\prime})$. Since each eigenvalue of a square matrix depends continuously upon its entries (see appendix $D$ of [23]), $\rho_K(\pi^s)$ is continuously and strictly decreasing in $\pi^s$ for $\pi^s > \underline{\pi}$.

[11]That is, there exists a set of users $\mathcal{U}$ such that $h_{ij} = 0$ for $i \in \mathcal{U}$ and $j \in \bar{\mathcal{U}}$ (complement of $\mathcal{U}$).

Now we show that there exist $\underline{\pi} < \pi_L^s < \pi_H^s$ such that $\rho_K(\pi_H^s) < 1 < \rho_K(\pi_L^s)$. First we consider $\pi_H^s$. From Theorem 8.1.22 of [23],

$$\rho_K(\pi^s) \leq \max_{j \in \{1,\ldots,M\}} \sum_{i \neq j}^{M} k_{ij}(\pi^s). \tag{32}$$

Thus it is sufficient to show that

$$\max_{i,j \in \{1,\ldots,M\}} k_{ij}(\pi_H^s) < \frac{1}{M-1}. \tag{33}$$

Using $(25)$ a sufficient condition for $(33)$ is

$$\pi_H^s > M\left(1 + \frac{P}{Bn_0} \max_{i,j \in \{1,\ldots,M\}} \left(\hat{h}_{ji}\right)\right)\underline{\pi} > \underline{\pi}. \tag{34}$$

To find a $\pi_L^s$, from $(29)$ it is sufficient to show that there exists an $\boldsymbol{x} > 0$ such that

$$\sum_{i=1}^{M} k_{ij}(\pi_L^s)\frac{x_i}{x_j} > 1, \forall j \in \{1,..,M\}. \tag{35}$$

Let $\pi_L^s = \underline{\pi} + \delta$, and $\tilde{\imath} = \arg\max_{i \in \{1,\ldots,M\}} \theta_i n_0 / \left(P\hat{h}_{ii}\right)$. First consider $j \neq \tilde{\imath}$, in which case it is sufficient to show that $k_{\tilde{\imath}j}(\pi_L^s) x_{\tilde{\imath}}/x_j > 1$, which is equivalent to

$$\frac{x_{\tilde{\imath}}}{x_j} > \frac{1}{k_{\tilde{\imath}j}(\pi_L^s)} = \frac{\delta}{\underline{\pi}\left(1 + \frac{P}{Bn_0}\hat{h}_{j\tilde{\imath}}\right)}, \forall j \neq \tilde{\imath}. \tag{36}$$

Next consider $j = \tilde{\imath}$; in this case it is sufficient to show that $k_{i\tilde{\imath}}(\pi_L^s) x_i/x_{\tilde{\imath}} > 1$ for some $i \neq \tilde{\imath}$. Defining $z = \min_{i \in \{1,\ldots,M\}, i \neq \tilde{\imath}} \left(\theta_i n_0 / \left(P\hat{h}_{ii}\right)\right)$, we have $k_{i\tilde{\imath}}(\pi_L^s) > z/(\underline{\pi} - z)$, and so it is sufficient to have

$$\frac{x_i}{x_{\tilde{\imath}}} > \frac{\underline{\pi} - z}{z}, \forall i \neq \tilde{\imath}. \tag{37}$$

An $\boldsymbol{x} > 0$ satisfying $(36)$ and $(37)$ can always be found for small enough $\delta$. One particular set of choices are $\hat{\delta} = \underline{\pi} z / (2(\underline{\pi} - z))$, $x_{\tilde{\imath}} = 1$ and $x_j = 2(\underline{\pi} - z)/z$ for all $j \neq \tilde{\imath}$, so that

$$\frac{\underline{\pi}\left(1 + \frac{P}{Bn_0}\hat{h}_{j\tilde{\imath}}\right)}{\hat{\delta}} > \frac{\underline{\pi}}{\hat{\delta}} = \frac{2(\underline{\pi} - z)}{z} = \frac{x_j}{x_{\tilde{\imath}}} > \frac{\underline{\pi} - z}{z}, \forall j \neq \tilde{\imath}. \tag{38}$$

This shows $\rho_K(\pi_L^s) = \rho_K\left(\underline{\pi} + \hat{\delta}\right) < 1$.

Since $\rho_K(\pi^s)$ is continuous and strictly decreasing with $\pi^s$, there must exist a $\pi_{th}^s \in (\pi_L^s, \pi_H^s)$ such that $\rho_K(\pi_{th}^s) = 1$. Thus the auction has a unique NE if $\pi^s > \pi_{th}^s > \underline{\pi}$, otherwise no NE exists.

**Case II ($\beta = 0$):** In this case, $(27)$ can be written as

$$(\mathbf{I} - \mathbf{K})\, \boldsymbol{b} = 0. \tag{39}$$

If $\mathbf{I} - \mathbf{K}$ is singular, then there are an infinite number of Nash equilibria associated with each (positive) solution for $\boldsymbol{b}$. If $\mathbf{I} - \mathbf{K}$ is nonsingular, then the unique solution is $\boldsymbol{b} = 0$. In that case, there is no NE, since $\underline{b}_i = 0$ cannot be a best response for user $i$.

Proposition 1 follows from setting $g(b_i)$ in $(23)$ to zero, in which case $\gamma_i^* = \frac{\theta_i}{\pi^s}$, and the payment of user $i$ at the NE is $C_i^* = \pi^s \gamma_i^* = \theta_i$.

### *B. Proof of Proposition 3*

With logarithmic utility and co-located receivers, the first-order conditions for surplus maximization for user $i$ gives

$$\pi^p = U_i'\left(\theta_i, \gamma_i\left(p_i^{r*}\right)\right) \gamma_i'\left(p_i^{r*}\right) = \frac{\theta_i\left(n_0 B + P\right)}{p_i^{r*}\left(n_0 B + P - p_i^{r*}\right)}. \tag{40}$$

Thus,

$$R^p = \sum_{i=1}^{M} \pi^p p_i^{r*} = \sum_{i=1}^{M} \frac{\theta_i\left(n_0 B + P\right)}{\left(n_0 B + P - p_i^{r*}\right)} > \sum_{i=1}^{M} \theta_i = R^s, \tag{41}$$

where the last equality is shown in the proof of Proposition 1. If $\theta_i \leq \bar{\theta}$ for each $i$, then as $M \to \infty$, $p_i^{r*} \to 0$ for each user $i$, and $R^p / R^s \to 1$.

### *C. Proof of Proposition 4*

We obtain $(14)$ by taking the limit of the conditions in Corollary $1$, under the assumed scaling. Let $Lim$ denote $\lim_{P,B,M\to\infty}$ with $P/B, P/M, \beta/M$ fixed. Thus,

$$Lim \sum_{i=1}^{M} \frac{k_i}{1+k_i} = Lim \sum_{i=1}^{M} \frac{\theta_i\left(P/B + n_0\right)}{P\left(\pi^s + \theta_i/B\right)} = \frac{1}{M} Lim \sum_{i=1}^{M} \frac{M\theta_i\left(P/B + n_0\right)}{P\pi^s} = \frac{P/B + n_0}{P/M\pi^s} E\left[\theta\right] \tag{42}$$

with probability $1$. The first equality follows from the definition of $k_i$ in $(10)$, the second follows from the limit $B \to \infty$, and the third follows from the strong law of large numbers. Condition $(14)$ then follows directly. The weighted max-min fair SINR allocation and payments stay fixed during the limiting process. Since every user sees the same noise plus interference at the NE, $n_0 + P/B$, we have $p_i^{r*} = \gamma_i^*(n_0 + P/B)$ for all $i$. This corresponds to a weighted max-min fair power allocation.

### D. Proof of Theorem 4 (Sketch)

In the limiting system, the maximum average utility per user is the solution to:

$$\begin{aligned} \underset{p(\theta)\geq 0}{\text{Max}} \quad & E_\theta \left[ U \left( \theta, \frac{p^r(\theta)}{n_0 + P/B} \right) \right] \\ \text{subject to:} \quad & E_\theta \left[ p^r(\theta) \right] = \frac{P}{M} (1 - \varepsilon) \end{aligned} \tag{43}$$

The objective is the average utility per user and the constraint corresponds to the total received power in the $\varepsilon$-system. In both cases, we have used the law of large numbers to express these in terms of expectations over $\theta$.

The optimization is over all received power allocations, $p^r : \left[\underline{\theta}, \overline{\theta}\right] \to \mathcal{R}^+$. It can be shown that each user receives a negligible fraction of the total power as the system scales, thus faces the same interference plus noise $n_0 + P/B$ at a solution to (43). This makes (43) a concave optimization problem. By using calculus of variations, we can solve for $p(\theta)$ in closed form, as well as the corresponding positive Lagrange multiplier $\lambda$ for the average power constraint. Letting $\pi^p = \lambda$ or $\pi^s = (n_0 + P/B)\lambda$ results in the same power allocation at the NE for the power- and SINR-based auctions, respectively.

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