Price of Anarchy for Routing Games with Incomplete Information

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***Abstract*— In this paper, we study a noncooperative traffic network where n users or agents need to send their traffic from a given source to a given destination over m identical, nonintersecting, parallel links. We consider a situation where each agent tries to send its traffic through the network in an entirely selfish manner, that is, aiming to minimize its individual delay. The model assumes that the amount of traffic which an agent wants to send through the network is known to the agent but not to the rest of the agents. However, each agent has a belief about the traffic loads of all other agents, expressed in terms of a probability distribution. For such a network, we wish to measure how bad selfish routing can be and this leads us to study the price of anarchy of such routing games. The model here turns out to be an incomplete information routing game, for which there are no studies yet on the price of anarchy. We develop an elegant framework for studying this by introducing the notion of traffic equations and traffic delay equations and using the solution concept of Bayesian Nash equilibrium. Our results show that the bounds on price of anarchy for incomplete information routing games are essentially the same as for complete information routing games. One possible interpretation of the results is that the worst case loss in performance due to selfish routing is not affected if the knowledge of agents about other agents is probabilistic rather than deterministic.**

***Index Terms*— Price of anarchy, routing games, complete information games, incomplete information games, Bayesian-Nash equilibrium**

I. Introduction

THe primary motivation for our work comes from several recent papers (for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]), where the authors study the degradation in network performance due to selfish behavior of noncooperative network users. For the purpose of measuring such a degradation, the authors use an index *price of anarchy* (so christened by Papadimitriou [12]). An important implicit assumption made by the authors while analyzing the underlying network game model is that all the users have complete information about the game, including information such as how much traffic is being sent through the network by the other users. That is, the traffic vector is common knowledge among the users.

The above assumption restricts the applicability of the model to real world traffic networks. In a traffic network, the users typically do not know how much traffic the other users are going to inject into the system at any given point. In this paper, we relax this assumption by saying that agents do not deterministically know the loads being injected by the other users. However, having observed the traffic pattern on the network over a sufficiently long time, each agent has a belief (that is, a probability distribution) about the loads of the other agents. In this more realistic scenario, the underlying network routing game becomes a *Bayesian game of incomplete information*, as opposed to the *game of complete information* considered in the existing literature. The obvious choice for solution of such a game is the *Bayesian-Nash equilibrium*, defined by Harsanyi [13]. In this paper, we ask and try to answer the question: By making the information of agents about other agents uncertain (read: probabilistic) rather than deterministic, what would happen to the worst case loss of performance due to selfish routing? Our results show that the bounds on price of anarchy for incomplete information routing games are essentially the same as those for complete information routing games. This would perhaps mean that the worst case loss in performance due to selfish routing is not affected by making the knowledge of agents about other agents probabilistic instead of deterministic.

A. Related Work

An attempt to maximize individual welfare by selfish users of any given traffic network (for example, transportation networks, fluid supply networks, electric networks, telephone networks, and more recently the Internet, etc.) may result in decreased performance of the network and hence poor utilization of the expensive infrastructure. This phenomenon has engaged the attention of users, service providers, and researchers since early 50's when the web of highways was being set up all around the globe. Since then it has been of immense interest for researchers to investigate the loss in social welfare due to the absence of a central authority that can route the traffic injected by selfish-noncooperative network users. The paper by Wardrop [14] is one of the early papers that shed light on this phenomenon. An equilibrium concept, the Wardrop equilibrium, emerged out of his investigation and has been extensively studied in transportation networks. Nash equilibrium [15] is another popular and extensively used solution concept that arises in game theoretic models of such networks. In fact, for a given network, these two equilibrium concepts are related to each other - see [16] for a rigorous discussion. In the recent years, researchers working in this area have focused on both types of equilibria. See for example

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[2] and [17] for Wardrop equilibria and [1], [12], [18], and [6] for Nash equilibria. Several recent papers (for example, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]), study the degradation in network performance due to selfish behavior of noncooperative network users. For the purpose of measuring such a degradation, the authors use an index *price of anarchy*.

Despite the vast literature available, to the best of our knowledge, traffic networks where users have incomplete information about the game have not been looked into yet from the point of view of price of anarchy. This is surprising considering the fact that games with incomplete information model the real-world much more accurately than games with complete information. A very recent paper by Beier, Czumaj, Krysta, and Vocking [19] does consider games with incomplete information, but is concerned with complexity and algorithmic issues in computing equilibria of such games.

B. Contributions

The main contribution of this paper is to investigate the effect of selfish routing in routing games with incomplete information. We propose a technical setup that works for games with complete information and generalizes naturally to games with incomplete information. The sequence in which we progress in this paper is as follows.

- *Traffic Equations:* First, we develop traffic equations for complete as well as incomplete information games. The idea behind these equations is to capture the dependency between agents' strategies for sending their traffic through the network and the ensuing congestion in the network.
- *Traffic Delay Equations:* Next, we develop traffic-delay equations for both the cases. These equations capture the dependency between agents' strategies for sending their traffic through the network and the ensuing delay to them.
- *Solution of the Game:* We formulate the above problem as an n-person noncooperative game and characterize the solution for it. In particular, we characterize Nash equilibria for the complete information case and Bayesian-Nash equilibria for the incomplete information case.
- *Bounds on Price of Anarchy:* Next, we consider price of anarchy for both the cases and compute upper bounds. The upper bound on price of anarchy for incomplete information games turns out to be the same as for complete information games.
- *Bounds on Price of Anarchy for Two Link Networks:* We show that for two link networks the price of anarchy is bounded above by $\frac{4}{3}$ when number of users are more than 2 and is independent of whether users have complete or incomplete information about the loads of the rival agents.

Our contribution is different from related work in the following way. We consider for the first time routing games with *incomplete information* in contrast to routing games with complete information discussed in the literature. In our work, we deal with the Bayesian Nash equilibrium which is the most natural solution concept for games with incomplete information. Our definition of social cost is the same as that employed by Roughgarden [2] and Lucking *et al* [3], but slightly different from that employed by Koutsoupias and Papadimitriou [12]. Our results for games with complete information consider the Nash equilibrium whereas Roughgarden [2] considers the Wardrop equilibrium.

II. The Model

Consider a network in which there are m identical, nonintersecting, parallel links, $\{L^1, L^2, \ldots, L^m\}$, to carry the traffic from source S to destination D. There are n users (agents) who have to send traffic from S to D. We identify each agent i by the symbol A_i and denote the set of all agents by $\mathcal{A}$, that is $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. The condition here is that the traffic injected by any agent A_i cannot be split over the links. We assume that each agent A_i can inject any amount of traffic from a given set $\mathcal{W}_i = \{1, 2, \ldots, K\}$. We shall use symbol w_i to denote the actual amount of traffic injected by A_i. We assume that before the show starts, the load w_i is the private information of agent A_i and is unknown to the rest of the agents. Sticking to standard phraseology used in the context of incomplete information games (Bayesian games) [20], we prefer to call w_i as the type of agent A_i and hence $\mathcal{W}_i$ becomes the set of all possible types of agent A_i. A few other symbols that will be used later are listed below.

$$
\begin{aligned}
\mathcal{W} &= \text{Set of type profiles of the agents} \\
&= \mathcal{W}_1 \times \mathcal{W}_2 \ldots \times \mathcal{W}_n \\
w &= \text{A type profile of the agents} \\
&= (w_1, w_2, \ldots, w_n); w \in \mathcal{W} \\
\mathcal{W}_{-i} &= \text{Set of type profiles of agents excluding } A_i \\
&= \mathcal{W}_1 \times \ldots \times \mathcal{W}_{i-1} \times \mathcal{W}_{i+1} \ldots \times \mathcal{W}_n \\
w_{-i} &= \text{A type profile of agents excluding } A_i \\
&= (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n); w_{-i} \in \mathcal{W}_{-i} \\
\triangle\mathcal{W} &= \text{Set of all probability distributions over } \mathcal{W} \\
\triangle\mathcal{W}_{-i} &= \text{Set of all probability distributions over } \mathcal{W}_{-i}
\end{aligned}
$$

We also assume that each agent A_i has a belief function $p_i : \mathcal{W}_i \mapsto \triangle\mathcal{W}_{-i}$. That is, for any possible type w_i of the agent A_i, the belief function specifies the probability distribution over the set $\mathcal{W}_{-i}$, representing what agent A_i would believe about the other agents' type if its own type were w_i. Thus for any $w_i \in \mathcal{W}_i$, $p_i(w_{-i}|w_i)$ denotes the subjective probability to the event that w_{-i} is the actual profile of the rival agents' type, if its own type were w_i. The beliefs $(p_i)_{\mathcal{A}}$ are said to be *consistent* [20] iff there is some common prior distribution over the set of type profiles w such that each agent's belief given its type is just the conditional probability distribution that can be computed from the prior distribution by Bayes formula. That is (in the finite case), beliefs are consistent iff there exists some probability distribution $P \in \triangle\mathcal{W}$ such that

$$p_i(w_{-i}\,|w_i\,) = \frac{P(w_{-i}, w_i)}{\sum\limits_{s_{-i} \in \mathcal{W}_{-i}} P(s_{-i}, w_i)};\ \forall A_i \in \mathcal{A}$$

In this paper, we will stick to this assumption of consistent beliefs. Under this assumption, the probability that agent A_i's type is w_i is given by the following relation.

$$t_i(w_i) = \sum_{w_{-i} \in \mathcal{W}_{-i}} P(w_{-i}, w_i)$$

Now consider the following problem with regard to this network. The agents are rational, intelligent, selfish, and non-cooperative, and this fact is common knowledge. Each agent *independently* tries to find out a strategy for routing its traffic that yields minimum expected delay. On the other hand, a central authority, if one exists, would like the agents to behave in a way that social welfare (that is, total delay) gets optimized. A conflicting situation of this type can be well studied by invoking the theory of noncooperative games with incomplete information. Before we can give the exact formulations, we need to define the notion of strategy of the agents and payoff to the agents.

Note that the decision problem of each agent is to choose the best link for sending its traffic through the network, hence the obvious choice for strategy set of any agent is the set of available links. Therefore, we define the set of all the links as a pure strategy set of any agent A_i and denote it by $\mathcal{L}_i$. That is, $\mathcal{L}_i = \{L^1, L^2, \ldots, L^m\}$. We shall be using the symbol l_i to denote a particular pure strategy of agent A_i. We also define a mixed strategy for agent A_i as any valid probability distribution over the set of pure strategies. We use the symbol $\mathcal{T}_i$ to denote the set of all the mixed strategies of agent A_i, and the symbol τ_i to denote a particular mixed strategy of agent A_i, that is, $\mathcal{T}_i = \triangle \mathcal{L}_i$, and $\tau_i \in \mathcal{T}_i$. A few more useful symbols are listed below.

$$\begin{aligned} \mathcal{L} &= \text{Set of pure strategy profiles of the agents} \\ &= \mathcal{L}_1 \times \mathcal{L}_2 \ldots \times \mathcal{L}_n \\ \mathcal{T} &= \text{Set of mixed strategy profiles of the agents} \\ &= \mathcal{T}_1 \times \mathcal{T}_2 \ldots \times \mathcal{T}_n \end{aligned}$$

We can define $\mathcal{L}_{-i}$ and $\mathcal{T}_{-i}$ in a similar way as we defined $\mathcal{W}_{-i}$ earlier. We use the lowercase letters to denote an element of the above sets, that is $l, l_{-i}, \tau, \tau_{-i}$ represent an element of the sets $\mathcal{L}, \mathcal{L}_{-i}, \mathcal{T}, \mathcal{T}_{-i}$, respectively.

The payoff to the agents here is the delay experienced by them. We assume that the delay experienced by any agent is equal to the total traffic passing through the link on which its own traffic is running. It is not difficult to see that in the case of incomplete information games, where each agent has a set of its possible types, the payoff of each agent not only depends on the strategy profile of all the agents but also on the types of all the agents. Therefore, we define $u_i : \mathcal{L} \times \mathcal{W} \mapsto \Re$ as a payoff function for agent A_i, where

$$\begin{aligned} u_i(l, w) &= \text{Payoff to agent } A_i \text{ while the agents have a} \\ &\quad \text{pure strategy profile } l \text{ and traffic profile } w \\ &= \sum_{j: l_j = l_i} w_j \\ u_i(\tau, w) &= \text{Payoff to agent } A_i \text{ while the agents have a} \\ &\quad \text{mixed strategy profile } \tau \text{ and traffic profile } w \\ &= \sum_{l \in \mathcal{L}} \left(\prod_{A_i \in \mathcal{A}} \tau_i(l_i) \right) u_i(l, w) \end{aligned}$$

The above model describes a Bayesian game, denoted by

$$\Gamma^b = \left\{ (\mathcal{A}), (\mathcal{L}_i)_{A_i \in \mathcal{A}}, (\mathcal{W}_i)_{A_i \in \mathcal{A}}, (p_i)_{A_i \in \mathcal{A}}, (u_i)_{A_i \in \mathcal{A}} \right\} \quad (1)$$

Harsanyi [13], proposed a way to represent such types of games in strategic form, which he called the *Selten Game*. Myerson [20] calls such a representation as *type-agent representation*. We also prefer calling the strategic form of such a game as type-agent representation. In the next section we first transform the Bayesian game Γ^b into a type-agent representation, denoted by Γ, and then start analyzing it.

A. Type Agent Representation for Bayesian Games of the Traffic Network

In the type-agent representation, there is one player or agent for every possible type of every player in the given Bayesian game. In order to differentiate these agents from the actual agents (i.e. network users), we prefer to call them as *traffic agents*. Thus, for the Bayesian game Γ^b given by (1), the set of agents in the type-agent representation becomes $\mathcal{A}^t = \bigcup_{i \in \mathcal{A}} \mathcal{A}_i^t$, where $\mathcal{A}_i^t = \{A_{i1}, A_{i2}, \ldots, A_{iK}\}$ represents the set of traffic agents for agent A_i.

The pure strategy and mixed strategy sets of traffic agent A_{ij} are the same as the pure and mixed strategies set of agent A_i. That is, $\mathcal{L}_{ij} = \mathcal{L}_i$ and $\mathcal{T}_{ij} = \mathcal{T}_i$. We shall use symbols l_{ij} and τ_{ij} to denote a particular pure and mixed strategy respectively for the traffic agent A_{ij}. A few more useful symbols are listed below for our future use.

$$\begin{aligned} \mathcal{L}^t &= \text{Set of pure strategy profiles of traffic agents} \\ &= \mathcal{L}_{11} \times \mathcal{L}_{12} \ldots \times \mathcal{L}_{nK} \\ \mathcal{T}^t &= \text{Set of mixed strategy profiles of traffic agents} \\ &= \mathcal{T}_{11} \times \mathcal{T}_{12} \times \ldots \times \mathcal{T}_{nK} \end{aligned}$$

Once again, $\mathcal{L}_{-ij}^t$, $\mathcal{L}_{-i}^t$, $\mathcal{T}_{-ij}^t$, and $\mathcal{T}_{-i}^t$ have their usual interpretations. Also, we shall use lowercase letters to denote an element of the above sets, that is $l^t, l_{-ij}^t, l_{-i}^t, \tau^t, \tau_{-ij}^t, \tau_{-i}^t$ represent an element of the sets $\mathcal{L}^t, \mathcal{L}_{-ij}^t, \mathcal{L}_{-i}^t, \mathcal{T}^t, \mathcal{T}_{-ij}^t, \mathcal{T}_{-i}^t$, respectively. Two other quantities that are of much use to us are $(l^t|w)$ and $(\tau^t|w)$. The first one represents the pure strategy profile of the agents for a given pure strategy profile of the traffic agents and a given type profile of the agents. The second quantity is a mixed strategy counterpart of the first one.

$$\begin{aligned} \left(l^t|w\right) &= \left(l_{1w_1}, l_{2w_2}, \ldots l_{nw_n}\right); \left(l^t|w\right) \in \mathcal{L} \\ \left(\tau^t|w\right) &= \left(\tau_{1w_1}, \tau_{1w_2}, \ldots \tau_{nw_n}\right); \left(\tau^t|w\right) \in \mathcal{T} \end{aligned}$$

In the type agent representation, the payoff to any traffic agent A_{ij} is defined to be the conditionally expected payoff to agent A_i in Γ^b given that j is A_i's actual type. Formally, for any agent A_i in $\mathcal{A}$ and any type w_i, the payoff function $v_{iw_i} : \mathcal{L}^t \mapsto \Re$ in the type-agent representation is defined in the following way.

$$v_{iw_i}(l^t) = \sum_{w_{-i} \in \mathcal{W}_{-i}} p_i\left(w_{-i}|w_i\right) u_i\left((l^t|w), (w_{-i}, w_i)\right) \quad (2)$$

Similarly, for the mixed strategy case, the payoff is given by the following equation.

$$v_{iw_i}(\tau^t) = \sum_{l^t \in \mathcal{L}^t} \left(\prod_{A_{pq} \in \mathcal{A}^t} \tau_{pq}(l_{pq}) \right) v_{iw_i}(l^t) \tag{3}$$

Substituting the value of equation (2) in equation (3) leads to the following alternative form of $v_{iw_i}(\tau^t)$:

$$v_{iw_i}(\tau^t) = \sum_{w_{-i} \in \mathcal{W}_{-i}} p_i\left(w_{-i}|w_i\right) u_i\left(\tau^t|w,w\right) \tag{4}$$

where $w = (w_{-i}, w_i)$.

With these definitions, the type-agent representation $\Gamma = \left\{(\mathcal{A}^t), (\mathcal{L}_{ij})_{A_{ij} \in \mathcal{A}^t}, (v_{ij}(.))_{A_{ij} \in \mathcal{A}^t}\right\}$ is indeed a game in strategic form and may be viewed as a representation of the given Bayesian game. Before moving to the next section, we would like to define the most important quantity, that is payoff to agent A_i in the incomplete information game. The following relations give the expected payoff to agent A_i when the pure strategy and mixed strategy profile of the traffic agents are l^t and τ^t, respectively.

$$\begin{aligned}
u_i(l^t) &= \text{Expected payoff to agent } A_i \text{ when pure strategy profile of the traffic agents is } l^t \\
&= \sum_{w \in \mathcal{W}} P(w) u_i(l^t|w,w) \\
u_i(\tau^t) &= \text{Expected payoff to agent } A_i \text{ when mixed strategy profile of the traffic agents is } \tau^t \\
&= \sum_{l^t \in \mathcal{L}^t} \left(\prod_{A_{pq} \in \mathcal{A}^t} \tau_{pq}(l_{pq}) \right) u_i(l^t) \\
&= \sum_{w \in \mathcal{W}} P(w) u_i(\tau^t|w,w) \qquad (5)
\end{aligned}$$

The second expression for $u_i(\tau^t)$ in equation (5) can be obtained by substituting the value of $u_i(l^t)$ in the first expression of $u_i(\tau^t)$. Also, by making use of equations (2) and (4), it is very simple to get the following alternative expressions for $u_i(l^t)$ and $u_i(\tau^t)$.

$$u_i(l^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(l^t) \tag{6}$$

$$u_i(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(\tau^t) \tag{7}$$

III. Traffic Equations

In this section, we compute the total traffic arising on any link due to a particular way in which agents behave. We call these equations *traffic equations*. We develop the traffic equations for both complete information and incomplete information games. First, we consider the complete information version of the above Bayesian game Γ^b, where we assume that the traffic of each agent is fixed and is known to every other agent before the game starts. Next we deal with the actual Bayesian game Γ^b.

A. The Complete Information Case

Let $M^j(l,w)$ be the traffic arising on link L^j when the agents' traffic profile is w, and they push it as suggested by strategy profile l. Similarly, $M^j(\tau,w)$ is the expected traffic arising on link L^j when the agents' traffic profile is w, and they push it as suggested by mixed strategy profile τ. The probabilities with which agents send traffic through the links will play a critical role in the development of the traffic equations. We call the probability of agent A_i choosing link L_j as contribution probability of agent A_i to traffic on link L^j and denote it by $q_i^j(\tau)$. That is, $q_i^j(\tau) = \tau_i(L^j)$. We define another useful quantity, $r_i^j(\tau,w)$, which is the contribution of agent A_i to the traffic on link L^j, provided that the agents' traffic profile is w and they push it as suggested by strategy profile τ. We define $r_i^j(\tau,w)$ to be equal to $q_i^j(\tau).w_i$, which is the expected traffic arising on link L^j due to agent A_i.

The following lemma summarizes the traffic equations for the complete information case.

Lemma 1: (Traffic equations for the Complete Information Case)[3]

$$\sum_{A_i \in \mathcal{A}} r_i^j(\tau,w) = M^j(\tau,w) = \left(r_i^j(\tau,w) + M^j(\tau_{-i},w_{-i})\right)$$

Remark: The essence of the above traffic equations is following. The expected traffic on link L^j, which is the sum of contributions of all agents towards the traffic on link L^j, is equal to the contribution of any agent A_i towards the traffic on link L^j plus the expected traffic on link L^j when agent A_i is removed from the scene.

Proof: The way we have defined the model, it is easy to see that the traffic on link L^j is given by the following relation.

$$M^j(l,w) = \sum_{k:l_k = L^j} w_k$$

Similarly, $M^j(\tau,w)$ is given by the following relation.

$$\begin{aligned}
M^j(\tau,w) &= \sum_{l \in \mathcal{L}} \left(\prod_{\mathcal{A}} \tau_i(l_i) \right) M^j(l,w) \\
&= \sum_{l_{-1} \in \mathcal{L}_{-1}} \left(\prod_{\mathcal{A} \setminus \{A_1\}} \tau_i(l_i) \right) \sum_{l_1} \tau_1(l_1) \left(M^j((l_{-1},l_1),w)\right) \qquad (8)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
M^j((l_{-1},l_1),w) &= M^j(l_1,w_1) + M^j(l_{-1},w_{-1}) \\
&= w_1 + M^j(l_{-1},w_{-1}) \qquad (9)
\end{aligned}$$

Substituting the value in equation (9) in equation (8) we get the following equation.

$$\begin{aligned}
M^j(\tau,w) &= \tau_1(L^j)w_1 + \sum_{l \in \mathcal{L}_{-1}} \left(\prod_{\mathcal{A} \setminus \{A_1\}} \tau_i(l_i) \right) M^j(l_{-1},w_{-1}) \\
&= \tau_1(L^j)w_1 + M^j(\tau_{-1},w_{-1})
\end{aligned}$$

[3]Note that l is a degenerate case of τ where each agent's mixed strategy is degenerate. Therefore, this lemma will go through even when τ is replaced by l with appropriate changes.

Unfolding the recursion in the above relation yields the first part of the statement. To get the second part of the statement, we just replace 1 by i in the above calculation. ∎

B. The Incomplete Information Case

Before we can write down the traffic equations for this case, we need to define the following quantities.

$$\begin{aligned} M^j(l^t) &= \text{Expected traffic on link } L^j \text{ when pure strategy profile of the traffic agents is } l^t \\ &= \sum_{w \in \mathcal{W}} P(w) M^j(l^t|w,w) \\ M^j(\tau^t) &= \text{Expected traffic on link } L^j \text{ when mixed strategy profile of the traffic agents is } \tau^t \\ &= \sum_{l^t \in \mathcal{L}^t} \left(\prod_{A_{pq} \in \mathcal{A}^t} \tau_{pq}(l_{pq}) \right) M^j(l^t) \\ &= \sum_{w \in \mathcal{W}} P(w) M^j(\tau^t|w,w) \end{aligned} \tag{10}$$

The second expression for $M^j(\tau^t)$ in the above equation (10) can be obtained by substituting the value of $M^j(l^t)$ in the first expression of $M^j(\tau^t)$.

$$\begin{aligned} M^j(l^t_{-uv}) &= \text{Expected traffic on link } L^j \text{ after removing the traffic agents } A_{uv} \text{ from the scene} \\ &= \sum_{w_{-u}} P(w_{-u}, w_u = v) M^j((l^t|w)_{-u}, w_{-u}) \\ &= t_u(w_u = v) \sum_{w_{-u}} p_u(w_{-u}|w_u = v) \\ &\quad M^j((l^t|w)_{-u}, w_{-u}) \\ M^j(l^t_{-u}) &= \text{Expected traffic on link } L^j \text{ after removing the agent } A_u \text{ from the scene} \\ &= \sum_{w_u} M^j(l^t_{-uw_u}) \\ M^j(\tau^t_{-uv}) &= \text{Expected traffic on link } L^j \text{ after removing the traffic agents } A_{uv} \text{ from the scene} \\ &= \sum_{l^t_{-uv} \in \mathcal{L}^t_{-uv}} \left(\prod_{A_{pq} \in \mathcal{A}^t \setminus A_{uv}} \tau_{pq}(l_{pq}) \right) M^j(l^t_{-uv}) \\ &= t_u(w_u = v) \sum_{w_{-u}} p_u(w_{-u}|w_u = v) \\ &\quad M^j((\tau^t|w)_{-u}, w_{-u}) \\ M^j(\tau^t_{-u}) &= \text{Expected traffic on link } L^j \text{ after removing the agent } A_u \text{ from the scene} \\ &= \sum_{w_u} M^j(\tau^t_{-uw_u}) \end{aligned} \tag{11}$$

The second expression for $M^j(\tau^t_{-uv})$ in the above definition (11) can be obtained by substituting the value of $M^j(l^t_{-uv})$ in the first expression of $M^j(\tau^t_{-uv})$.

$$\begin{aligned} q^j_{iw_i}(\tau^t) &= \text{contribution probability of traffic agent } A_{iw_i} \text{ to the traffic on link } L^j = t_i(w_i)\tau_{iw_i}(L^j) \\ Q^j_i(\tau^t) &= \text{contribution probability of agent } A_i \text{ to the traffic on link } L^j = \sum_{w_i} q^j_{iw_i}(\tau^t) \\ r^j_{iw_i}(\tau^t) &= \text{contribution of traffic agent } A_{iw_i} \text{ to the traffic on link } L^j = q^j_{iw_i}(\tau^t).w_i \\ R^j_i(\tau^t) &= \text{contribution of agent } A_i \text{ to the traffic on link } L^j = \sum_{w_i} r^j_{iw_i}(\tau^t) \end{aligned}$$

With the above definition, we can summarize the traffic equations for incomplete information case in the form of Lemma 2.

Lemma 2: (Traffic Equations for the Incomplete Information Case)

$$\begin{aligned} \sum_{A_i \in \mathcal{A}} R^j_i(\tau^t) &= M^j(\tau^t) = \left(R^j_i(\tau^t) + M^j(\tau^t_{-i}) \right) \\ \sum_{A_{pq} \in \mathcal{A}^t} r^j_{pq}(\tau^t) &= M^j(\tau^t) = \sum_{w_i} \left(r^j_{iw_i}(\tau^t) + M^j(\tau^t_{-iw_i}) \right) \end{aligned}$$

Remark: We prefer to call the first equation as the *agent version* and the second equation as the *traffic agent version* because of obvious reasons. The essence of the above traffic equations is following. The expected traffic on link L^j, which is the sum of contributions of all the agents towards the traffic on link L^j, is equal to the contribution of any agent A_i towards the traffic on link L^j plus the expected traffic on link L^j when the agent A_i is removed from the scene.

Proof: Recall from equation (10) that $M^j(\tau^t)$ is given by the following relation:

$$M^j(\tau^t) = \sum_{w \in \mathcal{W}} P(w) M^j(\tau^t|w,w)$$

Note that for a fixed traffic profile of the agents, a Bayesian game becomes a game of complete information and hence the value of $M^j(\tau^t|w,w)$, in the above equation, can be substituted using Lemma 1. This results in

$$M^j(\tau^t) = \sum_{w \in \mathcal{W}} P(w) \left(r^j_i(\tau^t|w,w) + M^j((\tau^t|w)_{-i}, w_{-i}) \right)$$

We will analyze the first and the second term of the above expression separately.

First Term: Recalling the way we have defined $r^j_i(\tau, w)$ in Section III-A, it is easy to see that $r^j_i(\tau^t|w,w) = \tau_{iw_i}(L^j)w_i$. Thus, the first term can be written as

$$\begin{aligned} &\sum_{w \in \mathcal{W}} P(w) r^j_i(\tau^t|w,w) \\ &= \sum_{w \in \mathcal{W}} P(w)\tau_{iw_i}(L^j)w_i = \sum_{w_i}\sum_{w_{-i}} P(w_i, w_{-i})\tau_{iw_i}(L^j)w_i \\ &= \sum_{w_i} t_i(w_i)\tau_{iw_i}(L^j)w_i = \sum_{w_i} r^j_{iw_i}(\tau^t) = R^j_i(\tau^t) \end{aligned}$$

Second Term:

$$
\begin{aligned}
& \sum_{w \in \mathcal{W}} P(w) M^j((\tau^t|w)_{-i}, w_{-i}) \\
= & \sum_{w_i} t_i(w_i) \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \\
= & \sum_{w_i} M^j(\tau^t_{-iw_i}) = M^j(\tau^t_{-i})
\end{aligned}
$$

Summing the first term and the second term gives the second part of the agent-version of the traffic equations. The first part of the same equation can be obtained by unfolding the recursion. The traffic agent version of the traffic equations is a direct consequence of the agent version. ∎

IV. Traffic-Delay Equations

The objective of this section is to compute the payoff (delay) of any agent due to a particular way in which agents behave. We call such an equation *traffic-delay equation*. On the lines of the traffic equations, we develop the traffic-delay equations for both the complete and the incomplete information cases. The description of these two cases is precisely the same as given in the previous section.

A. The Complete Information Case

Lemma 3: (Traffic-Delay Equations for Complete Information Case)

$$
u_i(\tau, w) \quad = \quad \sum_j \left(\tau_i(L^j)\right) \left(w_i + M^j(\tau_{-i}, w_{-i})\right)
$$

Remark: The essence of this equation is following. The expected delay of an agent A_i is equal to the summation, over all links, of the product of the probability that A_i sends its traffic on link L^j and the expected traffic on L^j that ensues when A_i sends its traffic on link L^j.

Proof: It is easy to see that $u_i(\tau, w)$ is given by following relation

$$
\begin{aligned}
& u_i(\tau, w) \\
&= \sum_{l \in \mathcal{L}} \left(\prod_{\mathcal{A}} \tau_k(l_k) \right) \left(\sum_{k : l_k = l_i} w_k \right) \\
&= \sum_{l_{-i} \in \mathcal{L}_{-i}} \left(\prod_{\mathcal{A} \setminus A_i} \tau_k(l_k) \right) \sum_{l_i \in \mathcal{L}_i} \tau_i(l_i) \left(\sum_{k : l_k = l_i} w_k \right) \\
&= \sum_{l_{-i} \in \mathcal{L}_{-i}} \left(\prod_{\mathcal{A} \setminus A_i} \tau_k(l_k) \right) \sum_j \tau_i(L^j) M^j((l_{-i}, l_i), w) \\
&= \sum_{l_{-i} \in \mathcal{L}_{-i}} \left(\prod_{\mathcal{A} \setminus A_i} \tau_k(l_k) \right) \sum_j \tau_i(L^j)(w_i + M^j(l_{-i}, w_{-i})) \\
&= \sum_j \left(\tau_i(L^j)\right) \left(w_i + M^j(\tau_{-i}, w_{-i})\right) \blacksquare
\end{aligned}
$$

B. The Incomplete Information Case

Lemma 4: (Traffic-Delay Equations for Incomplete Information Case - Traffic Agents Version)

$$
\begin{aligned}
& v_{iw_i}(\tau^t) \\
&= \sum_j \tau_{iw_i}(L^j) \left\{ w_i + \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \right\}
\end{aligned}
$$

Remark: The essence of this equation is following. The expected delay of a traffic agent $A_i w_i$ is equal to the sum, over all links, of the product of probability that traffic agent $A_i w_i$ sends its traffic on a particular link L^j and the expected value of traffic that ensues on L^j when $A_i w_i$ sends its traffic on L^j.

Proof: Recall from equation (4) that $v_{iw_i}(\tau^t)$ was defined in the following manner:

$$
v_{iw_i}(\tau^t) = \sum_{w_{-i} \in \mathcal{W}_{-i}} p_i\left(w_{-i}|w_i\right) u_i\left(\tau^t|w, w\right)
$$

Once again, we use the same argument that, for a fixed traffic profile of the agents, the Bayesian game becomes a game of complete information and hence the value of $u_i\left(\tau^t|w, w\right)$ in the above equation can be substituted from Lemma 3. This results in the following expression.

$$
\begin{aligned}
& v_{iw_i}(\tau^t) \\
&= \sum_{w_{-i}} p_i\left(w_{-i}|w_i\right) \sum_j \left(\tau_{iw_i}(L^j)\right) \left(w_i + M^j((\tau^t|w)_{-i}, w_{-i})\right)
\end{aligned}
$$

Some algebra on the above expression will prove the lemma. ∎

Lemma 5: (Traffic-Delay equations for Incomplete Information Case - Agents Version)

$$
u_i(\tau^t) \quad = \quad \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(\tau^t)
$$

Remark: The above lemma essentially conveys the same thing as equation (7). However, we have included it as a separate lemma because by invoking the previous lemma we can relate the expected delay of any agent A_i with the expected traffic on links. ∎

Note that in the above lemmas, we have made use of the fact that agents pick their types independently.

V. Characterizing the Solution of the Game

In this section, we characterize the game theoretic solution of the underlying routing game.

A. Nash Equilibria of Complete Information Games

Let us consider the Bayesian game Γ^b with a fixed type profile of the agent. In this situation, the game reduces to a game of complete information where each agent knows the type of every other agent, and Nash equilibrium is the standard solution concept. In this section we wish to characterize Nash equilibrium, if it exists, for such a routing game with complete information. The celebrated theorem of Nash [15] guarantees the existence of such an equilibrium in the case players are allowed to play with mixed strategy. Koutsoupias and

Papadimitriou have already characterized such an equilibrium in their paper [1]. We, however, recall it here as it would be insightful when we characterize Bayesian-Nash equilibrium for the incomplete information case. Myerson [20] has presented the characteristic equations for a Nash equilibrium by using the notion of support and we will use this characterization. Let $w = (w_1, w_2, \ldots, w_n)$ be a given traffic profile of the agents. Consider the following strategic form of the underlying routing game $\Gamma^c = (\mathcal{A}, (\tau_i)_{\mathcal{A}}, (u_i(.))_{\mathcal{A}})$, where τ_i is the set of mixed strategies of agent A_i and $u_i()$ is the payoff function of agent A_i. Let $S_i \subset \mathcal{L}_i$ be some nonempty subset of agent A_i's pure strategy set $\mathcal{L}_i$. If there is a Nash equilibrium τ with support $S = S_1 \times S_2 \times \ldots \times S_n$, then for each agent $A_i \in \mathcal{A}$, there must exist a number η_i such that

$$\eta_i = u_i((L^j, \tau_{-i}), w) = w_i + M^j(\tau_{-i}, w_{-i}) \; \forall L^j \in S_i \quad (12)$$

$$\eta_i \leq u_i((L^j, \tau_{-i}), w) = w_i + M^j(\tau_{-i}, w_{-i}) \; \forall L^j \notin S_i \quad (13)$$

$$\sum_{L^j \in S_i} \tau_i(L^j) = 1 \quad (14)$$

$$\tau_i(L^j) = 0 \; \forall L^j \notin S_i \quad (15)$$

$$\tau_i(L^j) \geq 0 \; \forall L^j \in S_i \quad (16)$$

Condition (12) says that each agent must get the same payoff, denoted by η_i, by choosing any of its pure strategies having positive probability under τ_i. Condition (13) says that playing with any other pure strategy outside of S_i would do no better to agent A_i than playing with any pure strategy inside of S_i. Recall that payoffs of the agents are the delays experienced by them. Therefore, every agent wants to minimize its payoff. Conditions (14) and (15) follow from the assumption that τ is a mixed strategy profile with support S. Condition (16) is obvious. Conditions (12)-(16) together imply that η_i is agent A_i's expected payoff under τ, because

$$u_i(\tau, w) = \sum_{l_i \in \mathcal{L}_i} \tau_i(l_i) u_i((\tau_{-i}, l_i), w) = \eta_i \quad (17)$$

Substituting the value of η_i from equations (12) and (13), we get the following relations:

$$\begin{aligned} u_i(\tau, w) &= w_i + M^j(\tau_{-i}, w_{-i}) \text{ if } L^j \in S_i \\ u_i(\tau, w) &\leq w_i + M^j(\tau_{-i}, w_{-i}) \text{ if } L^j \notin S_i \end{aligned}$$

Substituting the value of $M^j(\tau_{-i}, w_{-i})$ from Lemma 1, the above relations become:

$$\begin{aligned} u_i(\tau, w) &= M^j(\tau, w) + w_i \left(1 - \tau_i(L^j)\right) \text{ if } L^j \in S_i \\ u_i(\tau, w) &\leq M^j(\tau, w) + w_i \left(1 - \tau_i(L^j)\right) \text{ if } L^j \notin S_i \end{aligned}$$

The above relations can be used to bound from above the expected payoff of agent A_i in the case of Nash equilibria.

$$u_i(\tau, w) \leq \frac{1}{m} \left\{ (m-1) w_i + \sum_j M^j(\tau, w) \right\}$$

where m is the number of links. Further, it is easy to see that $\sum_j M^j(\tau, w) = \sum_i w_i$. Thus the above bound can be given by the following inequality

$$u_i(\tau, w) \leq \frac{1}{m} \left\{ (m-1) w_i + \sum_i w_i \right\} \quad (18)$$

B. Bayesian Nash Equilibria of Incomplete Information Games

For a Bayesian game with incomplete information, the most popular solution concept is a Bayesian-Nash (BN) equilibrium, proposed by Harsanyi [13]. Harsanyi defined a BN equilibrium to be any Nash equilibrium of the type agent representation in strategic form. That is, a BN equilibrium specifies a mixed strategy for each traffic agent, such that each agent would be maximizing (or minimizing) its own expected payoff when it knows its own type but does not know the other agents' types.

We consider the Bayesian routing game Γ^b defined earlier and characterize the Bayesian Nash equilibria for it. Recall that BN equilibria of Γ^b are basically Nash equilibria of the corresponding type agent representation Γ defined in Section II-A. If Γ has a Nash equilibrium τ^t with support $S^t = S_{11} \times S_{12} \times \ldots \times S_{nK}$, where $S_{ij} \subset \mathcal{L}_{ij}$, then for each traffic agent A_{iw_i}, there must exist η_{iw_i} such that

$$\eta_{iw_i} = v_{iw_i}((L^j, \tau^t_{-iw_i})) \; \forall L^j \in S_i w_i \quad (19)$$

$$\eta_{iw_i} \leq v_{iw_i}((L^j, \tau^t_{-iw_i})) \; \forall L^j \notin S_i w_i \quad (20)$$

$$\sum_{L^j \in S_{iw_i}} \tau_{iw_i}(L^j) = 1 \quad (21)$$

$$\tau_{iw_i}(L^j) = 0 \; \forall L^j \notin S_{iw_i} \quad (22)$$

$$\tau_{iw_i}(L^j) \geq 0 \; \forall L^j \in S_{iw_i} \quad (23)$$

It is easy to show that η_{iw_i} that we obtain by solving the above system of equations is indeed the expected payoff of traffic agent A_{iw_i} under τ^t. Further, by making use of Lemma 5, we get the following expression for the expected payoff of the agent A_i under Bayesian Nash equilibrium τ^t.

$$\sum_{w_i \in \mathcal{W}_i} t_i(w_i) u_i(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) \eta_{iw_i}$$

Taking the value of η_{iw_i} from relation (19) and (20), the above relation reduces to the following relations:

$$\begin{aligned} u_i(\tau^t) &= \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}((L^j, \tau^t_{-iw_i})) \text{ if } L^j \in S_{iw_i} \forall w_i \\ u_i(\tau^t) &\leq \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}((L^j, \tau^t_{-iw_i})) \text{ otherwise} \end{aligned}$$

Substituting the value of $v_{iw_i}((L^j, \tau^t_{-iw_i}))$ from Lemma 4, we get the following form of the above relations:

If $L^j \in S_{iw_i} \; \forall w_i \in \mathcal{W}_i$

$$u_i(\tau^t) = \sum_{w_i} t_i(w_i) \left\{ w_i + \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \right\}$$

otherwise

$$u_i(\tau^t) \leq \sum_{w_i} t_i(w_i) \left\{ w_i + \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \right\}$$

Further by substituting the value of $M^j((\tau^t|w)_{-i}, w_{-i})$ from Lemma 1, we get the following alternative form of the above expressions:

If $L^j \in S_{iw_i}\ \forall w_i \in \mathcal{W}_i$

$$u_i(\tau^t) = \sum_{w_i} t_i(w_i)w_i + \sum_{w} P(w)\left\{M^j((\tau^t|w),w) - \tau_{iw_i}(L^j)w_i\right\} \quad (24)$$

Otherwise

$$u_i(\tau^t) \leq \sum_{w_i} t_i(w_i)w_i + \sum_{w} P(w)\left\{M^j((\tau^t|w),w) - \tau_{iw_i}(L^j)w_i\right\} \quad (25)$$

If we sum up $u_i(\tau^t)$ over all the links and make use of the above relations then it is easy to get the following bound:

$$u_i(\tau^t) \leq \frac{1}{m}\left\{(m-1)\sum_{w_i} t_i(w_i)w_i + \sum_{w} P(w)\sum_{i} w_i\right\} \quad (26)$$

Note that the above expression is analogous to the expression (18) in following sense.

- The load w_i of agent A_i in expression (18) is replaced by $\sum_{w_i \in \mathcal{W}_i} t_i(w_i)w_i$, which is the expected load of agent A_i in the case of the incomplete information game.
- The total load $\sum_i w_i$ in expression (18) is replaced by $\sum_w P(w)\sum_i w_i$, which is the expected total load of all the agents.

VI. Price of Anarchy

In this section, we define the price of anarchy for the routing game discussed. We also derive an upper bound for it under both the complete and the incomplete information cases. Note that Koutsoupias and Papadimitriou [1] have already computed the bounds for the complete information case. However, the way we define social cost here is different than the way they have defined.

A. Price of Anarchy for the Complete Information Case

Once again we consider the Bayesian game Γ^b with a fixed type profile of the agents. Let $w = (w_1, w_2, \ldots, w_n)$ be a given type profile of the agents. Now the underlying routing game becomes a game of complete information for which the strategic form is given by $\Gamma^c = (\mathcal{A}, (\tau_i)_{\mathcal{A}}, (u_i(.))_{\mathcal{A}})$, where τ_i and $u_i(.)$ have their usual interpretation. For the game Γ^c, we define the following quantities.

$$\begin{aligned}
S(\tau) &= \text{Social cost under mixed strategy profile } \tau \\
&= \sum_i u_i(\tau, w) \\
\underline{S} &= \text{Optimal social cost} = \min_{\tau} S(\tau) \\
\underline{S^*} &= \text{Social cost under the best Nash equilibrium} \\
&= \min_{\tau:\tau \text{ is a NE}} \{S(\tau)\} \\
\overline{S^*} &= \text{Social cost under the worst Nash equilibrium} \\
&= \max_{\tau:\tau \text{ is a NE}} \{S(\tau)\} \\
\overline{S} &= \max_{\tau} S(\tau)
\end{aligned}$$

The following inequality is a trivial consequence of the above definitions.

$$\underline{S} \leq \underline{S^*} \leq \overline{S^*} \leq \overline{S}$$

We define the price of anarchy for the game Γ^c by the ratio ϕ in the following way.

$$\phi = \frac{\overline{S^*}}{\underline{S}} = \frac{\text{Social cost under the worst Nash equilibrium}}{\text{Optimal social cost}}$$

The motivation behind the above ratio comes from the following questions: (a) How badly can the user's rational pursuit (minimizing individual expected delay) affect the social welfare (minimizing total expected delay)? (b) How much improvement can one gain in social welfare by switching to a centralized control of traffic routing as opposed to the case where each agent is free to send its traffic on any link it wishes? We wish to compute the upper bound for the above ratio. By making use of the upper bound for $u_i(\tau, w)$, given in (18), it is easy to show that

$$\text{If } m = 1 \text{ then } \overline{S^*} = \underline{S} = n\sum_i w_i$$

$$\text{Otherwise } \overline{S^*} \leq \frac{n + (m-1)}{m}\sum_i w_i\ ; \underline{S} \geq \sum_i w_i$$

The above relations result in the following theorem about bounds on price of anarchy for complete information routing games:

Theorem 1: For a complete information routing game with m identical parallel links and n users, the price of anarchy ϕ can be bounded in the following way:

$$\begin{aligned}
&\text{If } m = 1 \text{ then} && \phi = 1 \\
&\text{Otherwise} && 1 \leq \phi \leq \left\{\frac{n + (m-1)}{m}\right\}
\end{aligned}$$

Remarks:

- The above bound is not a tight one for the case when $1 < m$. However, given a value of m, one can use the structure of the problem and come up with a better approximation of $\underline{S}$ and get a tighter bound. The case $m = 2$ is very interesting and we shed some light on it in Section VII.
- Another interesting case is when $m = n$. For this case, it is easy to see that $1 \leq \phi \leq 2 - \frac{1}{m}$.
- For $m = (n-1)$, $1 \leq \phi \leq 2$.

B. Price of Anarchy for the Incomplete Information Case

Now we consider the generalized version of the game Γ^c, that is the Bayesian game Γ^b, and define the price of anarchy for it. First, recall that the type agent representation of Bayesian game
$\Gamma^b = \left\{(\mathcal{A}), (\mathcal{L}_i)_{A_i \in \mathcal{A}}, (\mathcal{W}_i)_{A_i \in \mathcal{A}}, (p_i)_{A_i \in \mathcal{A}}, (u_i)_{A_i \in \mathcal{A}}\right\}$
is indeed the strategic form of the game and is given by
$\Gamma = \left\{(\mathcal{A}^t), (\mathcal{L}_{ij})_{A_{ij} \in \mathcal{A}^t}, (v_{ij})_{A_{ij} \in \mathcal{A}^t}\right\}$. With regard to the above type-agent representation Γ, we define the following

quantities.

$$\begin{aligned} S(\tau^t) &= \text{Social cost under mixed strategy profile } \tau^t \\ &= \sum_i u_i(\tau^t) \\ \underline{S} &= \text{Optimal social cost} = \min_{\tau^t} S(\tau^t) \\ \underline{S^*} &= \text{Social cost under the best possible Bayesian} \\ &\quad \text{Nash equilibrium} = \min_{\tau^t:\tau^t \text{ is a BNE}} \left\{S(\tau^t)\right\} \\ \overline{S^*} &= \text{Social cost under the worst possible Bayesian} \\ &\quad \text{Nash equilibrium} = \max_{\tau^t:\tau^t \text{ is a BNE}} \left\{S(\tau^t)\right\} \\ \overline{S} &= \max_{\tau^t} S(\tau^t) \end{aligned}$$

The following inequality is a direct consequence of the above definitions:

$$\underline{S} \leq \underline{S^*} \leq \overline{S^*} \leq \overline{S}$$

We define the price of anarchy for the Bayesian game Γ^b by the following ratio ψ:

$$\psi = \frac{\overline{S^*}}{\underline{S}} = \frac{\text{Social cost under the worst BN equilibrium}}{\text{Optimal social cost}}$$

In what follows, we compute an upper bound on ψ. By making use of the upper bound for $u_i(\tau^t)$, given in (26), it is easy to show that

$$\text{If } m = 1 \text{ then } \overline{S^*} = \underline{S} = n \sum_w P(w) \sum_i w_i$$

Otherwise

$$\overline{S^*} \leq \frac{(m-1)\sum_{i=1}^{n}\sum_{w_i \in \mathcal{W}_i} t_i(w_i)w_i + n\sum_w P(w)\sum_i w_i}{m}$$

$$\underline{S} \geq \sum_i^n \sum_{w_i \in \mathcal{W}_i} t_i(w_i)w_i$$

We have made use of equation (24) to bound $\underline{S}$ from below when $1 < m$. The above relation results in the following bounds on price of anarchy for incomplete information routing games.

$$\text{If } m = 1 \text{ then } \psi = 1$$

Otherwise

$$1 \leq \psi \leq \frac{1}{m}\left\{(m-1) + n\frac{\sum_w P(w)\sum_i w_i}{\sum_i^n \sum_{w_i \in \mathcal{W}_i} t_i(w_i)w_i}\right\}$$

A little algebra shows that

$$\sum_w P(w) \sum_i w_i = \sum_i^n \sum_{w_i \in \mathcal{W}_i} t_i(w_i)w_i$$

The following theorem captures the bounds on the price of anarchy for the incomplete information case.

Theorem 2: For an incomplete information routing game with m identical parallel links and n users, the price of anarchy ψ is bounded by

$$\text{If } m = 1 \text{ then} \quad \psi = 1$$

$$\text{Otherwise} \quad 1 \leq \psi \leq \left\{\frac{n + (m-1)}{m}\right\}$$

Remarks:

- Note that the bounds on price of anarchy is the same in both the complete information and incomplete information cases.
- As mentioned earlier, the above bound is not a tight one for the case when $1 < m$. However, for a particular value of m, one can find a better approximation of $\underline{S}$ and get a tighter bound. The case with $m = 2$ is treated in Section VII.
- For $m = n$, $1 \leq \psi \leq 2 - \frac{1}{m}$
- For $m = (n-1)$, $1 \leq \psi \leq 2$.

VII. Price of Anarchy for Two Link Networks

A. The Complete Information Case

Lemma 6: For a complete information routing game with two identical parallel links and n users, the optimal social cost $\underline{S}$ is bounded below by $\frac{1}{2}n\sum_i w_i$.

Proof: The outline for the proof is the following. First we show that $S(l) = \sum_i u_i(l,w) \geq \frac{1}{2}n\sum_i w_i$ for any $l \in \mathcal{L}$. This would imply that $S(\tau) = \sum_i u_i(\tau,w) \geq \frac{1}{2}n\sum_i w_i$ for any $\tau \in \mathcal{T}$. This indeed will imply that $\underline{S} = \min_\tau S(\tau) \geq \frac{1}{2}n\sum_i w_i$.

By the definition of $u_i(l,w)$, it is easy to see that for $m = 2$, $S(l)$ can be viewed as a total weight of two bins packed with n objects with individual object weights given by $w_1, w_2, \ldots, w_n$. The way the items are assigned to the bins depends on l. The weight of each bin is defined to be the total weight of all the items in the bins multiplied with number of items in the bin. This is a combinatorial bin packing problem for which it can easily be shown that total weight of two bins cannot be less than $\frac{1}{2}n\sum_i w_i$. ∎

By making use of Lemma 6 and the an earlier upper bound for $\overline{S^*}$, we get the following result.

Theorem 3: For a complete information routing game with two identical parallel links and n users, the price of anarchy ψ can be bounded in following way:

$$\begin{aligned} &\text{If } n = 1 &&\text{then} \quad \phi = 1 \\ &\text{If } n = 2 &&\text{then} \quad 1 \leq \phi \leq \frac{3}{2} \\ &\text{If } 2 < n &&\text{then} \quad 1 \leq \phi \leq \left\{1 + \frac{1}{n}\right\} \leq \frac{4}{3} \end{aligned}$$

Remark: Roughgarden [2] has already proved a similar result for flow networks. However, Roughgarden works with Wardrop equilibria rather than Nash-equilibria.

B. The Incomplete Information Case

Lemma 7: For an incomplete information routing game with two identical parallel links, n users, and a common prior distribution $P(w)$, the optimal social cost $\underline{S}$ is bounded below by $\frac{1}{2}n\sum_w P(w)\sum_i w_i$.

Proof: The proof is similar to Lemma 6.

By making use of Lemma 6 and the an earlier upper bound for $\overline{S^*}$, we get the following result.

Theorem 4: For an incomplete information routing game with two identical parallel links and n users, the price of anarchy ψ can be bounded in the following way:

$$\begin{array}{lll} \text{If } n=1 & \text{then} & \psi = 1 \\ \text{If } n=2 & \text{then} & 1 \leq \psi \leq \frac{3}{2} \\ \text{If } 2<n & \text{then} & 1 \leq \psi \leq \left\{1+\frac{1}{n}\right\} \leq \frac{4}{3} \end{array}$$

VIII. Conclusions

In this paper we have studied a traffic network that consists of m identical, parallel, nonintersecting links between the source and destination. We showed that the bounds on price of anarchy for routing games with incomplete information are essentially the same as for games with complete information. This perhaps implies that, with only probabilistic information about other agents, the worst case loss in welfare is not worse than when every agent knows deterministically about other agents.

We feel the results are both intuitive and counter-intuitive. The results are intuitive because the price of anarchy looks at the worst case loss in performance. The results are counter-intuitive because in the incomplete information case, the knowledge of an agent about other agents is only probabilistic. With uncertain information about other agents, we would expect the agents to be less smart and less informed than when the knowledge about other agents is known with certainty. The notion of Bayesian Nash equilibrium plays a key role in settling this issue.

The results show that only the bounds on price of anarchy are the same for the two types of games. It does not still settle the more general question: how does selfish routing affect (improve or degrade) the actual performance when agents have probabilistic rather than deterministic information about other agents?

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